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A New Approximation Method to Solve Boundary Value Problems by Using Functional Perturbation Concepts

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ABSTRACT: Functional perturbation method (FPM) is presented for the solution of differential equations with boundary conditions. Some properties of FPM are utilized to reduce the differential equation with variable coefficients to the equations with constant coefficients. The FPM can be applied directly for many types of differential equations. The exact solution is obtained by only the first term of the Frechet series for polynomial cases. Four examples are included to demonstrate the method.

Key Words: Dirac operator, Frechet derivatives, Funct. perturbation method.

Contents

1. Introduction

Many real-world phenomena can be formalized in terms of differential equations. This equations should be supplied with boundary conditions to ensure that there is a unique solution. Thus boundary value problems have played a main role in the development of engineering and mathematics. They have many applications in fields such as mechanics [3], optimal control [21], geometric optics [16], oceanography [13], finance mathematics [5] etc. There are mainly three types of approaches for solving differential equations. One approach is solving equations by analytical techniques. The other approach is designing numerical algorithms to solve equations. For example step difference schemes [20], collocation method [22,10,12] and tau method [17] are of this types. The third class of methods is

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based on semi analytical approaches like Adomian decomposition method [8], variational iteration method [7], homotopy perturbation method [15], etc. But this paper proposes the functional perturbation method (FPM) which the main idea behind that is to use Frechet series. This idea is successfully and effectively applied in some papers. In 2003, the buckling load in equation is treated as a functional of the bending modulus field by Altus and et al. [4]. They have applied a functional perturbation to equation, therefore the buckling load was found analytically to any desired degree of accuracy. In the same year the FPM has been used for calculating the average deflections and reaction forces of stochastically heterogeneous beams in [1]. In 2006, a one dimensional stochastically heterogeneous rod embedded in a uniform shear resistant elastic medium is solved in [3]. The solution of natural frequencies and mode shapes of non-homogeneous rods and beams was studied based on the FPM in 2007 [14]. Also in the same year the buckling load of heterogeneous columns has been found by applying the FPM directly to the buckling differential equation in [18]. The FPM is generalized in [19] for solving eigenvalue functional differential equations in 2008. In the current study, we use the FPM to approximate the solution of linear and nonlinear ordinary differential equations. Despite some methods like Galerkin and Rayleigh-Ritz, the accuracy of the FPM dose not depend on the arbitrarily chosen shape functions. The solution for each problem is founded by only convoluting its functional derivatives. We expand the equations functionally, yielding some ordinary differential equations which have constant coefficients.

In a concise manner, our first aim is to find an approximate solution of the equation by considering $E_0 = \langle E \rangle$ as an average of function E. It is worth pointing out that finding the best choice of E_0 is an important subject of optimization which is under investigation. Our second aim is to improve the approximate solution obtained in the previous step by using properties of Dirac function and functional derivative rules.

Four examples are given to show the efficiency of the method. To the best of our knowledge this is the first time that the FPM is proposed to solve a nonlinear differential equation.

The remaining of this paper is organized as follows. Section 2 is outlined some necessary preliminaries. The FPM and theoretical aspects of the method are elaborated in Section 3. In Section 4, we employ FPM for four examples. Finally, major conclusions are drown in Section 5.

2. Preliminaries

First we introduce some mathematical definitions that will be used in the sequel. Derivatives of a function $u(x) \equiv u$ will be written as:

$$\frac{du}{dx} \equiv u_{,x}, \qquad \frac{d^2u}{dx^2} \equiv u_{,xx}.$$

Let E is a scalar function of x and $u[E]$ is a functional of E, i.e., a mapping from a normed linear space of functions (a Banach space) $M = \{E | E : \mathbb{R} \to \mathbb{R}\}$ to the field of real or complex numbers, $u : M \to \mathbb{R}$ or $\mathbb{C}$. The $\frac{\delta u[E(x)]}{\delta E(x_1)}$ tells how the value of the functional changes if the function $E(x)$ is changed at the point x_1. Thus the functional derivative (Frechet derivative) itself is an ordinary function depending on x_1. First order and higher orders Frechet derivatives of the functional will be written as:

$$\frac{\delta u}{\delta E(x_1)} \equiv u_{,E_1}, \qquad \frac{\delta^2 u}{\delta E(x_1)\delta E(x_2)} \equiv u_{,E_1E_2}.$$

