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Fuzzy rarely α continuity

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ABSTRACT: In this paper, we introduce the concepts of fuzzy rare α continuous,fuzzy rarely continuous, fuzzy rarely pre-continuous, fuzzy rarely semi-continuous are introduced and studied in light of the concept of rare set in a fuzzy setting. Some interesting properties are investigated besides giving some examples.

Key Words: fuzzy rare set, fuzzy rarely α continuous, fuzzy rarely pre- continuous, fuzzy almost α continuous, fuzzy weekly α continuous, fuzzy rarely semi continuous.

Contents

1. Introductions

The study of fuzzy sets was initiated by Zadeh [19] in 1965. Thereafter the paper of Chang [3] in 1968 paved the way for the subsequent tremendous growth of the numerous fuzzy topological concepts. Recently Fuzzy Topology has been found to be very useful in solving many practical problems.

Several mathematicians have tried almost all the pivotal concepts of General Topology for extension to the fuzzy settings. In 1981, Azad [2] gave fuzzy version of the concepts given by Levine [[10], [11]] and thus initiated the study of weak forms of several notions in fuzzy topological spaces.

Popa [18] introduced the notion of rare continuity as a generalization of weak continuity [10] which has been further investigated by Long and Herrington [12] and S.Jafari [[6], [7]]. Noiri [15] introduced and investigated weakly α-continuity as a generalization of weak continuity. He also introduced and investigated almost α- continuity [16] and almost continuity [4]. The concepts of Rarely α continuity was introduced by S.Jafari cite9.

The purpose of the present paper is to introduce the concepts of fuzzy rare α continuous,fuzzy rarely continuous, fuzzy rarely pre-continuous, fuzzy rarely semi-continuous are introduced and studied in light of the concept of rare set in a fuzzy setting. Some interesting properties are investigated besides giving some examples.

Definition 1.1. [4] A fuzzy topology on a set X is a collection δ of fuzzy set in X satisfying

i) $0 \in \delta$ and $1 \in \delta$

ii) if μ and ν belong to δ then so does $\mu \wedge \nu \in \delta$.

iii) if μ_i belongs to δ for each $i \in I$ then so does $\vee_{i\in I}\mu_i$.

If δ is a fuzzy topology on X then the pair (X, δ) is called a fuzzy topological space. Every member of δ is called fuzzy open set. A fuzzy set is fuzzy closed if and only if its complement is fuzzy open.

Definition 1.2. [4] Let λ be any fuzzy set in fuzzy topological space (X, δ). We define the closure of λ to be $\wedge\{\mu|\mu \geq \lambda, \mu,$ is fuzzy closed$\}$ and interior of λ to be $\vee\{\sigma|\sigma \leq \lambda, \sigma,$ is fuzzy open$\}$. The interior of the fuzzy set λ and the closure of the fuzzy set λ in X will be denoted by $int\lambda$ and $cl\lambda$ respectively.

Definition 1.3. A fuzzy set λ in fuzzy topological space (X, δ) is said to be

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1) fuzzy α open [3] if $\lambda \leq int(cl(int(\lambda)))$

2) fuzzy pre open [3]if $\lambda \leq int(cl(\lambda))$

3) fuzzy semi open [2] if $\lambda \leq cl(int(\lambda))$

4) fuzzy regular open [2] if $\lambda = int(cl(\lambda))$

Definition 1.4. [3] For a fuzzy set λ of X, the α-closure of λ and α-interior of λ are defined respectively
$int_\alpha(\lambda) = \vee\{\mu : \lambda \geq \mu$ and μ is a fuzzy α open set $\}$
$cl_\alpha(\lambda) = \wedge\{\nu : \lambda \leq \nu$ and ν is a fuzzy α closed set $\}$.

2. Main Results

Definition 2.1. A fuzzy set R is called fuzzy rare set if $int(R) = \phi$.

Definition 2.2. A fuzzy set R is called fuzzy nowhere dense set if $int(cl(R)) = \phi$.

Definition 2.3. Let (X, T) and (Y, S) be two fuzzy topological spaces. A function $f : (X, T) \to (Y, S)$ is called

(i) fuzzy α continuous if for each fuzzy point x_t in X and each fuzzy open set G in Y containing $f(x_t)$, there exists a fuzzy α open set U in X , such that $f(U) \leq G$.

