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An Asymptotic Expansion of Continuous Wavelet Transform for Large Dilation Parameter

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Abstract: In this paper , we derive asymptotic expansion of the wavelet transform for large values of the dilation parameter a by using Lopez and Pagola technique. Asymptotic expansion of Mexican Hat wavelet and Morlet wavelet transform are obtained as a special cases.

Key Words: Asymptotic expansion, Wavelet transform, Fourier transform, Mellin transform.

Contents

1. Introduction

The wavelet transform of g with respect to the wavelet ϕ is defined by

$$(W_\phi g)(b,a) = a^{-1/2}\int_{-\infty}^{\infty} g(t)\,\overline{\phi\left(\frac{t-b}{a}\right)}\,dt, b\in\mathbb{R}, a>0. \quad (1.1)$$

provided the integral exists [1]. Using Fourier transform it can also be expressed as

$$(W_\phi g)(b,a) = \frac{a^{1/2}}{2\pi}\int_{-\infty}^{\infty} e^{ib\omega}\hat{g}(\omega)\overline{\hat{\phi}}(a\omega)d\omega, \quad (1.2)$$

where

$$\hat{g}(\omega) = \int_{-\infty}^{\infty} e^{-it\omega} g(t)dt.$$

The Asymptotic expansion for the integral of the form

$$I(\xi) := \int_0^\infty e^{i\tau t} f(t)h(\xi t)dt, \quad \tau\in\mathbb{R},\ \tau\neq 0+,\ as\ \xi\to 0+, \quad (1.3)$$

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was obtained by Lopez and Pagola [[5],Theorem 2,3] under certain conditions on f and h. Then the asymptotic expansion of (1.2) for large a can be obtained by setting $f(t) = \hat{g}(t)$ for fixed $b \in \mathbb{R}$ and $h(t) = \overline{\hat{\phi}}(t)$. Here we assume that $f(t)$ and $h(t)$ has an expansion of the form

$$f(t) = \sum_{r=0}^{n-1} c_r t^{\eta_r} + f_n(t), \quad \text{as} \;\; t \to 0+, \tag{1.4}$$

and that as $t \to \infty$,

$$h(t) = \sum_{i=0}^{n-1} d_i t^{-\rho_i} + h_n(t), \tag{1.5}$$

where $\eta_r < \eta_{r+1}, \forall r \geq 0$ and $\rho_i < \rho_{i+1}, \forall\, i \geq 0$.
Also assume that

$$f(t) = O(t^{-\alpha}), \quad t \to \infty, \, \alpha \in \mathbb{R}, \tag{1.6}$$

and

$$h(t) = O(t^{-\beta}), \quad t \to 0+, \beta \in \mathbb{R}, \tag{1.7}$$

with α, β, η_0 and ρ_0 satisfying the relations: $\beta - \eta_0 < 1 < \alpha + \rho_0$ and $-\eta_0 < \alpha$ and $\beta < \rho_0$.

The aim of the present paper is to derive asymptotic expansion of the wavelet transform (1.2) for large value of dilation parameter a . In section 2, we assume that $\hat{g}(\omega)$ and $\overline{\hat{\phi}}(\omega)$ possess asymptotic expansion of the form (1.4) and (1.5) as $\omega \to 0+$ and $\omega \to \infty$ and derive the asymptotic expansion of $(W_\phi g)(b,a)$ as $a \to \infty+$. In section 3, we obtain asymptotic expansion of Mexican Hat and Morlet wavelet transform.

2. Asymptotic expansion of wavelet transform for large a

Let us rewrite (1.2) in the form:

$$\begin{aligned} (W_\phi g)(b,a) &= \frac{\sqrt{a}}{2\pi}\left\{ \int_0^\infty e^{ib\omega}\overline{\hat{\phi}(a\omega)}\hat{g}(\omega)d\omega \right. \\ &\quad \left. + \int_0^\infty e^{-ib\omega}\overline{\hat{\phi}(-a\omega)}\hat{g}(-\omega)d\omega \right\} && (2.1) \\ &= \left(W_\phi^+ g\right)(b,a) + \left(W_\phi^- g\right)(b,a), && (2.2) \end{aligned}$$

where

$$\left(W_\phi^+ g\right)(b,a) = \frac{\sqrt{a}}{2\pi}\int_0^\infty e^{ib\omega}\overline{\hat{\phi}(a\omega)}\hat{g}(\omega)d\omega, \tag{2.3}$$

$$\left(W_{\phi}^{-} g\right)(b,a) = \frac{\sqrt{a}}{2\pi}\int_0^\infty e^{-ib\omega}\overline{\hat{\phi}(-a\omega)}\hat{g}(-\omega)d\omega. \tag{2.4}$$