We consider a measure space $(\mathbb{R}^d, \Omega, \nu)$, where ν is a Borel measure, d is a positive integer. Also $u : L^p(\nu) \to \mathbb{R}$ be a real functional over the normed space $L^p(\nu)$ such that u maps functions that are L^p integrable with respect to ν to the real line. The bounded linear functional $u_{,E_1}$ is the Frechet derivative of u at $\langle E\rangle \in L^p(\nu)$ if

$$u[\langle E\rangle + E'] - u[\langle E\rangle] = u_{,E_1} + \epsilon[\langle E\rangle, E']\|E'\|_{L^p(\nu)},$$

for all $E' \in L^p(\nu)$, with $\epsilon[\langle E\rangle, E'] \to 0$ as $\|E'\|_{L^p(\nu)} \to 0$. Intuitively, what we are doing is perturbing the input function $\langle E\rangle$ by another function E', then shrinking the perturbing function E' to zero in terms or its L^p norm, and considering the difference $u[\langle E\rangle + E'] - u[\langle E\rangle]$ in this limit. For the second variation $u_{,E_1E_2}$, we have

$$u[\langle E\rangle + E'] - u[\langle E\rangle] = u_{,E_1} + \frac{1}{2}u_{,E_1E_2} + \epsilon[\langle E\rangle, E']\|E'\|_{L^p(\nu)},$$

where $\epsilon[\langle E\rangle, E'] \to 0$ as $\|E'\|_{L^p(\nu)} \to 0$ (see [9]). According to [6], the functional derivative is represented as the limit of divided differences:

$$u_{,E_1} = \lim_{\epsilon\to 0} \frac{u[E(x) + \epsilon\delta(x - x_1)] - u[E(x)]}{\epsilon}. \tag{2.1}$$

The x dependence on the right hand side of (2.1) is only a formal one. It can be written $\epsilon(\cdot - x_1)$ with the notation $E(\cdot)$ instead of $E(x)$. The Dirac function δ in (2.1) is (see [6]):

$$\delta(x - x_1) \equiv \delta_{xx_1} = \begin{cases} 1 & \text{if } x = x_1, \\ 0 & \text{if } x \neq x_1. \end{cases} \tag{2.2}$$

As a matter of fact, by considering $u[E] = E(x)$, we have:

$$u_{,E_1} = \frac{\delta u[E]}{\delta E(x_1)} = \lim_{\epsilon\to 0} \frac{1}{\epsilon}\big(E(x) + \epsilon\delta(x - x_1) - E(x)\big) = \delta(x - x_1) = \delta_{xx_1} = E_{,E_1}.$$

Therefore we can denote the derivative of Dirac function as:

$$\Big(\frac{\delta E(x)}{\delta E(x_1)}\Big)_{,x} = \delta_{xx_1,x}.$$

The average $\langle E\rangle$ and the deviation function $E'(x)$ of function E are defined as (see [2]):

$$\langle E\rangle = \int_0^1 E(x)dx = E * 1, \tag{2.3}$$

$$E'(x) = E(x) - \langle E \rangle. \tag{2.4}$$

In (2.3), sign $(*)$ is convolution and 1 is a unit function. The property of the Dirac function which we need them in FPM frequently is (see [11]):

$$\delta * E = E, \tag{2.5}$$

because:

$$\delta * E = \int \delta(x_1 - x_2)E(x_2)dx_2 = E(x_1). \tag{2.6}$$

It is worthwhile to know where $E(x_2)$ is a sufficiently smooth function, (2.6) is called the sifting property or reproducing property of the Dirac function [11]. For multiple convolutions we have:

$$u_{,E_1E_2} * *E_1E_2 = \int\int \frac{\delta}{\delta E(x_2)}(\frac{\delta u}{\delta E(x_1)})E(x_1)E(x_2)dx_1dx_2,$$

and since $u_{,E_1E_2}$ is symmetric, we can write also (see [4]):

$$u_{,E_1E_2} * *E_1E_2 = E_1 * u_{,E_1E_2} * E_2 = E_1 * \frac{\delta^2 u}{\delta E(x_1)\delta E(x_2)} * E_2.$$

Besides, the indispensable relation between the derivative of Dirac function δ and convolution is:

$$\frac{\partial \delta}{\partial x_i} * E = \delta * \frac{\partial E}{\partial x_i} = \frac{\partial E}{\partial x_i}. \tag{2.7}$$