(ii) fuzzy almost α continuous [1] if for each fuzzy point x_t in X and each fuzzy open set G containing $f(x_t)$, there exists a fuzzy α open set U, such that $f(U) \leq int(cl(G))$.

(iii) fuzzy weakly α continuous if for each fuzzy point x_t in X and each fuzzy open set G containing $f(x_t)$, there exists a fuzzy α open set U, such that $f(U) \leq cl(G)$.

Definition 2.4. Let (X, T) and (Y, S) be two fuzzy topological spaces. A function $f : (X, T) \to (Y, S)$ is called

(i) fuzzy rarely α continuous if for each fuzzy point x_t in X and each fuzzy open set G in (Y, S) containing $f(x_t)$, there exist a fuzzy rare set R with $G \cap cl(R) = \phi$ and fuzzy α open set U in (X, T),such that $f(U) \leq G \cup R$.

(ii) fuzzy rarely continuous if for each fuzzy point x_t in X and each fuzzy open set [3] G in (Y, S) containing $f(x_t)$, there exist a fuzzy rare set R with $G \cap cl(R) = \phi$ and fuzzy open set U in (X, T),such that $f(U) \leq G \cup R$.

(iii) fuzzy rarely pre continuous if for each fuzzy point x_t in X and each fuzzy open set G in (Y, S) containing $f(x_t)$, there exist a fuzzy rare set R with $G \cap cl(R) = \phi$ and fuzzy pre open set[16] U in (X, T),such that $f(U) \leq G \cup R$.

(iv) fuzzy rarely semi continuous if for each fuzzy point x_t in X and each fuzzy open set G in (Y, S) containing $f(x_t)$, there exist a fuzzy rare set R with $G \cap cl(R) = \phi$ and fuzzy semi open set [2] U in (X, T),such that $f(U) \leq G \cup R$.

Example 2.1. Let $X = \{a, b, c\}$. Define the fuzzy sets A, B and C as follows:
$A = \langle x, (\frac{a}{0}, \frac{b}{0}, \frac{c}{1})\rangle$, $B = \langle x, (\frac{a}{1}, \frac{b}{0}, \frac{c}{0})\rangle$ and $C = \langle x, (\frac{a}{0}, \frac{b}{1}, \frac{c}{0})\rangle$. Then $T = \{\phi, 1_X, A\}$ and $S = \{\phi, 1_X, A, B, A \cup B\}$ are fuzzy topologies on X. Let (X, T) and (X, S) are fuzzy topological spaces. Define $f : (X, T) \to (X, S)$ as a identity function. Clearly f is rarely α continuous.

Proposition 2.1. Let (X, T) and (Y, S) be any two fuzzy topological spaces. For a function $f : (X, T) \to (Y, S)$ the following statements are equivalents:

(i) The function f is fuzzy rarely α continuous at x_t in (X, T).

(ii) For each fuzzy open set G containing $f(x_t)$, there exists a fuzzy α open set U in (X,T) such that $int(f(U) \cap G^c) = \phi$.

(iii) For each fuzzy open set G containing $f(x_t)$, there exists a fuzzy α open set U in (X,T), such that $int(f(U)) \leq cl(G)$.

(iv) For each fuzzy open set G in (Y,S) containing $f(x_t)$, there exists a fuzzy rare set R with $G \cap cl(R) = \phi$, such that $x_t \in int_\alpha(f^{-1}(G \cup R))$.

(v) For each fuzzy open set G in (Y,S) containing $f(x_t)$, there exists a fuzzy rare set R with $cl(G) \cap R = \phi$, such that $x_t \in int_\alpha(f^{-1}(cl(G) \cup R))$

(vi) For each fuzzy regular open set G in (Y,S) containing $f(x_t)$, there exists a fuzzy rare set R with $cl(G) \cap R = \phi$, such that $x_t \in int_\alpha(f^{-1}(G \cup R))$

Proof. $(i) \Rightarrow (ii)$ Let G be a fuzzy open set in (Y,S) containing $f(x_t)$. By $f(x_t) \in G \leq int(cl(G))$ and $int(cl(G))$ containing $f(x_t)$, there exists a fuzzy rare set R with $int(cl(G)) \cap cl(R) = \phi$ and a fuzzy α open set U in (X,T) containing x_t, such that $f(U) \leq int(cl(G)) \cup R$. We have $int(f(U) \cap G^c) = int(f(U)) \cap int(G^c) \leq int(cl(G) \cup R) \cap (cl(G))^c \leq cl(G) \cup int(R) \cap (cl(G))^c = \phi$.

