Bol. Soc. Paran. Mat. (3s.) **v. 36** 4 (2018): 9–21.
©SPM –ISSN-2175-1188 ON LINE ISSN-00378712 IN PRESS
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v36i4.31139

Attractors and Commutation Sets in Hénon-like Diffeomorphisms *

Wissame Selmani and Ilham Djellit

ABSTRACT: In this work we display Hénon-like attractors that emerge and appear in diffeomorphisms generated by embedding of one-dimensional endomorphisms.We show the properties of basin of attraction, and identify various types of attractors and commutation sets which are associated with these diffeomorphisms. Numerically presented scenarii of the creation and destruction of these attractors via bifurcations are illustrated.

Key Words: Attractors, Commutation sets, Heteroclinic and homoclinic bifurcations.

Contents

1. Introduction

Hénon map is considered as the simplest diffeomorphism possessing important properties that contributed greatly to our understanding of complex and chaotic dynamics. We study a diffeomorphism in dependence of four parameters, and which can be considered as a purely artificial model coming from the embedding of a one-dimensional noninvertible map into a two-dimensional invertible one with constant Jacobian.

The dynamics involves various transitions by bifurcations. In this respect, it can be compared with classical examples such as the generalized Hénon maps. On the one hand, our study concerns a noninvertible map embedded into the invertible one and local bifurcations. On the other hand, we have to deal with global bifurcations of observable sets, such as crises of attractors or metamorphoses

* This work was partially supported by Research Project Grant B01120130023
2010 *Mathematics Subject Classification:* 34C28, 34C37, 37D45.
Submitted February 26, 2016. Published November 02, 2016

of basin boundaries, which are the most easily detected and probably the most often described in scientific literature.

There has been growing interest for homoclinic and heteroclinic phenomena, they have been the most studied objects in dynamical systems. From the qualitative point of view homoclinic phenomena are of interest because they represent a possible source for complex dynamics. It has been recognized that connecting orbits and their bifurcations play an important role in the qualitative theory of dynamical systems. Strong effort has gone into describing the different bifurcations that can occur in terms of genericity and into determining the different types of behavior in systems undergoing homoclinic and heteroclinic bifurcations.

A very influential work is used here, concerning the importance of commutation sets done by Mira and Gracio who have developed the fascinating role of these sets to delimit chaotic attractors (see [3]). Here the dynamical features are explored by numerical methods. Also, in several cases we find interesting dynamical objects predicted by the theory and global phenomena in the parameter plane. This kind of scanning has been made for giving a first idea about bifurcation organization.

Let's start with "the generalized logistic map " in dimension 1, the system presents a population evolution model, which generalizes the logistic models that are proportional to the beta densities with the shape parameters p and q, such that $p, q > 1$, and the growth rate r.

The complex dynamical behavior of these models is studied in the plane (r, p) using explicit methods when the parameter r increases. Anticipating the future evolution of population's dynamics is one of the most important issue in several domains, such as biological, ecological, social or economical sciences.

Rocha and *al.* in [4] introduced some basic concepts and results on probability density functions. They showed that the sequence

$$f_{r,2,2}(x) = rx(1-x)$$

is proportional to Beta density $Beta(2, 2)$ for $x \in [0, 1]$ and $r > 0$. This sequence is a simplified population model. For a small initial condition, that's mean a low population, the growth rate in years n is exponential. For a large initial condition, the population is more important for the same space and the same food, so it will be increase.

Also they have studied the complex dynamical behavior of some models of the main following form

$$f_{r,p,q}(x) = rx^{(p-1)}(1-x)^{(q-1)}$$

which are proportional to $Beta(p, q)$ densities, where the variable $x \in [0, 1]$ and the parameters $p, q > 1$.
In the particular case of $q = 2$, these models are typically used to study of whales population and forest fires. The parameter p measures the difficulty of the mating process.

