



Eigenvalues of the $p(x)$ –biharmonic operator with indefinite weight under Neumann boundary conditions

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ABSTRACT: In this paper we will study the existence of solutions for the nonhomogeneous elliptic equation with variable exponent $\Delta_{p(x)}^2 u = \lambda V(x)|u|^{q(x)-2}u$, in a smooth bounded domain, under Neumann boundary conditions, where λ is a positive real number, $p, q : \bar{\Omega} \rightarrow \mathbb{R}$, are continuous functions, and V is an indefinite weight function. Considering different situations concerning the growth rates involved in the above quoted problem, we will prove the existence of a continuous family of eigenvalues.

Key Words: Fourth order elliptic equation, variable exponent, Neumann boundary conditions, Ekeland variational principle.

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1. Introduction

We are concerned here with the eigenvalue problem:

$$\begin{cases} \Delta_{p(x)}^2 u = \lambda V(x)|u|^{q(x)-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial}{\partial \nu} (|\Delta u|^{p(x)-2} \Delta u) = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where Ω is a bounded domain in \mathbb{R}^N with smooth boundary $\partial\Omega$, $N \geq 1$, $\Delta_{p(x)}^2 u = \Delta(|\Delta u|^{p(x)-2} \Delta u)$, is the $p(x)$ –biharmonic operator, $\lambda \geq 0$, p, q are continuous functions on $\bar{\Omega}$, and V is an indefinite weight function.

The aim of this work is to study the existence of solutions for the nonhomogeneous eigenvalue problem (1.1), by considering different situations concerning the growth rates involved in the above quoted problem, we will prove the existence of a continuous family of eigenvalues.

In recent years, the study of differential equations and variational problems with $p(x)$ –growth conditions is an interesting topic, which arises from nonlinear electrorheological fluids and other phenomena related to image processing, elasticity and the flow in porous media. In this context we refer to ([10], [11], [6], [14],

[12], [13]).

This work is motivated by recent results in mathematical modeling of non Newtonian fluids and elastic mechanics, in particular, the electrorheological fluids (Smart fluids). This important class of fluids is characterized by change of viscosity, which is not easy to manipulate and depends on the electric field. These fluids, which are known under the name ER fluids, have many applications in electric mechanics, fluid dynamics etc...

The same problem, for $V(x) = 1$ and $p(x) = q(x)$ is studied by Ben Hadouch, El Allali, Ayoujl and Tsouli [2]. The authors established the existence of a continuous family of eigenvalues by using the mountain pass lemma and Ekeland variational principle.

Bin ge and Yuhu Wu in [15], studied the following nonhomogeneous eigenvalue problem

$$\begin{cases} \Delta_{p(x)}^2 u = \lambda V(x) |u|^{q(x)-2} u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.2)$$

They proved the existence of a continuous family of eigenvalues by considering different situations concerning the growth rates involved in the above quoted problem. In the case where $p(x) = q(x)$, the authors in [14] investigated the eigenvalues of the $p(x)$ -biharmonic with Navier boundary conditions. Ayoujl and El Amrouss [1], studied the same nonhomogeneous eigenvalue problem in the particular case when $V(x) = 1$.

In the case when $\max_{x \in \overline{\Omega}} q(x) < \min_{x \in \overline{\Omega}} p(x)$ it can be proved that the energy functional associated to problem (1.2) when $V(x) = 1$, has a nontrivial minimum for any positive λ (see Theorem 3.1 in [1]).

In the case when $\min_{x \in \overline{\Omega}} q(x) < \min_{x \in \overline{\Omega}} p(x)$ and $q(x)$ has a subcritical growth, Ayoujl and El Amrouss [1] used the Ekelands variational principle in order to prove the existence of a continuous family of eigenvalues which lies in a neighborhood of the origin.

In the case when $\max_{x \in \overline{\Omega}} p(x) < \min_{x \in \overline{\Omega}} q(x) \leq \max_{x \in \overline{\Omega}} q(x) < \frac{Np(x)}{N-2p(x)}$, by Theorem 3.8 in [1], for every $\lambda > 0$, the energy functional Φ_λ corresponding to (1.2) has a mountain pass type critical point which is nontrivial and nonnegative, and hence $\Lambda = (0, +\infty)$. The authors established the existence of infinity many eigenvalues for problem (1.2) if $q(x) = p(x)$ and $V(x) = 1$ by using an argument based on the Ljusternik-Schnirelman critical point theory. Denoting by Λ the set of all nonnegative eigenvalues, they showed that $\sup \Lambda = +\infty$.

Inspired by the above-mentioned paper, we will study the existence of solutions for the non-homogeneous elliptic eigenvalue problem

$$\begin{cases} \Delta_{p(x)}^2 u = \lambda V(x) |u|^{q(x)-2} u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial}{\partial \nu} (|\Delta u|^{p(x)-2} \Delta u) = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.3)$$

2. Preliminaries

In order to deal with $p(x)$ -biharmonic operator problems, we need some results on spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$ and some properties of $p(x)$ -biharmonic operator, which we will use later.

Define the generalized Lebesgue space by:

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \longrightarrow \mathbb{R}, \quad \text{measurable and} \quad \int_{\Omega} |u(x)|^{p(x)} dx < \infty \right\},$$

where $p \in C_+(\overline{\Omega})$ and

$$C_+(\overline{\Omega}) = \{ h \in C(\overline{\Omega}) : h(x) > 1, \quad \forall x \in \overline{\Omega} \}.$$

Denote

$$p^+ = \max_{x \in \overline{\Omega}} p(x), \quad p^- = \min_{x \in \overline{\Omega}} p(x),$$

and for all $x \in \overline{\Omega}$ and $k \geq 1$

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N, \end{cases}$$

and

$$p_k^*(x) = \begin{cases} \frac{Np(x)}{N-kp(x)} & \text{if } kp(x) < N, \\ +\infty & \text{if } kp(x) \geq N. \end{cases}$$

One introduces in $L^{p(x)}(\Omega)$ the following norm

$$|u|_{p(x)} = \inf \left\{ \mu > 0; \quad \int_{\Omega} \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \leq 1 \right\},$$

and the space $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ is a Banach.

Proposition 2.1. [28] *The space $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ is separable, uniformly convex, reflexive and its conjugate space is $L^{q(x)}(\Omega)$ where $q(x)$ is the conjugate function of $p(x)$ i.e*

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1, \quad \forall x \in \Omega.$$

For all $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$ the Hölder's type inequality

$$\left| \int_{\Omega} uv dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-} \right) |u|_{p(x)} |v|_{q(x)} \quad (2.1)$$

holds true.