Assume that $\hat{g}(\omega), \overline{\hat{\phi}}(\omega)$ are locally integrable in $(-\infty,\infty)$ and

$$\hat{g}(\omega) = \sum_{r=0}^{n-1} c_r \omega^{\eta_r} + \hat{g}_n(\omega), \quad \text{as } \omega \to 0+, \tag{2.5}$$

and that as $\omega \to \infty$,

$$\overline{\hat{\phi}}(\omega) = \sum_{i=0}^{n-1} d_i \omega^{-\rho_i} + \overline{\hat{\phi}}_n(\omega), \tag{2.6}$$

where $\rho_i < \rho_{i+1}, \forall\, i \geq 0$ and $\eta_r < \eta_{r+1}, \forall\, r \geq 0$.
Also assume that

$$\hat{g}(\omega) = O(\omega^{-\alpha}), \quad \omega \to \infty, \alpha \in \mathbb{R}, \alpha + \rho_0 > 1, \tag{2.7}$$

and

$$\overline{\hat{\phi}}(\omega) = O(\omega^{-\beta}), \quad \omega \to 0+, \beta \in \mathbb{R}, \beta - \eta_0 < 1. \tag{2.8}$$

Now by using [[5],Theorem 2,3], we can also prove the theorem for asymptotic expansion of wavelet transform for large value of dilation parameter a.

Theorem 2.1. . *Assume that (i) $\hat{g}(w)$ and $\overline{\hat{\phi}}(w)$ are locally integrable on $(0,\infty)$, (ii) $\hat{g}(w)$ satisfies (2.5) and (2.7),(iii) $\overline{\hat{\phi}}(w)$ satisfies (2.6) and (2.8). Then, for any $n, m \in \mathbb{N}$ such that $\rho_{n-1} - \eta_m < 1 < \rho_n - \eta_{m-1}$,*

$$\begin{aligned}\int_0^\infty e^{ib\omega}\hat{g}(\omega)\overline{\hat{\phi}}(\omega)d\omega &= \sum_{r=0}^{n-1} c_r\left[M[\overline{\hat{\phi}}(\omega)e^{ib\omega/a}; 1+\eta_r] - \sum_{p=0}^{R(r)-1} d_p B_p(1+\eta_r; a; b)\right] \\ &\times\ a^{-\eta_r - 1} + \sum_{i=0}^{m-1} d_i\left[M[\hat{g}(\omega)e^{ib}; 1-\rho_i]\right. \\ &\quad \left. - \sum_{j=0}^{I(i)-1} c_j\, A_j(1-\rho_i; b)\right] \\ &\times\ a^{-\rho_i} + R_{n,m}(a), \end{aligned} \tag{2.9}$$

$$\begin{aligned} B_p(x;a;b) &= e^{i\pi(x-\rho_p)/2}\,\Gamma(x-\rho_p)\left(\frac{b}{a}\right)^{x-\rho_p}, \\ A_j(x;b) &= e^{i\pi(x+\eta_j)/2}\frac{\Gamma(x+\eta_j)}{b^{x+\eta_j}}. \end{aligned}$$

$R(r)$ is the index r for which $\rho_i - \eta_r \leq 1 < \rho_i - \eta_{r-1}$ and $I(i)$ is the index i for which $\rho_{i-1} - \eta_r < 1 \leq \rho_i - \eta_r$. If $\rho_i - \eta_r = 1$ for some pair (i, r) then in (2.9), the sum of the terms

$$c_r\left[M[\overline{\hat{\phi}}(\omega)e^{ib\omega/a}; 1+\eta_r] - \sum_{p=0}^{R(r)-1} d_p B_p(1+\eta_r; a; b)\right]a^{-\eta_r - 1}$$
$$+ \quad d_i\left[M[\hat{g}(\omega)e^{ibw}; 1-\rho_i] - \sum_{j=0}^{I(i)-1} c_j A_j(1-\rho_i; b)\right]a^{-\rho_i}, \tag{2.10}$$

must be replaced by

$$a^{-\rho_i}\Bigg\{d_i\left[M[\hat{g}(\omega)e^{ib\omega}; 1-\rho_i] - \sum_{j=0}^{I(i)-1} c_j A_j(1-\rho_i; b) - c_r loga\right]$$
$$-c_r d_i[i\frac{\pi}{2} - logb - \gamma] + c_r \lim_{z\to 0}\Bigg[M[\overline{\hat{\phi}}(\omega)e^{ib\omega/a}; z+1+\eta_r] \tag{2.11}$$
$$-\sum_{p=0}^{r-1} d_p B_p(z+1+\eta_r; a; b)) - \frac{d_i}{z}\Bigg]\Bigg\},$$

and the remainder term $R_{n,m}(a)$ as $a \to \infty+$ is given by

$$R_{n,m}(a) = \left\{\begin{array}{ll} O(a^{-\rho_i} + a^{-\eta_r - 1}), & when \quad -\eta_r \neq -\rho_i + 1, \\ O(a^{-\rho_i}\log a), & when \quad -\eta_r = -\rho_i + 1. \end{array}\right\}. \tag{2.12}$$

Proof: Define $\hat{g_0}(\omega) = \hat{g}(\omega)$ and $\overline{\hat{\phi}}_0(\omega) + \overline{\hat{\phi}}_{\alpha-1}(\omega) = \overline{\hat{\phi}}(\omega)$. For any i there exists r such that $\rho_{i-1} - \eta_r \leq 1 < \rho_i - \eta_{r-1}$. For the given (i, r), the following integral exists:

$$\int_0^\infty e^{ib\omega}\, \hat{g}_r(\omega)\, \overline{\hat{\phi}}_i(a\omega). \tag{2.13}$$