(2.7) can be extended to the differential operator L of each order:

$$(L\delta) * E = \delta * L(E) = L(E). \tag{2.8}$$

We refer the interested reader to [6,11] for more discussion. The Frechet expansion (see [4,14,18,19]) of a function f around $\langle E \rangle$ is as:

$$\begin{aligned} f &= f(\langle E \rangle) + f_{,E_1}\big|_{E=\langle E \rangle} * E_1' + \frac{1}{2!} f_{,E_1E_2}\big|_{E=\langle E \rangle} * *E_1'E_2' + \cdots \\ &= f^{(0)} + f^{(1)} + \frac{1}{2!} f^{(2)} + \cdots . \end{aligned} \tag{2.9}$$

Expansion (2.9) is exact for a polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. As a matter of fact, by considering $E(x) = x^n$, (2.3) and (2.4) we have:

$$\langle E \rangle = \frac{1}{n+1}, \quad E' = x^n - \frac{1}{n+1}.$$

We rewrite $P(x)$ as:

$$P(x) = a_n E + \frac{a_{n-1}}{n} E_{,x} + \frac{a_{n-2}}{n(n-1)} E_{,xx} + \cdots + \frac{a_1}{n(n-1)\cdots \times 2} E_{,x^{n-1}} + \frac{a_0}{n!} E_{,x^n}.$$

Now, for the first term of the Frechet expansion of P, we have:

$$P^{(0)} \equiv P\big|_{E=\langle E\rangle} = a_n\langle E\rangle = \frac{a_n}{n+1}. \tag{2.10}$$

Also for the second term of the expansion of P:

$$\begin{aligned} P^{(1)} &\equiv P_{,E_1}\big|_{E=\langle E\rangle} * E_1' = (a_n E_{,E_1} + \frac{a_{n-1}}{n} E_{,xE_1} + \cdots + \frac{a_0}{n!} E_{,x^n E_1})\big|_{E=\langle E\rangle} * E_1' \\ &= (a_n\delta + \frac{a_{n-1}}{n}\delta_{,x} + \cdots + \frac{a_0}{n!}\delta_{,x^n}) * E_1' \\ &= a_n E_1' + \frac{a_{n-1}}{n}(E_1')_{,x} + \frac{a_{n-2}}{n(n-1)}(E_1')_{,xx} + \cdots + \frac{a_0}{n!}(E_1')_{,x^n} \\ &= a_n(x^n - \frac{1}{n+1}) + \frac{a_{n-1}}{n}(nx^{n-1}) \\ &\quad + \frac{a_{n-2}}{n(n-1)}(n(n-1)x^{n-2}) + \cdots + \frac{a_0}{n!}(n!) \\ &= a_n x^n - \frac{a_n}{n+1} + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0, \end{aligned} \tag{2.11}$$

and because of $E_{,E_1E_2} = 0, \quad E_{,E_1E_2E_3} = 0, \cdots$, etc:

$$P^{(2)} \equiv P_{,E_1E_2}\big|_{E=\langle E\rangle} ** E_1'E_2' = 0, \quad P^{(3)} = 0, \quad \cdots, \quad etc.$$

Therefore

$$P = P^{(0)} + P^{(1)} + \frac{1}{2!}P^{(2)} + \cdots = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

This shows that the Frechet expansion is exact for polynomials.

3. Functional Perturbation Method

In this section, we demonstrate the theoretical aspects of the FPM. For this purpose let us consider the differential equation:

$$\mathcal{L}u(x) = f(x), \quad x \in [0,1], \tag{3.1}$$

$\mathcal{L}$ is a general linear differential operator with boundary operator:

$$\mathcal{B}u(x_i) = u_i. \tag{3.2}$$

The Frechet expansion of the unknown function $u(x)$ is:

$$u(E(x)) = u\big|_{E=\langle E\rangle} + u_{,E_1}\big|_{E=\langle E\rangle} * E_1' + \frac{1}{2!}u_{,E_1E_2}\big|_{E=\langle E\rangle} ** E_1'E_2' + \cdots. \tag{3.3}$$

We denote:

$$u(\langle E\rangle) \equiv u^{(0)}, \quad u_{,E_1}\big|_{E=\langle E\rangle} * E_1' \equiv u^{(1)}, \quad u_{,E_1E_2}\big|_{E=\langle E\rangle} ** E_1'E_2' \equiv u^{(2)}, \quad \cdots.$$