Moreover, if $p_1, p_2, p_3 \in C_+(\overline{\Omega})$ and $\frac{1}{p_1(x)} + \frac{1}{p_2(x)} + \frac{1}{p_3(x)} = 1$, then for any

$u \in L^{p_1(x)}(\Omega)$, $v \in L^{p_2(x)}(\Omega)$ and $w \in L^{p_3(x)}(\Omega)$ the following inequality holds (see [24], proposition 2.5):

$$\int_{\Omega} |uvw| dx \leq \left(\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \right) |u|_{p_1(x)} |v|_{p_2(x)} |w|_{p_3(x)}. \quad (2.2)$$

Furthermore, if we define the mapping $\rho : L^{p(x)}(\Omega) \rightarrow \mathbb{R}$ by

$$\rho(u) = \int_{\Omega} |u|^{p(x)} dx,$$

then the following relations hold

$$|u|_{p(x)} < 1 (= 1, > 1) \Leftrightarrow \rho(u) < 1 (= 1, > 1), \quad (2.3)$$

$$|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \leq \rho(u) \leq |u|_{p(x)}^{p^+} \quad (2.4)$$

$$|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \leq \rho(u) \leq |u|_{p(x)}^{p^-} \quad (2.5)$$

$$|u_n - u|_{p(x)} \rightarrow 0 \Leftrightarrow \rho(u_n - u) \rightarrow 0. \quad (2.6)$$

We recall also the following proposition, which will be needed later:

Proposition 2.2. ([3]) *Let p and q be measurable functions such that $p \in L^{\infty}(\Omega)$ and $1 < p(x)q(x) \leq \infty$, for a.e. $x \in \Omega$. Let $u \in L^{q(x)}(\Omega)$, $u \neq 0$. Then*

$$\begin{aligned} |u|_{p(x)q(x)} \leq 1 &\Rightarrow |u|_{p(x)q(x)}^{p^+} \leq |u|_{q(x)}^{p(x)} \leq |u|_{p(x)q(x)}^{p^-}, \\ |u|_{p(x)q(x)} \geq 1 &\Rightarrow |u|_{p(x)q(x)}^{p^-} \leq |u|_{q(x)}^{p(x)} \leq |u|_{p(x)q(x)}^{p^+} \end{aligned} \quad (2.7)$$

The Sobolev space with variable exponent $W^{k,p(x)}(\Omega)$ is defined by

$$W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : D^{\alpha}u \in L^{p(x)}(\Omega), |\alpha| \leq k\},$$

where

$$D^{\alpha}u = \frac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_N^{\alpha_N}},$$

is the derivation in distribution sense, with $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$ is a multi-index

$$\text{and } |\alpha| = \sum_{i=1}^{i=N} \alpha_i.$$

The space $W^{k,p(x)}(\Omega)$, equipped with the norm

$$\|u\|_{k,p(x)} = \sum_{|\alpha| \leq k} |D^{\alpha}u|_{p(x)},$$

also becomes a Banach, separable and reflexive space. For more details, we refer to ([25], [4], [12], [28]).

Remark 2.1. ([29]) The norm $\|u\|_{2,p(x)}$ is equivalent to the norm $\|u\| = |\Delta u|_{p(x)}$ and $(W^{2,p(x)}(\Omega); \|\cdot\|)$ is a Banach, separable and reflexive space.

Through this paper, we will consider the following space

$$X = \{u \in W^{2,p(x)}(\Omega) : \frac{\partial u}{\partial \nu}|_{\partial\Omega} = 0\}.$$

which is considered in ([18]) and ([2]). They have proved that X is a nonempty, well defined and closed subspace of $W^{2,p(x)}(\Omega)$. For this they have showed the following boundary trace embedding theorem for variable exponent Sobolev spaces.

Theorem 2.3. ([18]) Let Ω be a bounded domain in \mathbb{R}^N with C^2 boundary. If $2p(x) \geq N \geq 2$ for all $x \in \overline{\Omega}$, then for all $q \in C_+(\Omega)$ there is a continuous boundary trace embedding

$$W^{2,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial\Omega), \quad (2.8)$$

and

$$W^{2,p(x)}(\Omega) \hookrightarrow W^{1,p(x)}(\partial\Omega). \quad (2.9)$$

Proposition 2.4. ([18]) If $2p(x) \geq N$ for all $x \in \overline{\Omega}$, then the set

$$X = \{u \in W^{2,p(x)}(\Omega) : \frac{\partial u}{\partial \nu}|_{\partial\Omega} = 0\}$$

is a closed subspace of $W^{2,p(x)}(\Omega)$.

Remark 2.2. $(X; \|\cdot\|)$ is a Banach, separable and reflexive space.

Proposition 2.5. If we put

$$I(u) = \int_{\Omega} |\Delta u|^{p(x)} dx,$$

then for all $u \in X$ then the following relations hold true

- (i) $\|u\| < 1$ ($= 1; > 1$) $\iff I(u) < 1$ ($= 1; > 1$),
- (ii) $\|u\| \leq 1 \implies \|u\|^{p^+} \leq I(u) \leq \|u\|^{p^-}$,
- (iii) $\|u\| \geq 1 \implies \|u\|^{p^-} \leq I(u) \leq \|u\|^{p^+}$,
for all $u_n \in X$, we have
- (iv) $\|u_n\| \longrightarrow 0 \iff I(u_n) \longrightarrow 0$,
- (v) $\|u_n\| \longrightarrow \infty \iff I(u_n) \longrightarrow \infty$

A pair $(u, \lambda) \in X \times \mathbb{R}$ is a weak solution of (1.3) provided that

$$\int_{\Omega} |\Delta u|^{p(x)-2} \Delta u \Delta v dx = \lambda \int_{\Omega} V(x) |u|^{q(x)-2} u v dx, \quad \forall v \in X.$$

In the case where u is a nontrivial solution, such a pair (u, λ) is called an eigenpair, λ is an eigenvalue and u is called an associated eigenfunction.