(a). For a given (i, r) satisfying $\rho_{i-1} - \eta_r \leq 1 < \rho_i - \eta_{r-1}$, do the following: If $\rho_i - \eta_r < 1$ go to (b). If $\rho_i - \eta_r > 1$ go to (c). If $\rho_i - \eta_r = 1$ go to (d).
(b). Use $\overline{\hat{\phi}}_i(a\omega) = d_i(a\omega)^{-\rho_i} + \overline{\hat{\phi}}_i(a\omega)$ in (2.13) and (iii) of Lemma(2) of [5], we get

$$\int_0^\infty e^{ib\omega}\, \hat{g}_r(\omega)\, \overline{\hat{\phi}}_i(a\omega) \quad = \quad d_i a^{-\rho_i}\left[M[\hat{g}(\omega)e^{ib\omega}; 1-\rho_i] - \sum_{j=0}^{I(i)-1} c_j A_j(1-\rho_i; b)\right]$$
$$+ \quad \int_0^\infty e^{ib\omega}\, \hat{g}_r(\omega)\, \overline{\hat{\phi}}_{i+1}(a\omega)d\omega, \tag{2.14}$$

where $A_j(1-\rho_i;b) = e^{\frac{i\pi(1-\rho_i+\eta_j)}{2}} \frac{\Gamma(1-\rho_i+\eta_j)}{b^{1-\rho_i+\eta_j}}$.
Go to (a) with i replaced by $i+1$.

(c). use $\hat{g}_r(\omega) = c_r\omega^{\eta_r} + \hat{g}_{r+1}(\omega)$ in (2.13) and (iii) of lemma (3) of [5],we get

$$\begin{aligned}\int_0^\infty e^{ib\omega}\,\hat{g}_r(\omega)\,\overline{\hat{\phi}}_i(a\omega) &= c_r a^{-\eta_r-1}\Big[M[\overline{\hat{\phi}}(\omega)e^{ib\omega/a};1+\eta_r] \\ &\quad - \sum_{p=0}^{R(r)-1} d_p B_p(1+\eta_r;a;b)\Big] \\ &\quad + \int_0^\infty e^{ib\omega}\,\hat{g}_{r+1}(\omega)\,\overline{\hat{\phi}}_i(a\omega)d\omega, \qquad (2.15)\end{aligned}$$

where, $B_p(1+\eta_r;a;b) = e^{\frac{i\pi(1+\eta_r-\rho_p)}{2}}\,\Gamma(1+\eta_r-\rho_p)\left(\frac{a}{b}\right)^{(1+\eta_r-\rho_p)}$.

Go to (a) with r replaced by $r+1$.

(d). use first $\overline{\hat{\phi}}_i(a\omega) = d_i(aw)^{-\rho_i} + \overline{\hat{\phi}}_{i+1}(a\omega)$ and then $\hat{g}_r(\omega) = c_r\omega^{\eta_r} + \hat{g}_{r+1}(\omega)$ in (2.13) , we get

$$\begin{aligned}\int_0^\infty e^{ib\omega}\,\hat{g}_r(\omega)\,\overline{\hat{\phi}}_i(a\omega) &= \int_0^\infty e^{ib\omega}[d_i(a\omega)^{-\rho_i}\hat{g}_r(\omega) + c_r\omega^{\eta_r}\overline{\hat{\phi}}_{i+1}(a\omega)]d\omega \\ &+ \int_0^\infty e^{ib\omega}\,\hat{g}_r(\omega)\,\overline{\hat{\phi}}_{i+1}(a\omega)d\omega.\end{aligned}$$

Define the function

$$E_{i,r}(z,\omega) = \omega^z\Big[d_i a^{-\rho_i}\omega^{-\rho_i}\hat{g}_r(\omega) + c_r\omega^{\eta_r}\overline{\hat{\phi}}_{i+1}(a\omega)\Big] z\in C.$$

Then

$$\int_0^\infty e^{ib\omega}\,\hat{g}_r(\omega)\,\overline{\hat{\phi}}_i(a\omega)d\omega = \int_0^\infty e^{ib\omega}\,E_{i,r}(0,\omega)d\omega + \int_0^\infty e^{ib\omega}\,\hat{g}_r(\omega)\,\overline{\hat{\phi}}_{i+1}(a\omega)d\omega \qquad (2.16)$$

On the one hand $\hat{g}_r(\omega) = c_r\omega^{\eta_r} + O(\omega^{\eta_r+1})$; when $\omega\to 0+$. On the other hand $\overline{\hat{\phi}}_{i+1}(\omega) = -d_i\omega^{-\rho_i} + O(\omega^{-\rho_i-1})$; when $\omega\to 0+$. Hence,$E_{i,r}(z,.)\in L^1[0,\infty)$ for Max $(\rho_i-\eta_r+1,\rho_i-1-\eta_r)-1 < Re(z) < Min(\rho_i-\eta_r-1,\rho_i+1-\eta_r)-1$. Choose two numbers z_0 and z_1, satisfying Max $(\rho_i-\eta_r+1,\rho_i-1-\eta_r)-1 < z_0 < 0$ and $0 < z_1 < Min(\rho_i-\eta_r-1,\rho_i+1-\eta_r)-1$. Then we have that for $z_0 < Re(z) < z_1$: $|E_{i,r}(\omega,z)| \le H_{i,r}(\omega) = \Big\{|E_{i,r}(\omega,z_0)| \text{ for } \omega\in[0,1] \text{ and } |E_{i,r}(\omega,z_1)| \text{ for } \omega\in[1,\infty)\Big\}$ and that $H_{i,r}(\omega)\in[0,\infty)$. using the dominated convergence theorem, we get