They have considered an extension of the function Beta and the density Beta to approach the dynamical system of Verhulst, which symbolizes the study of the birth and death processes of a population of one species, represented for $p \in N - \{1, 2\}$ by the map:

$$N(t_{n+1}) = r^* N(t_n)^{p-1} \left(1 - \frac{N(t_n)}{K}\right)$$

Considering that $x_n = \frac{N(t_n)}{K}$ and $r = r^* K^{p-2}$, we obtained:

$$x_{n+1} = r x_n^{p-1} (1 - x_n)$$

In other way, Verhulst in [7] has considered $N(t_n)$ the number of individuals at time t_n, $N(t_n) \geq 0$ with, $N(t_{n+1}) = f(N(t_n))$ where f determined by the birth and death rates in the population. We put $f(0) = 0$, $N(t_{n+1}) > N(t_n)$ if $N(t_n)$ is small and $N(t_{n+1}) < N(t_n)$ if $N(t_n)$ is large because of natural bounds on the amount of available space and food. We present a simple model

$$N(t_{n+1}) = N(t_n) + rN(t_n) - \frac{r}{k} N(t_n)^2 \tag{1.1}$$

r is the growth coefficient and k a positive constant. We introduce some basic rescaling $x_n = rN(t_n)/(k(1 + r))$ and $a = 1 + r$ the equation (1.1) becomes

$$x_{n+1} = a x_n (1 - x_n). \tag{1.2}$$

The equation (1.2) is called the quadratic equation, logistic equation or Verhulst equation.

This paper is devoted to present a numerical investigation of a two-dimensional diffeomorphism on the dynamical properties of its basins of attractions, regular and chaotic attractors, the bifurcation structures and the mechanisms that assure chaotic dynamics by extending works. We focuse on the topological structure of trajectories around eventual cycles and fixed points and to illustrate its phase-parameter portraits.

2. Endomorphisms depending of two parameters (r, p)

In this section we provide a few overviews of the analysis of the family of endomorphisms $f_{r,p} : [0, 1] \to [0, 1]$ depending of two parameters $p \in \ N - \{1, 2\}$ and $r > 0$, and defined by:

$$f_{r,p}(x) = r x^{p-1} (1 - x) \tag{2.1}$$

p and r are chosen such that , $p \in]1, p_M]$ and $r \in]0, r(p_M)]$, with p_M and $r(p_M)$ correspond to the maxima values of p and r.

Let c be a critical point of $f_{r,p}$ which satisfies the following conditions:

- $f'_{r,p}(c) = 0$ and $f''_{r,p}(c) < 0$ meaning that $f_{r,p}$ is strictly increasing in $[0, c[$ and strictly decreasing in $]c, 1]$;
  and $f'_{r,p}(x) \neq 0, \forall x \neq c,$;

- $f_{0,p}(c) = 0$ and $f_{r(p_M),p}(c) = 1$; with $c = \frac{p-1}{p}$;
- $f_{r,p}(0) = f_{r,p}(1) = 0$;

Singer in [6] used the concept of negative Schwartzian derivative to discuss the existence of stable periodic orbits.The Schwarz derivative of $f_{r,p}(x)$ is

$$S(f_{r,p}(x)) = \frac{f'''_{r,p}(x)}{f'_{r,p}(x)} - \frac{3}{2}\left(\frac{f''_{r,p}(x)}{f'_{r,p}(x)}\right)^2 < 0 \quad ; \quad \forall x \in]0,1[-\{c\}$$

Singer concluded that a function f which is C^1 -unimodal and for which $S(f(x)) < 0$ for all $x \in]0,1[-\{c\}$ has at most one stable periodic orbit plus possibly a stable fixed point in the interval $[0,1]$.

Thereby $f_{r,p}$of the interval $[0,1]$ into itself is C^1 -unimodal if it is continuous; $f_{r,p}(c) = 1$; $f_{r,p}$ is strictly decreasing on $[0,c]$ and strictly increasing on $[c,1]$; and $f_{r,p}$ is once continuously differentiable with $f'_{r,p}(x) \neq 0$ when $x \neq c$.

The maximum value of the parameter p is $p_M = 20$, it is the largest value for which we consider that the model can be realistic. The value $r(p_M)$ is the value of the parameter r corresponding to the full shift for $p = p_M$. We consider that $1 < p \leq p_M = 20$ and $0 < r \leq 53.001$. For any fixed value $p > 1$, if $r = 0$, then there is no curves.

For example, we can verify that unimodal maps $f_{r,p}$ have the fixed point $x^* = 0$ for $r > 0$ and $p > 1$. However, for $p = 1.1$ and $p = 1.5$, we can verify that these maps have another positive fixed point besides 0. For $p \in [2, p_M]$ and $r > 6.721$, there are the fixed point zero and two other fixed points (cf. Figure 1).