Proposition 2.6. *If $u \in X$ is a weak solution of (1.3) and $u \in C^4(\overline{\Omega})$ then u is a classical solution of (1.3).*

Proof:

Let $u \in C^4(\overline{\Omega})$ be a weak solution of problem (1.3) then for every $\varphi \in X$, we have

$$\int_{\Omega} |\Delta u|^{p(x)-2} \Delta u \Delta \varphi dx = \lambda \int_{\Omega} V(x) |u|^{q(x)-2} u \varphi dx. \quad (2.10)$$

By applying Green formula, we have:

$$\begin{aligned} \int_{\Omega} \Delta(|\Delta u|^{p(x)-2} \Delta u) \varphi dx &= - \int_{\Omega} \nabla(|\Delta u|^{p(x)-2} \Delta u) \cdot \nabla \varphi dx \\ &\quad + \int_{\partial\Omega} \varphi \frac{\partial}{\partial \nu} (|\Delta u|^{p(x)-2} \Delta u) dx, \end{aligned} \quad (2.11)$$

and

$$\begin{aligned} \int_{\Omega} |\Delta u|^{p(x)-2} \Delta u \Delta \varphi dx &= - \int_{\Omega} \nabla(|\Delta u|^{p(x)-2} \Delta u) \cdot \nabla \varphi dx \\ &\quad + \int_{\partial\Omega} |\Delta u|^{p(x)-2} \Delta u \frac{\partial}{\partial \nu} (\varphi) dx, \end{aligned} \quad (2.12)$$

As $\varphi \in X$, then $\frac{\partial}{\partial \nu} (\varphi) = 0$. For $\varphi \in \mathcal{D}(\overline{\Omega})$, we have

$$\Delta(|\Delta u|^{p(x)-2} \Delta u) = \lambda V(x) |u|^{q(x)-2} u \quad \text{a.e } x \in \Omega.$$

For each $\varphi \in X$, we have

$$\int_{\partial\Omega} \frac{\partial}{\partial \nu} (|\Delta u|^{p(x)-2} \Delta u) \varphi dx = 0,$$

then for $\varphi \in \mathcal{D}(\overline{\Omega})$, we have

$$\int_{\partial\Omega} \frac{\partial}{\partial \nu} (|\Delta u|^{p(x)-2} \Delta u) \varphi dx = 0,$$

which implies that

$$\frac{\partial}{\partial \nu} (|\Delta u|^{p(x)-2} \Delta u) = 0 \quad \text{a.e } x \in \Omega$$

the result follows.

Proposition 2.7. [2] *Let $p \in C_+(\overline{\Omega})$ such that $2p(x) > N$ for all $x \in \overline{\Omega}$, then*

(1) *there exists a continuous and compact embedding of $W^{2,p(x)}(\Omega)$ into $L^{q(x)}(\Omega)$, for all $q \in C_+(\Omega)$.*

(2) *there exists a continuous embedding of $W^{2,p(x)}(\Omega)$ into $C(\overline{\Omega})$.*

In what follows, we assume that the functions $p, q \in C_+(\overline{\Omega})$.

3. Main results and proofs

In this section we prove two theorems for problem (1.1). First, we prove the existence of a continuous family of eigenvalues for problem (1.1), in a neighborhood of the origin.

Theorem 3.1. *If*

$$H_1(p, q, s): \quad q^+ < p^- < \frac{N}{2} < s(x), \forall x \in \overline{\Omega}, \text{ where } s(x) \in C_+(\overline{\Omega}).$$

$H_1(V) : \quad V(x) \in L^{s(x)}(\Omega)$ and there exists a measurable set $\Omega_0 \subset \Omega$ of positive measure such that $V(x) > 0$, a.e. $x \in \Omega_0$.

Then any $\lambda > 0$ is an eigenvalue for problem (1.1). Moreover, for any $\lambda > 0$ there exists a sequence (u_n) of nontrivial weak solutions for problem (1.1) such that $u_n \rightarrow 0$ in X .

In order to formulate the variational problem (1.1), let us introduce the functionals $F, G, \Phi_\lambda : X \rightarrow \mathbb{R}$ defined by

$$F(u) = \int_{\Omega} \frac{1}{p(x)} |\Delta u|^{p(x)} dx, \quad G(u) = \int_{\Omega} \frac{1}{q(x)} V(x) |u|^{q(x)} dx$$

and

$$\Phi_\lambda(u) = F(u) - \lambda G(u).$$

Denote by $s'(x)$ the conjugate exponent of the function $s(x)$ and put $\alpha(x) = \frac{s(x)q(x)}{s(x)-q(x)}$. Thus, by the proposition 2.7 the embeddings $X \hookrightarrow L^{s'(x)q(x)}(\Omega)$ and $X \hookrightarrow L^{\alpha(x)}(\Omega)$ are compact and continuous.

The Euler-Lagrange functional associated with (1.1) is defined as $\Phi_\lambda : X \rightarrow \mathbb{R}$,

$$\Phi_\lambda(u) = \int_{\Omega} \frac{1}{p(x)} |\Delta u|^{p(x)} dx - \lambda \int_{\Omega} \frac{1}{q(x)} V(x) |u|^{q(x)} dx.$$

We will show that $\Phi_\lambda \in C^1(X, \mathbb{R})$ and

$$\langle \Phi'_\lambda(u), v \rangle = \int_{\Omega} |\Delta u|^{p(x)-2} \Delta u \Delta v dx - \lambda \int_{\Omega} V(x) |u|^{q(x)-2} u v dx, \forall v \in X.$$

We only need to prove that $G \in C^1(X, \mathbb{R})$, that is, we show for all $h \in X$,

$$\lim_{t \rightarrow 0} \frac{G(u+th) - G(u)}{t} = \langle dG(u), h \rangle,$$

and $dG : X \rightarrow X'$ is continuous, where we denote by X' the dual space of X . For all $h \in X$, we have

$$\begin{aligned}
\lim_{t \rightarrow 0} \frac{G(u+th) - G(u)}{t} &= \frac{d}{dt} G(u+th)|_{t=0} \\
&= \left(\frac{d}{dt} \int_{\Omega} \frac{V(x)}{q(x)} |u+th|^{q(x)} dx \right) |_{t=0} \\
&= \int_{\Omega} \frac{d}{dt} \left(\frac{V(x)}{q(x)} |u+th|^{q(x)} \right) |_{t=0} dx \\
&= \int_{\Omega} V(x) |u+th|^{q(x)-2} (u+th)h |_{t=0} dx \\
&= \int_{\Omega} V(x) |u|^{q(x)-2} u h dx \\
&= \langle dG(u), h \rangle.
\end{aligned}$$