$$\begin{aligned}\int_0^\infty e^{ib\omega}\,E_{i,r}(0,\omega)d\omega &= \lim_{z\to 0}\int_0^\infty e^{ib\omega}E_{i,r}(z,\omega)d\omega \\ &= a^{-\rho_i}\lim_{z\to 0}\int_0^\infty [d_i\omega^{z-\rho_i}e^{ib\omega}\hat{g}_r(\omega) \\ &\quad + c_r a^{-z}\omega^{z+\eta_r}e^{\frac{ib\omega}{a}}\overline{\hat{\phi}}_{i+1}(\omega)]d\omega. \qquad (2.17)\end{aligned}$$

From $\rho_i = 1+\eta_r$ and from Lemma 2 and 3 of [5], we have that $M[\overline{\hat{\phi}}(\omega); z+1+\eta_r]$ and $M[\hat{g}(\omega); z+1-\rho_i]$ have a common strip of analyticity: $a-\eta_r-1 < Re(z) < \eta_r+b$. we have that $a-\eta_0-1 < 0 < \eta_0+b$ and then the point $z=0$ belongs to that strip of analytic. Therefore,

$$\begin{aligned}\int_0^\infty e^{ib\omega}\hat{g}_r(\omega)\overline{\hat{\phi}}_i(a\omega)d\omega &= a^{-\rho_i}\lim_{z\to 0}\Bigg\{d_i\Bigg[M[\hat{g}_r(\omega)e^{ib\omega}; z+1-\rho_i] \\ &\quad - \sum_{j=0}^{I(i)-1} c_j A_j(z+1-\rho_i; b) - d_j\ e^{\frac{i\pi z}{2}}\ \frac{\Gamma(z)}{b^z}\Bigg] \\ &\quad + c_r a^{-z}\Bigg[M[\overline{\hat{\phi}}(\omega)e^{\frac{ib\omega}{a}}; z+1+\eta_r] \\ &\quad - \sum_{p=0}^{R(r)-1} d_p B_p(z+1+\eta_r; a; b)\Bigg]\Bigg\} \\ &\quad + \int_0^\infty e^{ib\omega}\hat{g}_r(\omega)\overline{\hat{\phi}}_{i+1}(a\omega)d\omega. \end{aligned} \tag{2.18}$$

Using $a^{-z} = 1 - z\log(a) + O(z^2)$; when $z \to 0$ and

$$\begin{aligned} M[\overline{\hat{\phi}}(\omega)e^{\frac{ib\omega}{a}}; z+1+\eta_r] &= \int_0^\infty e^{\frac{ib\omega}{a}}\hat{g}_r(\omega)\overline{\hat{\phi}}_i(\omega)\omega^{z+\eta_r}d\omega \\ &= \frac{d_i}{z} + 0(1); when\ z\to 0. \end{aligned} \tag{2.19}$$

We find that the above expression can be rewritten as

$$\begin{aligned}\int_0^\infty e^{ib\omega}\hat{g}_r(\omega)\overline{\hat{\phi}}_i(a\omega)d\omega &= a^{-\rho_i}\Bigg\{d_i\Bigg[M[\hat{g}_r(\omega)e^{ib\omega}; 1-\rho_i] - \sum_{j=0}^{I(i)-1} c_j A_j(1-\rho_i; b) \\ &- c_r\log a\Bigg] + c_r\lim_{z\to 0}\Bigg[M[\overline{\hat{\phi}}(\omega)e^{\frac{ib\omega}{a}}; z+1+\eta_r] \\ &- \sum_{p=0}^{R(r)-1} d_p B_p(z+1+\eta_r; a; b) - \frac{d_i}{z}\Bigg] \\ &- c_r d_i[\frac{i\pi}{2} - \log b - \gamma]\Bigg\} + \int_0^\infty e^{ib\omega}\hat{g}_r(\omega)\overline{\hat{\phi}}_{i+1}(a\omega)d\omega. \end{aligned} \tag{2.20}$$

Hence,

$$
\begin{aligned}
(W_\phi^+ g)(b,a) &= \frac{a^{1/2}}{2\pi}\left\{\int_0^\infty e^{ib\omega}\hat{g}(\omega)\overline{\hat{\phi}}(a\omega)d\omega\right\} \\
&= \frac{1}{2\pi}\left\{\sum_{r=0}^{n-1} c_r\left[M[\overline{\hat{\phi}}(\omega)e^{\frac{ib\omega}{a}};1+\eta_r]\right.\right. \\
&- \left.\sum_{p=0}^{R(r)-1} d_p B_p(1+\eta_r;a;b)\right]a^{-\eta_r-1/2} \\
&+ \sum_{i=0}^{m-1} d_i\left[M[\hat{g}(\omega)e^{ib\omega};1-\rho_i]\right. \\
&- \left.\left.\sum_{j=0}^{I(i)-1} c_j A_j(1-\rho_i;b)\right]a^{-\rho_i+1/2}+R_{n,m}\right\}.
\end{aligned}
\tag{2.21}
$$