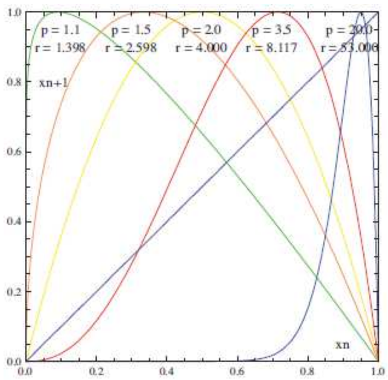

Figure 1: The fixed points for different values of (r, p) according to [1].

Rocha and *al.* in [1,5] have shown that there exists a relation between the parameters p, q and c such that for all $r \in]r_1, r_2[$, we can assure the existence of a unique attractor for $x \in [0;1]$ with

$$r_1 = \frac{{x_f}^{2-p}}{1 - x_f}$$

$$r_2 = \frac{2p^2 - p^2 - 4p + 2p + 1}{p^3 c^p (1-c)^2}$$

then x_f is the only one positive fixed point, $c = \frac{p-1}{p}$ is the critical point of order 0.

2.1. Embedding of one-dimensional noninvertible map into a two-dimensional invertible map

Let T be a two-dimensional Hénon-like map:

$$T : \begin{cases} x' = f_{r,p}(x) + y \\ y' = bx \end{cases} \tag{2.2}$$

The endomorphism $f_{r,p} = rx^{p-1}(1-x)$ is embedding into the diffeomorphism T with $b \neq 0$, and jacobian is $J = -b$. For a family of recurrence (2.2), the continuously passing of properties, for $b = 0$ to $b \neq 0$ and b sufficiently small, is obtained in the sense that we find identical cycles associated with the structures of bifurcation as in [3,4]. Figure 2 presents information on stability region for the fixed point (blue domain), and the existence region for attracting cycles of order k exists $(k \leq 14)$. The black regions $(k = 15)$ correspond to chaotic behavior.

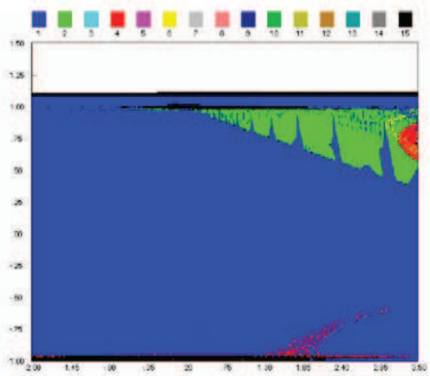

Figure 2: Scanning for $p = 3$, $q = 2$, $r \in [-2, 3.5]$, $b \in [-1, 1.5]$.

At once b is not small enough, an attractor type fixed point appears and coexists with a cycle of order two.

2.2. bidimensional map depending of three parameters (r, p, q)

In this part, we consider three parameters (r, p, q). The system T_b will be:

$$T_b : \begin{cases} x' = rx^{p-1}(1-x)^{q-1} + y \\ y' = bx \end{cases} \tag{2.3}$$

Taking into account [3], Mira describes some properties of the two-dimensional invertible systems T_b that can be expressed as

$$T_b : \begin{cases} x' = f_{r,p,q}(x) + y \\ y' = bx \end{cases} \tag{2.4}$$

associated with noninvertible one-dimensional maps T_0 such that when the parameter b is equal to the critical value, i.e.: $b = 0$ then $T_0 : x' = f_{r,p,q}(x)$. These properties are also about the stable and the instable fixed points and the concept of commutation set in invertible maps which are useful for interpreting such problems and fundamental in the definition of bifurcations leading to important modification of attractors and their basins of attraction.

Recall that a closed and invariant set A, is called an attracting set if some neighborhood U of A exists such that $T(U) \subset U$, and $T^n(x) \to A$ as $n \to \infty$, $\forall x \in U$. The set $D = \cup_{n\geq 0} T^{-n}(U)$ is the total basin (or simply: basin of attraction, or influence domain) of the attracting set A. In general, several types of attractors, e.g. fixed points, invariant closed curves, chaotic attractors, may coexist in the same mapping. This non-uniqueness also indicates that the routes to chaos depend on initial conditions and are therefore non-unique and depend on the values of the parameters. The basins of attraction D, defining the initial conditions leading to a certain attractor, may be a complex set.