The differentiation under the integral is allowed for t close to zero. indeed, for $|t| < 1$, using inequalities (2.2), (2.7) and condition $H_1(p, q, s)$, we have

$$\begin{aligned}
\int_{\Omega} |V(x)| |u+th|^{q(x)-2} (u+th)h dx &\leq \int_{\Omega} |V(x)| |u+th|^{q(x)-1} |h| dx \\
&\leq \int_{\Omega} |V(x)| (|u| + |h|)^{q(x)-1} |h| dx \\
&\leq 3|V|_{s(x)} \|u\| + |h|_{q(x)}^{q^i-1} \|h\|_{\alpha(x)} \\
&< +\infty,
\end{aligned}$$

where $i = +$ if $\|u\| + \|h\|_{q(x)} > 1$ and $i = -$ if $\|u\| + \|h\|_{q(x)} \leq 1$. Since $X \hookrightarrow L^{\alpha(x)}(\Omega)$, $X \hookrightarrow L^{q(x)}(\Omega)$ and $V \in L^{s(x)}(\Omega)$.

On the other hand, we have $X \hookrightarrow L^{\alpha(x)}(\Omega)$ (compact embedding). Furthermore, there exists c_1 such that $\|h\|_{\alpha(x)} \leq c_1 \|h\|$. Therefore, by condition $H_1(p, q, s)$, we have

$$\begin{aligned}
|\langle dG(h), h \rangle| &= \left| \int_{\Omega} V(x) |u|^{q(x)-2} u h dx \right| \\
&\leq \int_{\Omega} |V(x)| |u|^{q(x)-1} |h| dx \\
&\leq \left(\frac{1}{s^-} + \frac{q^+}{q^+ - 1} + \frac{1}{\alpha^-} \right) |V|_{s(x)} \|u\|^{q(x)-1} \left| \frac{q(x)}{q(x)-1} \right| \|h\|_{\alpha(x)} \\
&\leq \left(\frac{1}{s^-} + \frac{q^+}{q^+ - 1} + \frac{1}{\alpha^-} \right) |V|_{s(x)} \|u\|_{q(x)}^{q^i-1} \|h\|_{\alpha(x)} \\
&\leq c_1 \left(\frac{1}{s^-} + \frac{q^+}{q^+ - 1} + \frac{1}{\alpha^-} \right) |V|_{s(x)} \|u\|_{q(x)}^{q^i-1} \|h\|,
\end{aligned}$$

for any $h \in X$. Thus there exists $c_2 = c_1 \left(\frac{1}{s^-} + \frac{q^+}{q^+ - 1} + \frac{1}{\alpha^-} \right) |V|_{s(x)} \|u\|_{q(x)}^{q^i-1}$ such that

$$|\langle dG(u), h \rangle| \leq c_2 \|h\|.$$

Using the linearity of $dG(u)$ and the above inequality we deduce that $dG(u) \in X^*$. The map defined in $L^{q(x)}(\Omega)$ by $u \mapsto |u|^{q(x)-2}u \in L^{\frac{q(x)}{q(x)-1}}(\Omega)$ is continuous. For the Fréchet differentiability, we conclude that G is Fréchet differentiable. Furthermore,

$$\langle G'(u), v \rangle = \int_{\Omega} V(x) |u|^{q(x)-2} u v dx,$$

for all $u, v \in X$. Similarly, we can also show that $F \in C^1(X, \mathbb{R})$. Which implies that $\Phi_{\lambda} \in C^1(X, \mathbb{R})$ and

$$\langle \Phi'_{\lambda}(u), v \rangle = \int_{\Omega} |\Delta u|^{p(x)-2} \Delta u \Delta v dx - \lambda \int_{\Omega} V(x) |u|^{q(x)-2} u v dx.$$

for all $u, v \in X$. Thus the weak solutions of (1.1) coincide with the critical points of Φ_{λ} . If such a weak solution exists and is nontrivial, then the corresponding λ is an eigenvalue of problem (1.1).

Next, we write Φ'_{λ} as

$$\Phi'_{\lambda} = F' - \lambda G',$$

where $F', G' : X \rightarrow X'$ are defined by

$$\begin{aligned} \langle F'(u), v \rangle &= \int_{\Omega} |\Delta u|^{p(x)-2} \Delta u \Delta v dx, \\ \langle G'(u), v \rangle &= \int_{\Omega} V(x) |u|^{q(x)-2} u v dx. \end{aligned}$$

We have

Proposition 3.2. [17, Proposition 2.5]

- (i) G' is completely continuous, namely, $u_n \rightharpoonup u$ in X implies $G'(u_n) \rightarrow G'(u)$ in X' .
- (ii) F' satisfies condition (S^+) , namely, $u_n \rightharpoonup u$ in X and $\limsup \langle F'(u_n), u_n - u \rangle \leq 0$, imply $u_n \rightarrow u$ in X .

We want to apply the symmetric mountain pass lemma in [8] to prove the Theorem 3.1.

Theorem 3.3. (Symmetric mountain pass lemma) Let E be an infinite dimensional Banach space and $I \in C^1(E, \mathbb{R})$ satisfy the following two assumptions:

- (A1) $I(u)$ is even, bounded from below, $I(0) = 0$ and $I(u)$ satisfies the Palais-Smale condition (PS), namely, any sequence u_n in E such that $I(u_n)$ is bounded and $I'(u_n) \rightarrow 0$ in E as $n \rightarrow \infty$ has a convergent subsequence.
- (A2) For each $k \in \mathbb{N}$, there exists an $A_k \in \Gamma_k$ such that $\sup_{u \in A_k} I(u) < 0$.

Then, $I(u)$ admits a sequence of critical points u_k such that

$$I(u_k) < 0, u_k \neq 0 \text{ and } \lim_k u_k = 0,$$

where Γ_k denote the family of closed symmetric subsets A of E such that $0 \notin A$ and $\gamma(A) \geq k$ with $\gamma(A)$ is the genus of A , i.e.,

$$\gamma(K) = \inf\{k \in \mathbb{N} : \exists h : K \rightarrow \mathbb{R}^k \setminus \{0\} \text{ such that } h \text{ is continuous and odd}\}.$$

We start with two auxiliary results.