If $\rho_i - \eta_r = 1$ for some pair (i,r) then in (2.21), the sum of the terms

$$
\begin{aligned}
& c_r\left[M[\overline{\hat{\phi}}(\omega)e^{ib\omega/a};1+\eta_r]-\sum_{p=0}^{R(r)-1} d_p B_p(1+\eta_r;a;b)\right]a^{-\eta_r-1/2} \\
+\ & d_i\left[M[\hat{g}(\omega)e^{ib\omega};1-\rho_i]-\sum_{j=0}^{I(i)-1} c_j A_j(1-\rho_i;b)\right]a^{-\rho_i+1/2},
\end{aligned}
\tag{2.22}
$$

must be replaced by

$$
\begin{aligned}
& a^{-\rho_i+1/2}\left\{d_i\left[M[\hat{g}(\omega)e^{ib\omega};1-\rho_i]-\sum_{j=0}^{I(i)-1} c_j A_j(1-\rho_i;b)-c_r loga\right]\right. \\
& -c_r d_i[i\frac{\pi}{2}-logb-\gamma]+c_r\lim_{z\to 0}\left[M[\overline{\hat{\phi}}(\omega)e^{ib\omega/a};z+1+\eta_r]\right. \\
& \left.\left.-\sum_{p=0}^{r-1} d_p B_p(z+1+\eta_r;a;b)-\frac{d_i}{z}\right]\right\},
\end{aligned}
\tag{2.23}
$$

and the remainder term $R_{n,m}(a)$ as $a\to\infty+$ is given by

$$
R_{n,m}(a)=\left\{\begin{array}{ll} O(a^{-\rho_i+1/2}+a^{-\eta_r-1/2}), & when\ \ -\eta_r\neq-\rho_i+1, \\ O(a^{-\rho_i+1/2}\log a), & when\ \ -\eta_r=-\rho_i+1. \end{array}\right\}. \tag{2.24}
$$

Similarly by setting $h(\omega)=\overline{\hat{\phi}}(-\omega)$ and $f(\omega)=\hat{g}(-\omega)$, we get the asymptotic expansion of $\left(W_\phi^- g\right)(b,a)$, as $a\to\infty+$,

$$\begin{aligned}
\left(W_\phi^- g\right)(b,a) &= \frac{a^{1/2}}{2\pi}\left\{\int_0^\infty e^{-ib\omega}\hat{g}(-\omega)\overline{\hat{\phi}}(-a\omega)d\omega\right\} \\
&= \frac{1}{2\pi}\left\{\sum_{r=0}^{n-1} c_r(-1)^{\eta_r}\left[M[\overline{\hat{\phi}}(-\omega)e^{\frac{-ib\omega}{a}};1+\eta_r]\right.\right. \\
&- \left.\sum_{p=0}^{R(r)-1}(-1)^{-\rho_p}\, d_p\,\overline{B_p(1+\eta_r;a;b)}\right] \\
&\times\ a^{-\eta_r-1/2}+\sum_{i=0}^{m-1}(-1)^{-\rho_i}d_i\left[M[\hat{g}(-\omega)e^{-ib\omega};1-\rho_i]\right. \\
&- \left.\sum_{j=0}^{I(i)-1}(-1)^{\eta_j}c_j\overline{A_j(1-\rho_i;b)}\right] \\
&\times\ \left. a^{-\rho_i+1/2}+R_{n,m}\right\}. \qquad (2.25)
\end{aligned}$$

If $\rho_i-\eta_r=1$ for some pair (i,r) then in (2.25), the sum of the terms

$$\begin{aligned}
& c_r(-1)^{\eta_r}\left[M[\overline{\hat{\phi}}(-\omega)e^{\frac{-ib\omega}{a}};1+\eta_r]\right. \\
- & \left.\sum_{p=0}^{R(r)-1}(-1)^{-\rho_p}d_p\overline{B_p(1+\eta_r;a;b)}\right]a^{-\eta_r-1/2} \qquad (2.26) \\
+ & (-1)^{-\rho_i}d_i\left[M[\hat{g}(-\omega)e^{-ib\omega};1-\rho_i]-\sum_{j=0}^{I(i)-1}(-1)^{\eta_j}\,c_j\overline{A_j(1-\rho_i;b)}\right]a^{-\rho_i+1/2}, \qquad (2.27)
\end{aligned}$$

must be replaced by

$$\begin{aligned}
& a^{-\rho_i+1/2}\left\{(-1)^{-\rho_i}d_i\left[M[\hat{g}(-\omega)e^{-ib\omega};1-\rho_i]\right.\right. \\
& -\left.\sum_{j=0}^{I(i)-1}(-1)^{\eta_j}c_j\overline{A_j(1-\rho_i;b)}-(-1)^{\eta_r}c_r loga\right] \\
& +(-1)^{\eta_r}c_r\lim_{z\to 0}\left[M[\overline{\hat{\phi}}(-\omega)e^{-ib\omega/a};z+1+\eta_r]\right. \\
& -\left.\sum_{p=0}^{r-1}(-1)^{-\rho_p}d_p\overline{B_p(z+1+\eta_r;a;b)}-(-1)^{-\rho_i}\frac{d_i}{z}\right] \\
& \left.-(-1)^{\eta_r-\rho_i}c_rd_i[i\frac{\pi}{2}-logb-\gamma]\right\}. \qquad (2.28)
\end{aligned}$$