Therefore (3.3) can be written as $u = u^{(0)} + u^{(1)} + \frac{1}{2!}u^{(2)} + \cdots$. Let us assume that Eq. (3.1) can be expressed as:

$$\mathcal{L}u = \phi_{(0)}(E)u + \phi_{(1)}(E)u_{,x} + \phi_{(2)}(E)u_{,xx} + \cdots = f(x). \tag{3.4}$$

For functions $\phi_{(i)}, i = 0, 1, \cdots$, the Frechet expansion around $\langle E \rangle$ is:

$$\phi_{(i)} = \phi_{(i)}(\langle E \rangle) + \phi_{(i),E_1} * E'_1 + \frac{1}{2!}\phi_{(i),E_1E_2} * *E'_1E'_2 + \cdots, \tag{3.5}$$

by considering

$$\phi_{(i)}(\langle E \rangle) \equiv \phi^{(0)}_{(i)}, \quad \phi_{(i),E_1} * E'_1 \equiv \phi^{(1)}_{(i)}, \quad \phi_{(i),E_1E_2} * *E'_1E'_2 \equiv \phi^{(2)}_{(i)}, \quad etc. \tag{3.6}$$

(3.5) will be written as $\phi_{(i)} = \phi^{(0)}_{(i)} + \phi^{(1)}_{(i)} + \frac{1}{2!}\phi^{(2)}_{(i)} + \cdots$. Now, by using the Frechet expansion for the differential operator L, we will have:

$$\mathcal{L}u = \mathcal{L}u(\langle E \rangle) + \mathcal{L}(u)_{,E_1}\big|_{E=\langle E \rangle} * E'_1 + \frac{1}{2}\mathcal{L}(u)_{,E_1E_2}\big|_{E=\langle E \rangle} * * E'_1E'_2 + \cdots = f(x). \tag{3.7}$$

As the first step, by the special case $E = \langle E \rangle$, we suppose:

$$\mathcal{L}u(\langle E \rangle) = f(x).$$

Therefore we should have:

$$\mathcal{L}(u)_{,E_1}\big|_{E=\langle E \rangle} * E'_1 = 0, \quad \mathcal{L}(u)_{,E_1E_2}\big|_{E=\langle E \rangle} * *E'_1E'_2 = 0, \quad etc. \tag{3.8}$$

For the product of two functionals, the ordinary product rule applies:

$$\mathcal{L}u_{,E_1} = \phi_{(0),E_1}u + \phi_{(0)}u_{,E_1} + \phi_{(1),E_1}u_{,x} + \phi_{(1)}u_{,xE_1} + \phi_{(2),E_1}u_{,xx} + \phi_{(2)}u_{,xxE_1} + \cdots. \tag{3.9}$$

For brevity, we denote (3.9) as:

$$\mathcal{L}u_{,E_1} = \phi_{(i),E_1}.u_{,x^i} + \phi_{(i)}.u_{,x^iE_1}, \qquad i = 0, 1, \cdots, \tag{3.10}$$

which $(\cdot)$ is inner product. Now when we use $E = \langle E \rangle$, Eq. (3.4) can be shown as:

$$\phi^{(0)}_{(0)}u^{(0)} + \phi^{(0)}_{(1)}u^{(0)}_{,x} + \phi^{(0)}_{(2)}u^{(0)}_{,xx} + \cdots = f(x), \tag{3.11}$$

$u^{(0)}$ will be known from solving Eq. (3.11) subject to (3.2). By using (3.10) for $\langle E \rangle$ and inner integral product (convolution) E'_1 and then considering the first equality in (3.8) we have:

$$\begin{aligned} \mathcal{L}u_{,E_1}|_{\langle E \rangle} * E'_1 &= \phi_{(i),E_1}.u_{,x^i}\big|_{E=\langle E \rangle} * E'_1 + \phi_{(i)}.u_{,x^iE_1}\big|_{E=\langle E \rangle} * E'_1 \\ &= \phi^{(1)}_{(i)}.u^{(0)}_{,x^i} + \phi^{(0)}_{(i)}.u^{(1)}_{,x^i} = 0. \end{aligned} \tag{3.12}$$

Inasmuch as $u^{(0)}$ is known from the solving of Eq. (3.11), we obtain $u^{(1)}$ by solving (3.12) subject to homogeneous boundary conditions. Also from (3.10) and the product rule for functionals, we will have:

$$\mathcal{L}u_{,E_1E_2} = \phi_{(i),E_1E_2}.u_{,x^i} + \phi_{(i),E_1}.u_{,x^iE_2} + \phi_{(i),E_2}.u_{,x^iE_1} + \phi_{(i)}.u_{,x^iE_1E_2}. \quad (3.13)$$