Lemma 3.4. *The functional Φ_λ is even, bounded from below, satisfies the (PS) condition and $\Phi_\lambda(0) = 0$.*

Proof:

It is clear that Φ_λ is even and $\Phi_\lambda(0) = 0$. Since the embedding $X \hookrightarrow L^{s'(x)q(x)}(\Omega)$ is continuous, we can find a constant $c_3 > 0$ such that

$$|u|_{s'(x)q(x)} \leq c_3 \|u\|, \quad \forall u \in X. \quad (3.1)$$

According to the fact that

$$|u(x)|^{q(x)} \leq |u(x)|^{q^+} + |u(x)|^{q^-}, \quad \forall x \in \overline{\Omega}. \quad (3.2)$$

From the (3.2), we obtain:

$$\int_{\Omega} V(x) |u|^{q(x)} dx \leq |V|_{s(x)} \left| |u|^{q(x)} \right|_{s'(x)} \leq |V|_{s(x)} \left(|u|_{q(x)s'(x)}^{q^+} + |u|_{q(x)s'(x)}^{q^-} \right). \quad (3.3)$$

Combining ((3.1)) and ((3.3)), we obtain

$$\int_{\Omega} V(x) |u|^{q(x)} dx \leq |V|_{s(x)} \left(c_3^{q^+} \|u\|^{q^+} + c_3^{q^-} \|u\|^{q^-} \right). \quad (3.4)$$

Hence, from (3.4), we deduce that for any $u \in X$, we have

$$\begin{aligned} \Phi_\lambda(u) &= \int_{\Omega} \frac{1}{p(x)} |\Delta u|^{p(x)} dx - \lambda \int_{\Omega} \frac{V(x)}{q(x)} |u|^{q(x)} dx \\ &\geq \frac{1}{p^+} \alpha(\|u\|) - \frac{\lambda}{q^-} |V|_{s(x)} \left(c_3^{q^+} \|u\|^{q^+} + c_3^{q^-} \|u\|^{q^-} \right), \end{aligned}$$

where $\alpha : [0, +\infty[\rightarrow \mathbb{R}$ is defined by

$$\alpha(t) = \begin{cases} t^{p^+}, & \text{if } t \leq 1, \\ t^{p^-}, & \text{if } t > 1. \end{cases} \quad (3.5)$$

As $q^+ < p^-$, Φ_λ is bounded from below and coercive because, that is, $\Phi_\lambda(u) \rightarrow \infty$ as $\|u\| \rightarrow \infty$.

It remains to show that the functional Φ_λ satisfies the (PS) condition to complete the proof. Let $(u_n) \subset X$ be a (PS) sequence of Φ_λ in X ; that is,

$$\Phi_\lambda(u_n) \text{ is bounded and } \Phi'_\lambda(u_n) \rightarrow 0 \text{ in } X'. \quad (3.6)$$

Then, by the coercivity of Φ_λ , the sequence (u_n) is bounded in X . By the reflexivity of X , for a subsequence still denoted (u_n) , we have

$$u_n \rightharpoonup u \text{ in } X.$$

Since $q^+ < p^-$, it follows from proposition 2.7 that $u_n \rightharpoonup u$ in $L^{q(x)}(\Omega)$. We will show that

$$\lim_{n \rightarrow \infty} \int_{\Omega} V(x) |u_n|^{q(x)-2} u_n (u_n - u) dx = 0. \quad (3.7)$$

In fact, from the Hölder type inequality, we have

$$\begin{aligned} \int_{\Omega} V(x) |u_n|^{q(x)-2} u_n (u_n - u) dx &\leq |V(x)|_{s(x)} \left| |u_n|^{q(x)-2} u_n (u_n - u) \right|_{s'(x)} \\ &\leq |V(x)|_{s(x)} \left| |u_n|^{q(x)-2} u_n \right|_{\frac{q(x)}{q(x)-1}} |u_n - u|_{\alpha(x)} \\ &\leq |V|_{s(x)} (1 + |u_n|_{q(x)}^{q^+-1}) |u_n - u|_{\alpha(x)}. \end{aligned}$$

Since X is continuously embedded in $L^{q(x)}(\Omega)$ and (u_n) is bounded in X , so u_n is bounded in $L^{q(x)}(\Omega)$. On the other hand, since the embedding $X \hookrightarrow L^{\alpha(x)}(\Omega)$ is compact where $\alpha(x) = \frac{s(x)q(x)}{s(x)-q(x)}$, we deduce that $|u_n - u|_{\alpha(x)} \rightarrow 0$ as $n \rightarrow +\infty$.

we deduce that

$$\langle G'(u_n), u_n - u \rangle = \int_{\Omega} V(x) |u_n|^{q(x)-2} u_n (u_n - u) dx \rightarrow 0. \quad (3.8)$$

In view of (3.6) and (3.8), we obtain

$$\Phi'_\lambda(u_n) + \lambda \langle G'(u_n), u_n - u \rangle = \langle F'(u_n), u_n - u \rangle \rightarrow 0 \text{ as } n \rightarrow \infty.$$

According to the fact that F' satisfies condition (S^+) , we have $u_n \rightarrow u$ in X . The proof is complete.

Lemma 3.5. *For each $n \in \mathbb{N}^*$, there exists an $H_n \in \Gamma_n$ such that*

$$\sup_{u \in H_n} \Phi_\lambda(u) < 0.$$

Proof: Let $v_1, v_2, \dots, v_n \in C_0^\infty(\Omega)$ such that $\text{supp}(v_i) \cap \text{supp}(v_j) = \emptyset$ if $i \neq j$ and $\text{meas}(\text{supp}(v_j)) > 0$ for $i, j \in \{1, 2, \dots, n\}$. Take $F_n = \text{span}\{v_1, v_2, \dots, v_n\}$, it is clear that $\dim F_n = n$, $\text{supp}(v_i) \subset \Omega_0$ and

$$\int_{\Omega} |v(x)|^{q(x)} dx > 0 \text{ for all } v \in F_n \setminus \{0\}.$$

Denote $S = \{v \in X : \|v\| = 1\}$ and $H_n(t) = t(S \cap F_n)$ for $0 < t \leq 1$. Obviously, $\gamma(H_n(t)) = n$, for all $t \in]0, 1]$.