Therefore by using (2.21) and (2.25) in (2.2), we get the required asymptotic expansion of wavelet transform for large $a\to\infty+$ is

$$\begin{aligned}(W_\phi g)(b,a) =&\frac{1}{2\pi}\Bigg\{\sum_{r=0}^{n-1} c_r\Bigg[M[\bar{\hat{\phi}}(\omega)e^{ib\omega/a};1+\eta_r]+(-1)^{\eta_r}M[\bar{\hat{\phi}}(-\omega)e^{-ib\omega/a};1+\eta_r]\\ &-\sum_{p=0}^{R(r)-1} d_p\ B_p(1+\eta_r;a;b)\Big(1+(-1)^{\eta_r-\rho_p}\ e^{-i\pi(1+\eta_r-\rho_p)}\Big)\Bigg]a^{-\eta_r-\frac{1}{2}}\\ &+\sum_{i=0}^{m-1} d_i\Bigg[M[\hat{g}(\omega)e^{ib\omega};1-\rho_i]+(-1)^{-\rho_i}M[\hat{g}(-\omega)e^{-ib\omega};1-\rho_i]\\ &-\sum_{j=0}^{I(i)-1} c_j\ A_j(1-\rho_i;b)\Big(1+(-1)^{\eta_j-\rho_i}\ e^{-i\pi(1+\eta_j-\rho_i)}\Big)\Bigg]a^{-\rho_i+\frac{1}{2}}\\ &+R_{n,m}(a)\Bigg\}. \end{aligned}\tag{2.29}$$

If $\rho_i-\eta_r=1$ for some pair (i,r) then in (2.29), the corresponding sum of (2.22) and (2.26) must be replace by

$$\begin{aligned} a^{-\rho_i+\frac{1}{2}}&\Bigg\{d_i\Bigg[M[\hat{g}(\omega)e^{ib\omega};1-\rho_i]+(-1)^{-\rho_i}M[\hat{g}(-\omega)e^{-ib\omega};1-\rho_i]\\ &-\sum_{j=0}^{I(i)-1} c_jA_j(1-\rho_i;b)\Big(1+(-1)^{\eta_j-\rho_i}\ e^{-i\pi(1+\eta_j-\rho_i)}\Big)\Bigg]\\ &+c_r\lim_{z\to 0}\Bigg[M[\bar{\hat{\phi}}(\omega)e^{ib\omega/a};z+1+\eta_r]+(-1)^{\eta_r}M[\bar{\hat{\phi}}(-\omega)e^{-ib\omega/a};z+1+\eta_r]\\ &-\sum_{p=0}^{r-1} d_pB_p(z+1+\eta_r;a;b)\Big(1+(-1)^{\eta_r-\rho_p}\ e^{-i\pi(z+1+\eta_r-\rho_p)}\Big)\Bigg]\Bigg\}. \end{aligned}\tag{2.30}$$

□

3. Application

Using the aforesaid technique, we find the asymptotic expansions of Mexican Hat wavelet and Morlet wavelet transform.

3.1. Asymptotic Expansion of Mexican Hat Wavelet Transform

We consider $\phi(t)=(1-t^2)\,e^{\frac{-t^2}{2}}$ to be a Mexican Hat wavelet.Since Fourier transform of Mexican Hat $\hat{\phi}(\omega)=\sqrt{2\pi}\,\omega^2\,e^{\frac{-\omega^2}{2}}$ is locally integrable in $(-\infty,\infty)$ and has an asymptotic expansion of $\bar{\hat{\phi}}(\omega)$ as $\omega\to\infty+$ [1]

$$\bar{\hat{\phi}}(\omega)=\sum_{r=0}^{\infty} d_r\omega^{-2r+1},\tag{3.1}$$

where, $d_r = a_r e^{\frac{i\pi}{2}(2r-1)}\,\Gamma(2r-1)$ and $a_r = \frac{(-1)^{r+1}\,r\,(2r-1)}{r!\;2^{r-1}}$.