Definition 2.1. *[3] Let T be a continuous noninvertible map $x' = Tx$, $dimx = n$. The critical set of rank-one, said CS, is the geometrical locus of points x having at least two coincident preimages. The critical set CS_i of rank-(i+1), $i > 0$, is the rank-i image of the set $CS_0 \equiv CS$.*

Definition 2.2. *Let S be a saddle fixed point and U a neighborhood of S. The local unstable set $W^u_{loc}(S)$ of $S \in U$ is given by:*

$$W^u_{loc}(S) = \{x \in U : x_{-k} \in T^{-k}(x) \to S, x_{-k} \in U, \forall k\}$$

and the global unstable set $W^u(S)$ of S is given by:

$$W^u(S) = \cup_{k\geq 0} T^{-k}[W^u_{loc}(S)].$$

The local stable set $W^s_{loc}(S)$ of $S \in U$ is given by:

$$W^s_{loc}(S) = \{x \in U : x_k \in T^k(x) \to S, x_k \in U, \forall k\},$$

and the global stable set $W^s(S)$ of S is given by:$W^s(S) = \cup_{k\geq 0} T^k[W^s{}_{loc}(S)]$.

Definition 2.3. *The point Q is said to be homoclinic to the nonattracting fixed point S (or homoclinic point of S) if $Q \in W^u(S) \cap W^s(S)$.*

Let M be another nonattracting fixed point. A point $Q \in U(S)$ is said heteroclinic from S to M, if $T^k(Q) \to M$, when k increases, and Q belongs to the local unstable set ${W^u}_{loc}(S)$.

Definition 2.4. *[3] We define the commutation sets E_i as*

$$E_0 = \{\textit{the line}\, x = c \textit{ such that } f'_{r,p}(c) = 0\}.$$

and

$$E_i = T_b(E_{i-1}), \forall i \geq 1.$$

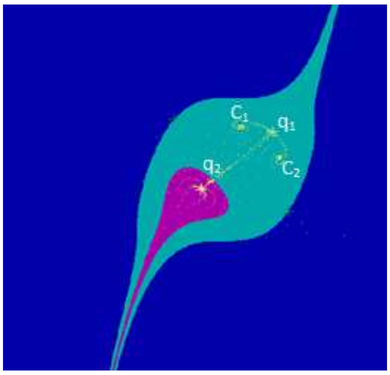

Figure 3: Basin of attraction for $p = 3$, $q = 2$, $r = 0.8$, $b = 0.9$

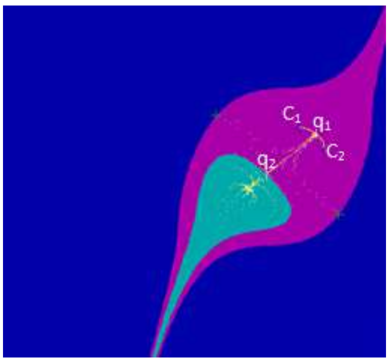

Figure 4: Existence of two attractors for $p = 3$, $q = 2$, $r = 0.6$, $b = 0.9$

Figures 3, 4 represent two attraction basins associated with two attractors for $q = 2$ and $p = 3$ in the phase space. In Figure 3, The red basin is associated with the fixed point and the green one is associated with the 2 -cycle represented by C_1 and C_2 points, q_1 and q_2 are two saddle points such that q_2 is on the boundary of the attraction basin of the attractive fixed point and q_1 is inside of the 2-cycle attraction basin.

3. Bifurcations and complexity

First, we consider the one-dimensional maps $T_0 : x' = f_{r,p,q}(x)$, and we study their dynamics. The complexity described by the family of $f_{r,p,q}(x)$ is analyzed in terms of the parameter $r > 0$, and the shape parameters $p > 1$ and $q > 1$, related with growth-retardation phenomena.

This family is subject to spontaneous existence which can be guaranteed for the value of the growth parameter

$$r_1 = x_f^{2-p}(1 - x_f)^{1-q}, \ \textit{with}\ p, q > 1$$

where x_f is the only positive fixed point (for more details see [5]). With r_2, we have a set of globally stable fixed points where the population growth remains stable, r_2 is given by

$$r_2 = \frac{p^2q - p^2 - 2pq + 2p + q - 1}{(p + q - 2)^3 c^p (1 - c)^q}, \ \textit{with}\ p, q > 1$$

where $c = \frac{p-1}{p+q-2}$ is the critical point. Globally, the iterates of the map $f_{r,p,q}(c)$ are always attracted to the positive fixed point sufficiently near of the attractive point, designated by stability region 1, where r satisfies $r_1 < r < r_2$.