Now, we show that, for any $n \in \mathbb{N}^*$, there exist $t_n \in]0, 1]$ such that

$$\sup_{u \in H_n(t_n)} \Phi_\lambda(u) < 0.$$

Indeed, for $0 < t \leq 1$, we have

$$\begin{aligned} \sup_{u \in H_n(t)} \Phi_\lambda(u) &\leq \sup_{v \in S \cap F_n} \Phi_\lambda(tv) \\ &= \sup_{v \in S \cap F_n} \left\{ \int_{\Omega} \frac{t^{p(x)}}{p(x)} |\Delta v(x)|^{p(x)} dx - \lambda \int_{\Omega} \frac{t^{q(x)}}{q(x)} V(x) |v(x)|^{q(x)} dx \right\} \\ &\leq \sup_{v \in S \cap F_n} \left\{ \frac{t^{p^-}}{p^-} \int_{\Omega} |\Delta v(x)|^{p(x)} dx - \frac{\lambda t^{q^+}}{q^+} \int_{\Omega_0} V(x) |v(x)|^{q(x)} dx \right\} \\ &\leq \sup_{v \in S \cap F_n} \left\{ t^{p^-} \left(\frac{1}{p^-} - \frac{\lambda}{q^+} \frac{1}{t^{p^- - q^+}} \int_{\Omega_0} V(x) |v(x)|^{q(x)} dx \right) \right\}. \end{aligned}$$

Since $m := \min_{v \in S \cap F_n} \int_{\Omega_0} V(x) |v(x)|^{q(x)} dx > 0$, we may choose $t_n \in]0, 1]$ which is small enough such that

$$\frac{1}{p^-} - \frac{\lambda}{q^+} \frac{1}{t_n^{p^- - q^+}} m < 0.$$

This completes the proof.

Proof:[Proof of Theorem 3.1]

By lemmas 3.4, 3.5 and theorem 3.3, Φ_λ admits a sequence of nontrivial weak solutions $(u_n)_n$ such that for any n , we have

$$u_n \neq 0, \quad \Phi'_\lambda(u_n) = 0, \quad \Phi_\lambda(u_n) \leq 0, \quad \lim_n u_n = 0. \quad (3.9)$$

Theorem 3.6. *If*

$H_2(p, q, s): \quad q^- < p^-$ and $q^+ < p_2^*(x)$, for all $x \in \overline{\Omega}$.

$H_2(V): \quad V \in L^{s(x)}(\Omega)$ and there exists a measurable set $\Omega_0 \subset \Omega$ of positive measure such that $V(x) > 0$ a.e. $x \in \Omega_0$.

Then there exists λ^* such that any $\lambda \in (0, \lambda^*)$ is an eigenvalue for problem (1.1).

For applying Ekeland's variational principle. We start with two auxiliary results.

Lemma 3.7. *There exists $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$ there exist $\rho, a > 0$ such that $\Phi_\lambda(u) \geq a > 0$ for any $u \in X$ with $\|u\| = \rho$.*

Proof:

Since the embedding $X \hookrightarrow L^{s'(x)q(x)}(\Omega)$ is continuous, we can find a constant $c_3 > 0$ such that:

$$|u|_{s'(x)q(x)} \leq c_3 \|u\|, \quad \forall u \in X. \quad (3.10)$$

Let us fixe $\rho \in (0, 1)$ such that $\rho < \frac{1}{c_3}$. Then relation (3.10) implies $|u|_{s'(x)q(x)} < 1$, for all $u \in X$ with $\|u\| = \rho$. Thus,

$$\int_{\Omega} V(x) |u|^{q(x)} dx \leq |V|_{s(x)} \left| |u|^{q(x)} \right|_{s'(x)} \leq |V|_{s(x)} |u|_{q(x)s'(x)}^{q^-}, \quad (3.11)$$

for all $u \in X$ with $\|u\| = \rho$.

Combining (3.10) and (3.11), we obtain

$$\int_{\Omega} V(x) |u|^{q(x)} dx \leq c_3^{q^-} |V|_{s(x)} \|u\|^{q^-}. \quad (3.12)$$

Hence, from (3.12) we deduce that for any $u \in X$, with $\|u\| = \rho < 1$ we have

$$\begin{aligned} \Phi_{\lambda}(u) &= \int_{\Omega} \frac{1}{p(x)} |\Delta u|^{p(x)} dx - \lambda \int_{\Omega} \frac{V(x)}{q(x)} |u|^{q(x)} dx \\ &\geq \frac{1}{p^+} \alpha(\|u\|) - \frac{\lambda c_3^{q^-}}{q^-} |V|_{s(x)} \|u\|^{q^-} \\ &= \frac{1}{p^+} \rho^{p^+} - \frac{\lambda c_3^{q^-}}{q^-} |V|_{s(x)} \rho^{q^-} \\ &= \rho^{q^-} \left(\frac{1}{p^+} \rho^{p^+ - q^-} - \frac{\lambda c_3^{q^-}}{q^-} |V|_{s(x)} \right). \end{aligned}$$

Putting

$$\lambda^* = \frac{\rho^{p^+ - q^-}}{2p^+} \frac{q^-}{c_3^{q^-} |V|_{s(x)}},$$

then for any $\lambda \in (0, \lambda^*)$ and $u \in X$ with $\|u\| = \rho$, there exists $a = \frac{\rho^{p^+ - q^-}}{2p^+}$ such that

$$\Phi_{\lambda}(u) \geq a \geq 0.$$

Lemma 3.8. *There exists $\psi \in X$ such that $\psi \geq 0$, $\psi \neq 0$ and $\Phi_{\lambda}(t\psi) < 0$, for $t > 0$ small enough.*

Proof:

Since $q^- < p^-$, there exist $\varepsilon_0 > 0$ such that

$$q^- + \varepsilon_0 < p^-.$$

Since $q \in C(\overline{\Omega}_0)$, there exist an open set $\Omega_1 \subset \Omega_0$ such that

$$|q(x) - q^-| < \varepsilon_0, \quad \text{for all } x \in \Omega_1 \cap \Omega_0.$$

Thus, we deduce

$$q(x) \leq q^- + \varepsilon_0 < p^-, \quad \text{for all } x \in \Omega_1 \cap \Omega_0. \quad (3.13)$$