$$\overline{\hat{\phi}}(\omega) = O(1)\ as\ \omega \to 0+\,. \tag{3.2}$$

Now, assume $\hat{g}(\omega)$ satisfies (2.5) and (2.7) with $\eta_0 > 1$. Then by using (2.29) and by means of formula [[2],(10,30),pp.318,320], we get the asymptotic expansion of Mexican Hat wavelet transform at $a \to \infty+$ is

$$\begin{aligned}
\Big(W_\phi g\Big)(b,a) \;=\; & \frac{1}{2\pi}\Big\{\sum_{r=0}^{n-1} c_r\Big[2^{1+\frac{\eta_r}{2}}\sqrt{\pi}\Big(-\frac{1}{a}\sqrt{2}\Big(-1+(-1)^{\eta_r}\Big)bi \\
\times\; & \Gamma[\frac{(4+\eta_r)}{2}]1F1\Big[\frac{(4+\eta_r)}{2},\frac{3}{2},\frac{b^2 i^2}{2a^2}\Big] + \Big(1+(-1)^{\eta_r}\Big)\Gamma[\frac{(3+\eta_r)}{2}] \\
\times\; & 1F1\Big[\frac{(3+\eta_r)}{2},\frac{1}{2},\frac{b^2 i^2}{2a^2}\Big]\Big)\Big] - \sum_{p=0}^{R(r)-1} a_p\; e^{\frac{i\pi}{2}(2p-1)}\,\Gamma[2p-1] \\
\times\; & B_p(1+\eta_r;a;b)\Big(1-(-1)^{-\eta_r}e^{-i\pi\eta_r}\Big)\Big]a^{-\eta_r-\frac{1}{2}} + \sum_{i=0}^{m-1} a_i \\
\times\; & e^{\frac{i\pi}{2}(2i-1)}\Gamma[2i-1]\Big[M[\hat{g}(\omega)e^{ib\omega};2r] - M[\hat{g}(-\omega)e^{-ib\omega};2r] \\
-\; & \sum_{j=0}^{I(i)-1} c_j\;A_j(2r;b)\;\Big(1-(-1)^{\eta_j}e^{-i\pi\eta_j}\Big)\Big]a^{2r-\frac{1}{2}} + R_{n,m}(a)\Big\},
\end{aligned} \tag{3.3}$$

where $R(r)$ is the index r for which $(1-2r)-\eta_r \leq 1 < (1-2r)-\eta_r-1$ and $I(i)$ is the index i for which $-2r-\eta_r < 1 \leq (1-2r)-\eta_r$.
If $(-2r+1)-\eta_r = 1$ for some pair (i,r) then, in (3.3), the corresponding sum of the terms must be replace by

$$\begin{aligned}
& a^{2r-\frac{1}{2}}\Big\{a_i\,e^{\frac{i\pi}{2}(2i-1)}\Gamma[2i-1]\Big[M[\hat{g}(\omega)e^{ib\omega};2r] - M[\hat{g}(-\omega)e^{-ib\omega};2r] \\
-\; & \sum_{j=0}^{I(i)-1} c_j\;A_j(2r;b)\;\Big(1-(-1)^{\eta_j}e^{-i\pi\eta_j}\Big)\Big] \\
+\; & c_r \lim_{z\to 0}\Big[2^{\frac{1}{2}\Big(2+z+\eta_r\Big)}\sqrt{\pi}\Big(\Big(1+(-1)^{\eta_r}\Big) \\
\times\; & \Gamma[\frac{(3+z+\eta_r)}{2}]\;1F1\Big[\frac{1}{2}\Big(3+z+\eta_r\Big),\frac{1}{2},\frac{b^2 i^2}{2a^2}\Big] - \frac{1}{a}\sqrt{2}\Big(-1+(-1)^{\eta_r}\Big) \\
\times\; & bi\,\Gamma[\frac{(4+z+\eta_r)}{2}]\;1F1\Big[\frac{1}{2}\Big(4+z+\eta_r\Big),\frac{3}{2},\frac{b^2 i^2}{2a^2}\Big]\Big)\Big]
\end{aligned}$$

$$- \sum_{p=0}^{r-1} a_p \, e^{\frac{i\pi}{2}(2p-1)} \Gamma[2p-1] \; B_p(1+\eta_r; a; b)\left(1-(-1)^{-\eta_r} e^{-i\pi\eta_r}\right)\Big] a^{-\eta_r-\frac{1}{2}}$$

$$+ \sum_{i=0}^{m-1} a_i \, e^{\frac{i\pi}{2}(2i-1)} \Gamma[2i-1] \Big[M[\hat{g}(\omega)e^{ib\omega}; 2r] - M[\hat{g}(-\omega)e^{-ib\omega}; 2r]$$

$$- \sum_{j=0}^{I(i)-1} c_j \; A_j(2r; b) \; \left(1-(-1)^{\eta_j} e^{-i\pi\eta_j}\right)\Big] a^{2r-\frac{1}{2}} \Big\}, \tag{3.4}$$

and the remainder term $R_{n,m}(a)$ as $a \to \infty+$ is given by

$$R_{n,m}(a) = \left\{ \begin{array}{ll} O(a^{2r-\frac{1}{2}} + a^{-\eta_r-\frac{1}{2}}), & when \quad -\eta_r \neq 2r, \\ O(a^{2r-\frac{1}{2}} \log a), & when \quad -\eta_r = 2r. \end{array} \right\}. \tag{3.5}$$

3.2. Asymptotic Expansion of Morlet Wavelet Transform

In this application, we consider Morlet wavelet $\phi(t)= e^{i\omega_o t-\frac{t^2}{2}}$. Since $\hat{\phi}(\omega) = \sqrt{2\pi}\, e^{\frac{-(\omega-\omega_o)^2}{2}}$ is locally integrable in $(-\infty, \infty)$ and has an asymptotic expansion as $\omega \to \infty+$ [4].