Otherwise for the values of the parameter r such that $r_2 < r < r_3$, the stability of the region 2 is characterized by the beginning of period-doubling. The value r_3 satisfies this condition

$$r(c^{p-1}(1 - c)^{q-1}r)^{p-1}(1 - c^{p-1}(1 - c)^{q-1}r)^{q-1} = c, \ \textit{with}\ p, q > 1.$$

For $r = r_4$ we can observe complex dynamical behavior. It is defined by

$$r_4 = c^{1-p}(1 - c)^{1-q}, \ \textit{with}\ p, q > 1$$

For other values of the parameters p and q, due to certain factors, the population can take the risk of extinction.

In two-dimensional diffeomorphisms T_b, we consider other tools for analyzing the complexity and chaotic patterns of behavior. The notion of commutation sets instead of critical sets helps us to acquire such information about extension and period-doubling.

We fix $p = q = 2$, the system (6) is considered in [3], by combining of properties of T_b and T_0

$$T_b : \begin{cases} x' = rx(1 - x) + y \\ y' = bx, \end{cases} \tag{3.1}$$

with $f_{r,2,2}(x) = rx(1-x)$, $2 < r < 4$, $0 < b < 1$.

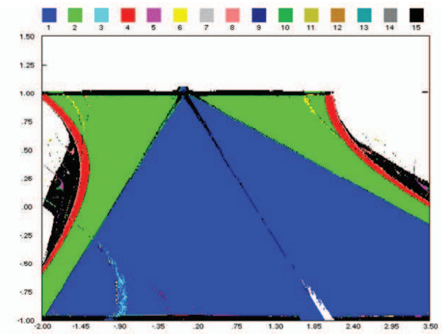

Figure 5: Scanning for $p = q = 2$, $r \in [-2, 3.5]$, $b \in [-1, 1.5]$.

The extremum of $f_{r,p,q}(x)$ is obtained for $x = 1/2$. The first three commutation sets are : E_0 is the line $x = 1/2$, E_1 is the line $y = b/2$, and E_2 is the parabola $x = ry/b - ry^2/b^2 + b/2$. The intersection point $P_1 = E_1 \cap E_2$ has the coordinates $x_{P_1} = (r/2 + b)/2, y_{P_1} = b/2$.
This point can be considered as $P_1 = T_b(P_0)$, with $P_0 = E_0 \cap E_1$, whose its coordinates are $x_{P_0} = 1/2, y_{P_0} = b/2$.

Let P_n be the intersection points defined as $P_n = T_b^n(P_{n-1}) = E_n \cap E_{n+1}$, $n = 1, 2, ...,$ which give rise to principal E-fold points of E_{n+2} in the neighborhood of P_n. The other fold points of E_{n+2} resulting from $T_b[E_{n+1} \cap E_k]$, $k = 0, 1, ..., n-1$, are the secondary E-fold points of E_{n+2}.

Proposition 3.1. *Consider the system* (3.1) *with* $b \to 0$*; we have*

(a) *The commutation sets* E_n*,* $n \geq 2$*, are crushed on the axe* $y = 0$*, and* $lim_{n\to\infty}(E_2) = [y = 0, -\infty < x \leq x(c)] = lim_{n\to\infty}(E_n)$*.*

(b) *The principal E-fold points are such that* $lim_{n\to\infty}(P_n) = c_n$*,* $c_n = T_b^n(c)$ *rank-*$(n+1)$ *critical point* $(C \equiv C_0)$ *de* T_0 *on* $y = 0$*.*

(c) *The secondary fold points on* E_1 *tend toward the rank-one critical point* c*.*

As mentioned in proposition 3.1, commutation sets are of primordial importance for the study of critical singularities. This method consists to approach attractors due to principal and secondary fold points (see Figure 6).

The inverse map T_b^{-1} is defined by the relations:

$$x = y'/b, y = x' - ry'(1 - y'/b)/b \tag{3.2}$$

Let E'_0 be the line $y = b/2$, which is given by $J = 0$ and by the same method applied to T_b. The commutation curves are defined by $E'_n = T_b^{-n}(E'_0)$.