$(ii) \Rightarrow (iii)$ It is straight forward.

$(iii) \Rightarrow (i)$ Let G be a fuzzy open set in (Y,S) containing $f(x_t)$. Then by (iii), there exists a fuzzy α open set U containing x_t, such that $int(f(U) \leq cl(G)$. We have $f(U) = (f(U) \cap (int(f(U)))^c) \cup int(f(U)) < (f(U) \cap (int(f(U)))^c) \cup cl(G) = (f(U) \cap (int(f(U)))^c) \cup G \cup (cl(G) \cap G^c) = (f(U) \cap (int(f(U)))^c \cap G^c) \cup G \cup (cl(G) \cap G^c)$. Set $R_1 = f(U) \cap (int(f(U)))^c \cap G^c$ and $R_2 = cl(G) \cap G^c$. Then R_1 and R_2 are fuzzy rare sets. More $R = R_1 \cup R_2$ is a fuzzy set such that $cl(R) \cap G = \phi$ and $f(U) \leq G \cup R$. This show that f is fuzzy rarely α continuous.

$(i) \Rightarrow (iv)$ Suppose that G be a fuzzy open set in (Y,S) containing $f(x_t)$. Then there exists a fuzzy rare set R with $G \cap cl(R) = \phi$ and U be a fuzzy α open set in (X,T) containing x_t such that $f(U) \leq G \cup R$. It follows that $x_t \in U \leq f^{-1}(G \cup R)$. This implies that $x_t \in int_\alpha(f^{-1}(G \cup R))$.

$(iv) \Rightarrow (v)$ Suppose that G be a fuzzy open set in (Y,S) containing $f(x_t)$. Then there exists a fuzzy rare set R with $G \cap cl(R) = \phi$ such that $x_t \in int_\alpha(f^{-1}(G \cup R))$. Since $G \cap cl(R) = \phi$,$R \leq G^c$, where $G^c = (cl(G))^c \cup (cl(G) \cap G^c)$. Now, we have $R \leq R \cup (cl(G))^c \cup (cl(G) \cap G^c)$. Now, $R_1 = R \cap (cl(G))^c$. It follows that R_1 is a fuzzy rare set with $cl(G) \cap R_1 = \phi$. Therefore $x_t \in int_\alpha(f^{-1}(G \cup R)) \leq int_\alpha(f^{-1}(G \cup R_1))$.

$(v) \Rightarrow (vi)$ Assume that G be a fuzzy regular open set [2] in (Y,S) containing $f(x_t)$. Then there exists a fuzzy rare set R with $cl(G) \cap R = \phi$, such that $x_t \in int_\alpha(f^{-1}(cl(G) \cup R))$. Now $R_1 = R \cup (cl(G) \cup G^c)$. It follows that R_1 is a fuzzy rare set and $(G \cap cl(R_1)) = \phi$. Hence $x_t \in int_\alpha(f^{-1}(cl(G) \cup R)) = int_\alpha(f^{-1}(G \cup (cl(G) \cap G^c)) \cup R) = int_\alpha(f^{-1}(G \cup R_1))$. Therefore $x_t \in \int_\alpha(f^{-1}(G \cup R_1))$.

$(vi) \Rightarrow (ii)$ Let G be a fuzzy open set in (Y,S) containing $f(x_t)$. By $f(x_t) \in G \leq int(cl(G))$ and the fact that $int(cl(G))$ is a fuzzy regular open in (Y,S), there exists a fuzzy rare set R and $int(cl(G)) \cap cl(R) = \phi$,such that $x_t \in int_\alpha(f^{-1}(int(cl(G)) \cup R)$. Let $U = int_\alpha(f^{-1}(int(cl(G)) \cup R)$. Hence U is a fuzzy α open set in (X,T) containing x_t and therefore $f(U) \leq int(cl(G)) \cup R$. Hence, we have $int(f(U) \cap G^c) = \phi$. ☐

Proposition 2.2. Let (X,T) and (Y,S) be any two fuzzy topological space. Then a function $f : (X,T) \to (Y,S)$ is a fuzzy rarely α continuous if and only if $f^{-1}(G) \leq int_\alpha(f^{-1}(G \cup R))$ for every fuzzy open set G in (Y,S), where R is a fuzzy rare set with $cl(R) \cap G = \phi$.