Take $\psi \in C_0^\infty(\Omega_0)$ such that $\overline{\Omega}_1 \subset \text{supp } \psi$, $\psi(x) = 1$ for $x \in \overline{\Omega}_1$ and $0 \leq \psi \leq 1$ in Ω_0 . Without loss of generality, we may assume $\|\psi\| = 1$, that is

$$\int_{\Omega_0} |\Delta \psi|^{p(x)} dx = 1. \quad (3.14)$$

By using (3.13), (3.14), for all $t \in]0, 1[$, we obtain

$$\begin{aligned} \Phi_\lambda(t\psi) &= \int_{\Omega} \frac{t^{p(x)}}{p(x)} |\Delta \psi|^{p(x)} dx - \lambda \int_{\Omega} \frac{t^{q(x)}}{q(x)} V(x) |\psi|^{q(x)} dx \\ &\leq \frac{t^{p^-}}{p^-} \int_{\Omega} |\Delta \psi|^{p(x)} dx - \frac{\lambda}{q^+} \int_{\Omega_1} t^{q(x)} V(x) |\psi|^{q(x)} dx \\ &\leq \frac{t^{p^-}}{p^-} - \frac{\lambda}{q^+} \int_{\Omega_1} t^{q(x)} V(x) |\psi|^{q(x)} dx \\ &\leq \frac{t^{p^-}}{p^-} - \frac{\lambda t^{q^- + \varepsilon_0}}{q^+} \int_{\Omega_1} V(x) |\psi|^{q(x)} dx. \end{aligned}$$

Then, for any $t < \delta^{\frac{1}{p^- - q^- - \varepsilon_0}}$, with $0 < \delta < \min\{1, \frac{\lambda p^-}{q^+} \int_{\Omega_1} V(x) |\psi|^{q(x)} dx\}$, we conclude that

$$\Phi_\lambda(t\psi) < 0.$$

The proof is complete.

Proof: [Proof of theorem 3.6]

By lemma 3.7, we have

$$\inf_{\partial B_\rho(0)} \Phi_\lambda(u) > 0. \quad (3.15)$$

On the other hand, from lemma (3.8), there exist $\psi \in X$ such that $\Phi_\lambda(t\psi) < 0$ for $t > 0$ small enough. Using 3.12, it follows that

$$\Phi_\lambda(u) \geq \frac{1}{p^+} \|u\|^{p^+} - \frac{\lambda}{q^-} c_3^{q^-} |V|_{s(x)} \|u\|^{q^-} \quad \text{for } u \in B_\rho(0).$$

Thus,

$$-\infty < \underline{c}_\lambda := \inf_{B_\rho(o)} \Phi_\lambda < 0,$$

Let

$$0 < \varepsilon < \inf_{\partial B_\rho(0)} \Phi_\lambda - \inf_{B_\rho(0)} \Phi_\lambda.$$

Then, by applying Ekeland's variational principle to the functional

$$\Phi_\lambda : \overline{B_\rho(0)} \rightarrow \mathbb{R},$$

there exists $u_\varepsilon \in \overline{B_\rho(0)}$ such that

$$\begin{aligned}\Phi_\lambda(u_\varepsilon) &\leq \inf_{\overline{B_\rho(0)}} \Phi_\lambda + \varepsilon, \\ \Phi_\lambda(u_\varepsilon) &< \Phi_\lambda(u) + \varepsilon \|u - u_\varepsilon\| \quad \text{for } u \neq u_\varepsilon.\end{aligned}$$

Since $\Phi_\lambda(u_\varepsilon) < \inf_{\overline{B_\rho(0)}} \Phi_\lambda + \varepsilon < \inf_{\partial B_\rho(0)} \Phi_\lambda$, we deduce $u_\varepsilon \in B_\rho(0)$. Now, define $I_\lambda : \overline{B_\rho(0)} \rightarrow \mathbb{R}$ by

$$I_\lambda(u) = \Phi_\lambda(u) + \varepsilon \|u - u_\varepsilon\|.$$

It is clear that u_ε is an minimum of I_λ . Therefore, for $t > 0$ and $v \in B_1(0)$, we have

$$\frac{I_\lambda(u_\varepsilon + tv) - I_\lambda(u_\varepsilon)}{t} \geq 0$$

for $t > 0$ small enough and $v \in B_1(0)$; that is,

$$\frac{\Phi_\lambda(u_\varepsilon + tv) - \Phi_\lambda(u_\varepsilon)}{t} + \varepsilon \|v\| \geq 0$$

for t positive and small enough, and $v \in B_1(0)$. As $t \rightarrow 0$, we obtain

$$\langle \Phi'_\lambda(u_\varepsilon), v \rangle + \varepsilon \|v\| \geq 0 \quad \text{for all } v \in B_1(0).$$

Hence, $\|\Phi'_\lambda(u_\varepsilon)\|_{X'} \leq \varepsilon$. We deduce that there exists a sequence $(u_n)_n \subset B_\rho(0)$ such that

$$\Phi_\lambda(u_n) \rightarrow \underline{c}_\lambda \quad \text{and} \quad \Phi'_\lambda(u_n) \rightarrow 0. \quad (3.16)$$

It is clear that (u_n) is bounded in X . Thus, there exists u in X such that $u_n \rightharpoonup u$ in X , we will show that $u_n \rightarrow u$ in X .

We have $\lim_{n \rightarrow \infty} \int_{\Omega} V(x) |u_n|^{q(x)-2} u_n (u_n - u) dx = 0$. Moreover, since $\Phi'_\lambda(u_n) \rightarrow 0$ and $(u_n)_n$ is bounded in X , we have

$$\begin{aligned}|\langle \Phi'_\lambda(u_n), u_n - u \rangle| &\leq |\langle \Phi'_\lambda(u_n), u_n \rangle| + |\langle \Phi'_\lambda(u_n), u \rangle| \\ &\leq \|\Phi'_\lambda(u_n)\|_{X^*} \|u_n\| + \|\Phi'_\lambda(u_n)\|_{X^*} \|u\|,\end{aligned}$$

that is,

$$\lim_{n \rightarrow +\infty} \langle \Phi'_\lambda(u_n), u_n - u \rangle = 0.$$

Using the last relation we deduce that

$$\lim_{n \rightarrow +\infty} \int_{\Omega} |\Delta u_n|^{p(x)-2} \Delta u_n \Delta(u_n - u) dx = 0. \quad (3.17)$$

From (3.17) and the fact that $u_n \rightharpoonup u$ in X it follows that

$$\lim_{n \rightarrow +\infty} \langle F'(u_n), u_n - u \rangle = 0,$$

and by proposition 2.5 (ii) [1], we deduce that $u_n \rightarrow u$ in X . Thus, in view of (3.16), we obtain

$$\Phi_\lambda(u) = c_\lambda < 0 \quad \text{and} \quad \Phi'_\lambda(u) = 0.$$

The proof is complete.