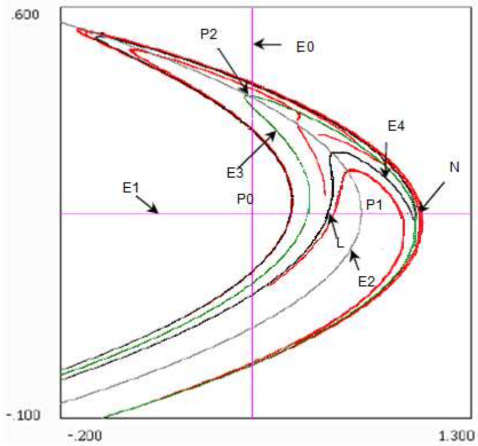

Figure 6: The commutation sets E_i, $r = 2.2$, $b = 0.7$.

3.1. Invariant manifolds of fixed points

The stable invariant manifold $W^s(S)$ constitutes in general the frontier of the basin of attraction of an associated attractor and the unstable one converges to this attractor. Contact bifurcations between the attractor and its attraction basin may correspond to homoclinic or heteroclinic bifurcations. Homoclinic orbits and heteroclinic orbits are important concepts in the study of bifurcation of structures and chaos. Many chaotic behaviors of a complex system are related to these kinds of trajectories in the system. A homoclinic orbit is an orbit that is doubly asymptotic to a fixed point, or is a closed trajectory asymptotic to itself. A heteroclinic orbit is a trajectory that connects two fixed points or cycles of saddle type. In this part, we are interested in the existence or nonexistence of homoclinic orbits and heteroclinic orbits of this considered system.

The fixed points of (3.1) are $q_1(x = y = 0)$ and $q_2(x_{q_2} = (r + b - 1)/r, y_{q_2} = bx_{q_2})$. Both fixed points are saddles for fixed parameters (r, b), and since $W^s(q_1) \cap W^u(q_1) = \emptyset$, there is no homoclinic point from q_1 (but heteroclinic points from q_1 to q_2 may exist).

When $b \to 0$, the condition to have q_1 and q_2 two saddles, is satisfied for $3 < r \leq 4$. The multipliers S_1 and S_2 of the saddle point q_1 are:

$$S_1(q_1) = [r - \sqrt{r^2 + 4b}]/2, S_2(q_1) = [r + \sqrt{r^2 + 4b}]/2$$

due to $3 < r \leq 4$, $0 < b < 1$, q_1 a saddle with $-1 < S_1(q_1) < 0$, $S_2(q_1) > 1$. The slope of the eigenvector related to $W^u(q_1)$ at q_1 is $s_1(q_1) = 2b/S_2(q_1) = 2b/[r + \sqrt{r^2 + 4b}]$.
The multipliers of q_2 are $S_1(q_2) = [2 - 2b - r - \sqrt{\Delta}]/2$, $S_2(q_2) = [2 - 2b - r + \sqrt{\Delta}]/2$ with $\Delta = (r + 2b)^2 + 4(1 - r - b)$, q_2 a saddle with $S_1(q_2) < -1$, $0 < S_2(q_2) < 1$.The slope of the eigenvector related to $W^u(q_2)$ at q_2 is : $s_1(q_2) = 2b/S_1(q_2)$.

For fixed parameter values, attractors's basins are illustrated in the phase space. When there exist several attractors, it is possible to define a global basin, that means the set of initial conditions giving rise to bounded iterated sequences, independently of the fact that they converge to one attractor or another.

We fix $b = 0.2$, we draw the invariant manifolds of a saddle point which is inside the basin for $r = 2$.

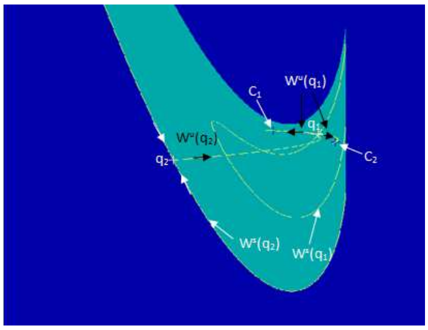

Figure 7: Attractor and invariant manifolds $r = 2$, $p = 2$, $q = 1.5$, $b = 0.2$.

When we investigate the heteroclinic tangency between unstable and instable manifolds by numerical way, we use saddle points.