$$\overline{\hat{\phi}}(\omega) = \sum_{r=0}^{\infty} a_r \, e^{\frac{i\pi}{2}(r+1)} \Gamma[r+1] \omega^{-(r+1)}, \tag{3.6}$$

where

$$a_r = \sum_{j=0}^{[r/2]} \frac{(-1)^j (i\omega_o)^{r-2j}}{(2)^j j!(r-2j)!}, \tag{3.7}$$

and

$$\overline{\hat{\phi}}(\omega) = O(1) \; as \; \omega \to 0+. \tag{3.8}$$

Now, assume $\hat{g}(\omega)$ satisfies (2.5) and (2.7) with $\eta_0 > 1$. Hence, using (2.29) and formula [[2],(11, 31), pp.318, 320], we get the following asymptotic expansion of

Morlet wavelet transform for $a \to \infty+$ as

$$
\begin{aligned}
(W_\phi g)(b,a) =& \frac{1}{2\pi}\Bigg\{\sum_{r=0}^{n-1} c_r\Bigg[2^{\frac{\eta_r}{2}} e^{\frac{-w_o^2}{2}}\Bigg(\Big(1+(-1)^{\eta_r}\Big)\Gamma[\frac{(1+\eta_r)}{2}] \\
&\times 1F1\Big[\frac{(1+\eta_r)}{2},\frac{1}{2},\frac{(bi+aw_o)^2}{2a^2}\Big]\Bigg] - \frac{1}{a}\sqrt{2}\Big(-1+(-1)^{\eta_r}\Big) \\
&\times \Gamma[\frac{(2+\eta_r)}{2}]\, 1F1\Big[\frac{(2+\eta_r)}{2},\frac{3}{2},\frac{(bi+aw_o)^2}{2a^2}\Big](bi+aw_o)\Bigg)\Bigg] \\
&+ \sum_{i=0}^{m-1} a_i\; e^{\frac{i\pi}{2}(i+1)}\Gamma[i+1]\Big[M[\hat{g}(\omega)e^{ib\omega};-r] \\
&- (-1)^{-r}M[\hat{g}(-\omega)e^{-ib\omega};-r] \qquad (3.9) \\
&- \sum_{j=0}^{I(i)-1} c_j A_j(-r;b)\Big(1-(-1)^{-\eta_j-r}e^{-i\pi(\eta_j-r)}\Big)\Big]a^{-r-\frac{1}{2}} + R_{n,m}\Bigg\},
\end{aligned}
\tag{3.10}
$$

where $R(r)$ is the index r for which $(r+1)-\eta_r \le 1 < (r+1)-\eta_r-1$ and $I(i)$ is the index i for which $r-\eta_r < 1 \le (r+1)-\eta_r$.
If $(r+1)-\eta_r = 1$ for some pair (i,r) then in (3.9), the corresponding sum of the terms must be replace by

$$
\begin{aligned}
& a^{-r-\frac{1}{2}}\Bigg\{a_i\; e^{\frac{i\pi}{2}(i+1)}\Gamma[i+1]\Big[M[\hat{g}(\omega)e^{ib\omega};-r] - (-1)^{-r}M[\hat{g}(-\omega)e^{-ib\omega};-r] \\
- & \sum_{j=0}^{I(i)-1} c_j A_j(-r;b)\Big(1-(-1)^{-\eta_j-r}e^{-i\pi(\eta_j-r)}\Big)\Big] \\
+ & c_r \lim_{z\to 0} 2^{\frac{1}{2}(z+\eta_r)}e^{-\frac{w_o^2}{2}}\sqrt{\pi}\Bigg(\Big(1+(-1)^{\eta_r}\Big)\Gamma\Big[\frac{1}{2}\Big(1+z+\eta_r\Big)\Big] \\
\times & 1F1\Big[\frac{1}{2}\Big(1+z+\eta_r\Big),\frac{1}{2},\frac{(bi+aw_o)^2}{2a^2}\Big] - \frac{1}{a}\sqrt{2}\Big(-1+(-1)^{\eta_r}\Big) \\
\times & \Gamma\Big[\frac{1}{2}\Big(2+z+\eta_r\Big)\Big]1F1\Big[\frac{1}{2}\Big(2+z+\eta_r\Big),\frac{3}{2},\frac{\Big(bi+aw_o\Big)^2}{2a^2}\Big]\Big(bi+aw_o\Big)\Bigg) \\
- & \sum_{p=0}^{R(r)-1} a_p\; e^{\frac{i\pi}{2}(p+1)}\Gamma[p+1]B_p(z+\eta_r+1;a;b)\Big(1-(-1)^{\eta_r-p}e^{-i\pi(z+\eta_r-p)}\Big)\Bigg\},
\end{aligned}
\tag{3.11}
$$

and the remainder term $R_{n,m}(a)$ as $a \to \infty+$ is given by

$$
R_{n,m}(a) = \left\{\begin{array}{ll} O(a^{-r-\frac{1}{2}} + a^{-\eta_r-\frac{1}{2}}), & when \quad -\eta_r \neq -r, \\ O(a^{-r-\frac{1}{2}}\log a), & when \quad -\eta_r = -r. \end{array}\right\}. \tag{3.12}
$$

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