Theorem 3.9. *If*

$H_3(p, q): \quad p^+ < q^- \leq q^+ < p_2^*(x) \quad \text{for all } x \in \overline{\Omega},$
 $H_3(V) : \quad V(x) \in L^{s(x)} \text{ and there exists a measurable set } \Omega_0 \subset \Omega \text{ of positive measure such that } V(x) > 0, \text{ a.e. } x \in \overline{\Omega}_0,$
then for any } \lambda > 0, \text{ problem (1.1) possesses a nontrivial weak solution.}

We want to construct a mountain geometry, and first need two lemmas.

Lemma 3.10. *There exist } \eta, b > 0 \text{ such that } \Phi_\lambda(u) \geq b, \text{ for } u \in X \text{ with } \|u\| = \eta.*

Proof:

Since the embedding $X \hookrightarrow L^{s'(x)q(x)}(\Omega)$ is continuous, we can find a constant $c_3 > 0$ such that

$$|u|_{s'(x)q(x)} \leq c_3 \|u\|, \quad \forall u \in X. \quad (3.18)$$

According to the fact that

$$|u(x)|^{q(x)} \leq |u(x)|^{q^+} + |u(x)|^{q^-}, \quad \forall x \in \overline{\Omega}. \quad (3.19)$$

From the (3.19), we obtain :

$$\int_{\Omega} V(x) |u|^{q(x)} dx \leq |V|_{s(x)} \left| |u|^{q(x)} \right|_{s'(x)} \leq |V|_{s(x)} \left(|u|_{q(x)s'(x)}^{q^+} + |u|_{q(x)s'(x)}^{q^-} \right). \quad (3.20)$$

Combining 3.18 and 3.20, we obtain

$$\int_{\Omega} V(x) |u|^{q(x)} dx \leq |V|_{s(x)} \left(c_3^{q^+} \|u\|^{q^+} + c_3^{q^-} \|u\|^{q^-} \right). \quad (3.21)$$

Hence, from (3.21), we deduce that for any $u \in X$, we have

$$\begin{aligned} \Phi_\lambda(u) &= \int_{\Omega} \frac{1}{p(x)} |\Delta u|^{p(x)} dx - \lambda \int_{\Omega} \frac{V(x)}{q(x)} |u|^{q(x)} dx \\ &\geq \frac{1}{p^+} \alpha(\|u\|) - \frac{\lambda}{q^-} |V|_{s(x)} \left(c_3^{q^+} \|u\|^{q^+} + c_3^{q^-} \|u\|^{q^-} \right), \\ &= \begin{cases} \left(\frac{1}{p^+} - \frac{\lambda}{q^-} M_1 \left(c_3^{q^+} \|u\|^{q^+ - p^+} + c_3^{q^-} \|u\|^{q^- - p^+} \right) \right) \|u\|^{p^+} & \text{if } \|u\| \leq 1, \\ \left(\frac{1}{p^+} - \frac{\lambda}{q^-} M_1 \left(c_3^{q^+} \|u\|^{q^+ - p^-} + c_3^{q^-} \|u\|^{q^- - p^-} \right) \right) \|u\|^{p^-} & \text{if } \|u\| > 1. \end{cases} \end{aligned}$$

Since $p^+ < q^- \leq q^+$, the functional $g : [0, 1] \rightarrow \mathbb{R}$ defined by

$$g(s) = \frac{1}{p^+} - \frac{\lambda}{q^-} M_1 \left(c_3^{q^+} s^{q^+ - p^+} + c_3^{q^-} s^{q^- - p^+} \right)$$

is positive on neighborhood of the origin. So, the result of lemma 3.10 follows.

Lemma 3.11. *There exists $e \in X$ with $\|e\| \geq \eta$ such that $\Phi_\lambda(e) < 0$, where η is given in lemma 3.10.*

Proof:

Choose $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$ and $\varphi \neq 0$. For $t > 1$, we have

$$\Phi_\lambda(t\varphi) \leq \frac{t^{p^+}}{p^-} \int_\Omega |\Delta \varphi(x)|^{p(x)} dx - \frac{\lambda t^{q^-}}{q^+} \int_\Omega V(x) |\varphi(x)|^{q(x)} dx.$$

Then, since $p^+ < q^-$, we deduce that

$$\lim_{t \rightarrow \infty} \Phi_\lambda(t\varphi) = -\infty.$$

Therefore, for $t > 1$ large enough, there is $e = t\varphi$ such that $\|e\| \geq \eta$ and $\Phi_\lambda(e) < 0$. This completes the proof.

Lemma 3.12. *The functional Φ_λ satisfies the condition (PS).*

Proof:

Let $(u_n) \subset X$ be a sequence such that $d := \sup_n \Phi_\lambda(u_n) < \infty$ and $\Phi'_\lambda(u_n) \rightarrow 0$ in X' . By contradiction suppose that

$$\|u_n\| \rightarrow +\infty \text{ as } n \rightarrow \infty \quad \text{and} \quad \|u_n\| > 1 \quad \text{for any } n.$$

Thus, for sufficiently large n we have

$$\begin{aligned} d + 1 + \|u_n\| &\geq \Phi_\lambda(u_n) - \frac{1}{q^-} \langle \Phi'_\lambda(u_n), u_n \rangle \\ &= \int_\Omega \frac{1}{p(x)} |\Delta u_n|^{p(x)} dx - \frac{\lambda}{q^-} \int_\Omega |\Delta u_n|^{p(x)} dx + \lambda \int_\Omega \left(\frac{1}{q^-} - \frac{1}{q(x)} \right) V(x) |u_n|^{q(x)} dx \\ &\geq \left(\frac{1}{p^+} - \frac{1}{q^-} \right) \int_\Omega |\Delta u_n|^{p(x)} dx \\ &\geq \left(\frac{1}{p^+} - \frac{1}{q^-} \right) \|u_n\|^{p^-}. \end{aligned}$$

This contradicts the fact that $p^- > 1$. So, the sequence (u_n) is bounded in X and similar arguments as those used in the proof of lemma 3.5 completes the proof.

Proof:[Proof of theorem 3.9]

From Lemmas 3.10 and 3.11, we deduce

$$\max(\Phi_\lambda(0), \Phi_\lambda(e)) = \Phi_\lambda(0) < \inf_{\|u\|=\eta} \Phi_\lambda(u) =: \beta.$$

By lemma 3.12 and the mountain pass theorem, we deduce the existence of critical points u of Φ_λ associated of the critical value given by

$$c := \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} \Phi_\lambda(\gamma(t)) \geq \beta, \quad (3.22)$$

where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0 \text{ and } \gamma(1) = e\}$. This completes the proof.

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