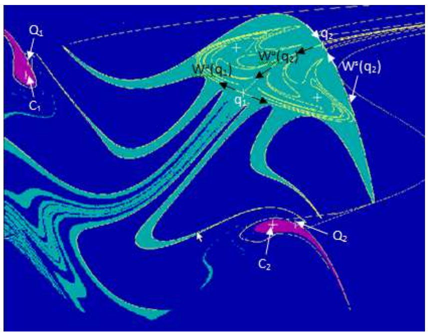

Figure 8: Attractors and invariant manifolds $r = 5$, $p = 2.5$, $q = 3$, $b = 0.9$,$x \in [0; 1.54]$, $y \in [-0.5; 1.23]$

Figure 7 for $r = 2$ and Figure 8 for $r = 5$ show how of the stable manifold of q_1 and the unstable manifold of q_2 intersect and then heteroclinic points appear. In Figure 9, only the invariant manifolds of saddle points are shown.

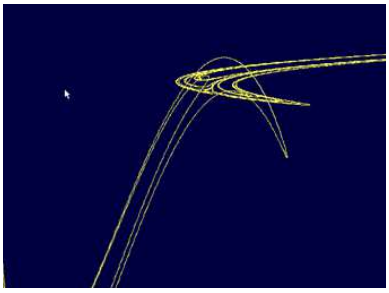

Figure 9: Heteroclinic curve of saddle points $r = 10$, $x \in [0; 1.8]$, $y \in [1; 1.7]$

For $b = 0.21$, we find Hénon attractor that is growing with r , has a contact with the frontier of its basin and disappears by contact bifurcation, $p = 3.5$ and $q = 2.5$ (see Figure 10).

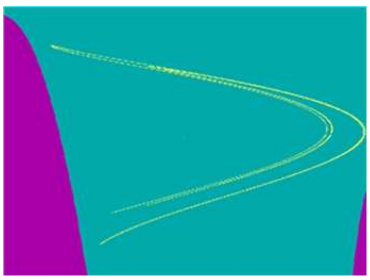

Figure 10: Strange attractor $r = 11.5$, $p = 3.5$, $q = 2.5$ and $b = 0.21$.

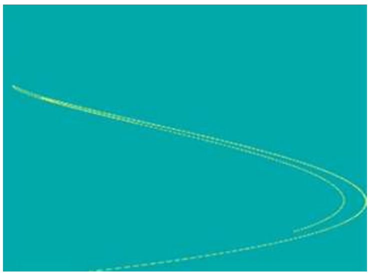

Figure 11: Strange attractor $r = 5$, $p = 2.5$ and $b = 0.084$, $x \in [0; 1]$, $y \in [0; 0.12]$

In this figure, we prove the abundance of strange attractors in this family around parameter values close to $r = 5$, $p = 2.5$ and $b = 0.084$. These attractors are similar to Hénon attractor.

4. Conclusion

After embedding a one-dimensional noninvertible map into a larger system which becomes invertible, we reviewed and used some analysis techniques as commutation sets and heteroclinic solutions together to detect chaotic behaviors.

References

1. S. M. Aleixo, J. L. Rocha, D. D. Pestana, *Populational growth models proportional to Beta densities with Allee effect*,1124 (2009), 3-12.

2. I. Gumowski, C. Mira, *Dynamique chaotique: Transformations ponctuelles: Transition ordre-désordre,* Cepadeus Editions 1980.

3. C. Mira, C. Gracio, *On the embedding of a (p-1)-dimensional noninvertible map into a p-dimentional invertible map (p=2,3),* Int. J. Bifurcation Chaos 13 (2003) 1787-1810.

4. J. L. Rocha, S. M. Aleixo, *An extension of gompertzian growth dynamics: Weibull and Fréchet models,* Mathematical Biosciences and Engineering (MBE), 10 (2013), 379-398.

5. J. L. Rocha, S. M. Aleixo, *Dynamical analysis in growth models : Blumberg's equation,* Discrete and Continuous Dynamical Systems - Series B (DCDS-B), 18 (2013), 783-795.

6. D. Singer, *Stable orbits and bifurcation of maps of the interval,* SIAM J. Appl. Math. 35 (1980) 260–267.

7. F. Verhulst, *Nonlinear Differential Equations and Dynamical Systems,* Springer-Verlag 1990.

Selmani Wissame, Djellit Ilham,
Laboratory of Mathematics, Dynamics and Modelization
University Badji Mokhtar Annaba,
Algeria.
E-mail address: selmani-wissame@hotmail.fr
E-mail address: ilhem.djelit@univ-annaba.org