Proof. Suppose that G be a fuzzy rarely α open set in (Y,S) containing $f(x_t)$. Then $G \cap cl(R) = \phi$ and U be a fuzzy α open set in (X,T) containing x_t, such that $f(U) \leq G \cup R$. It follows that $x_t \in U \leq f^{-1}(G \cup R)$. This implies that $f^{-1}(G) \leq int_\alpha(f^{-1}(G \cup R))$. ☐

Definition 2.5. A function $f : (X,T) \to (Y,S)$ is fuzzy $I\alpha$ continuous at x_t in (X,T) if for each fuzzy open set G in (Y,S) containing $f(x_t)$, there exists a fuzzy α open set U containing x_t, such that $int(f(U)) \leq G$.

If f has this property at each fuzzy point x_t in (X,T), then we say that f is fuzzy $I\alpha$ continuous on (X,T).

Example 2.2. Let $X = \{a, b, c\}$. Define the fuzzy sets A and B as follows:
$A = \langle x, (\frac{a}{0}, \frac{b}{1}, \frac{c}{0})\rangle$ and $B = \langle x, (\frac{a}{1}, \frac{b}{0}, \frac{c}{0})\rangle$. Then $T = \{\phi, 1_X, A\}$ and $S = \{\phi, 1_X, B\}$ are fuzzy topologies on X. Let (X, T) and (X, S) are fuzzy topological spaces. Let $f : (X, T) \to (Y, S)$ as defined by $f(a) = f(b) = b$ and $f(c) = c$ is fuzzy $I\alpha$ continuous.

Proposition 2.3. Let (Y, S) be a fuzzy regular space [2]. Then the function $f : (X, T) \to (Y, S)$ is fuzzy $I\alpha$ continuous on X if and only if f is fuzzy rarely α continuous on X.

Proof. $\Rightarrow$ It is obvious.
$\Leftarrow$ Let f be fuzzy rarely α continuous on (X, T). Suppose that $f(x_t) \in G$, where G is a fuzzy open set in (Y, S) and a fuzzy point x_t in X. By the fuzzy regularity of (Y, S), there exists a fuzzy open set G_1 in (Y, S), such that G_1 containing $f(x_t)$ and $cl(G_1) \leq G$. Since f is fuzzy rarely α continuous, then there exists a fuzzy α open set U, such that $int(f(U)) \leq cl(G_1)$. This implies that $int(f(U)) \leq G$, which means that f is fuzzy $I\alpha$ continuous on X. □

Definition 2.6. A function $f : (X, T) \to (Y, S)$ is called fuzzy Pre α open if for every fuzzy α open set U in X such that $f(U)$ is an fuzzy α open in Y.

Proposition 2.4. If a function $f : (X, T) \to (Y, S)$ is a fuzzy pre α open, fuzzy rarely α continuous then f is fuzzy almost α continuous.

Proof. suppose that a fuzzy point x_t in X and a fuzzy open set G in Y, containing $f(x_t)$. Since f is fuzzy rarely α continuous at x_t, then there exists a fuzzy α open set U in X, such that $int(f(U)) \subset cl(G)$. Since f is fuzzy pre α open, we have $f(U)$ in Y. This implies that $f(U) \subset int(cl(int(f(U)))) \subset int(cl(G))$. Hence f is fuzzy almost α continuous. □

For a mapping $f : X \to Y$, the graph $g : X \to X \times Y$ of f is defined by $g(x) = (x, f(x))$, for each $x \in X$.

Proposition 2.5. Let $f : (X, T) \to (Y, S)$ be any mapping. If the $g : X \to X \times Y$ of f is fuzzy rarely α continuous then f is also fuzzy rarely α continuous.

Proof. Suppose that a fuzzy point x_t in X and a fuzzy open set W in Y, containing $g(x_t)$. It follows that there exists fuzzy open sets 1_X and V in X and Y respectively, such that $(x_t, f(x_t)) \in 1_X \times V \subset W$. Since f is fuzzy rarely α continuous, there exists a fuzzy α open set G, such that $int(f(G)) \subset cl(V)$. Let $E = 1_X \cap G$. It follows that E be a fuzzy α open set in X, and we have $int(g(E)) \subset int(1_X \times f(G)) \subset 1_X \times cl(V) \subset cl(W)$. Therefore g is fuzzy rarely α continuous. □

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