Applying multiple convolution on (3.13):

$$\begin{aligned}\mathcal{L}u_{,E_1E_2} **E_1'E_2' &= \phi_{(i),E_1E_2}.u_{,x^i} **E_1'E_2' + \phi_{(i),E_1}.u_{,x^iE_2} **E_1'E_2' \\ &+ \phi_{(i),E_2}.u_{,x^iE_1} **E_1'E_2' + \phi_{(i)}.u_{,x^iE_1E_2} **E_1'E_2' \\ &= \phi_{(i),E_1E_2}.u_{,x^i} **E_1'E_2' + \phi_{(i),E_1} * E_1'.u_{,x^iE_2} * E_2' \\ &+ \phi_{(i),E_2} * E_2'.u_{,x^iE_1} * E_1' + \phi_{(i)}.u_{,x^iE_1E_2} **E_1'E_2' \\ &= 0. \end{aligned} \quad (3.14)$$

Also

$$\phi_{(i)}^{(2)} \equiv \phi_{(i),E_1E_2} **E_1'E_2', \quad u_{,x^i}^{(2)} \equiv u_{,x^iE_1E_2} **E_1'E_2'. \quad (3.15)$$

By using (3.15), we can rewrite Eq. (3.14) as:

$$\phi_{(i)}^{(2)}.u_{,x^i}^{(0)} + 2\phi_{(i)}^{(1)}.u_{,x^i}^{(1)} + \phi_{(i)}^{(0)}.u_{,x^i}^{(2)} = 0. \quad (3.16)$$

We ponder homogeneous boundary conditions for obtaining $u^{(2)}$ by solving Eq. (3.16). The solution of the Eq. (3.1) is:

$$u = u^{(0)} + u^{(1)} + \frac{1}{2}u^{(2)} + \cdots,$$

which $u^{(i)}$, $i = 0, 1, 2, \cdots$ are obtained from solving respectively (3.11), (3.12), (3.16), etc.

4. Examples

In this section, we apply the above method for some examples. The first one has a polynomial solution and the FPM is exact for it by only one term of the expansion. Two other examples are linear ODEs and the last one is a nonlinear case which FPM solutions are compared with the exact solutions of them. We measure the accuracy by considering the root mean square error (RMSE) as follow:

$$RMSE = \frac{1}{n}\sqrt{\sum_{i=1}^{n} e_i^2},$$

where n is the number of interior nodes and e_i is the error. If $u_{exact}(x_i) = 0$; we use absolute error $e_i = u_{exact}(x_i) - u_{FPM}(x_i)$, otherwise we use relative error $e_i = \frac{u_{exact}(x_i) - u_{FPM}(x_i)}{u_{exact}(x_i)}$.

4.1. Example 1

As the first example, we consider

$$u+(x-1)u_{,x}-\frac{3}{2}x^2u_{,xx}=-1, \tag{4.1}$$

with $u(0)=0$ and $u_{,x}(0)=1$. The exact solution of this problem is $u(x)=x^2+x$. By $E(x)=x^2$ and considering Eq. (4.1) as Eq. (3.4), we have:

$$\phi_{(0)}(E)=1=\frac{1}{2}E_{,xx} \quad \phi_{(1)}(E)=x-1=\frac{1}{2}E_{,x}-\frac{1}{2}E_{,xx}, \quad \phi_{(2)}(E)=-\frac{3}{2}x^2=-\frac{3}{2}E. \tag{4.2}$$

Also

$$\langle E\rangle=\int_0^1 x^2dx=\frac{1}{3}, \quad E'=E-\langle E\rangle=x^2-\frac{1}{3}. \tag{4.3}$$

As the first step, by (4.2) and (4.3) we have:

$$\phi_{(0)}^{(0)}=\phi_{(0)}\big|_{E=\langle E\rangle}=\frac{1}{2}(\langle E\rangle)_{,xx}=0, \quad \phi_{(1)}^{(0)}=\frac{1}{2}(\langle E\rangle)_{,x}-\frac{1}{2}(\langle E\rangle)_{,xx}=0,$$
$$\phi_{(2)}^{(0)}=-\frac{3}{2}\langle E\rangle=-\frac{1}{2}. \tag{4.4}$$

In this step, the equation $\phi_{(0)}u+\phi_{(1)}u_{,x}+\phi_{(2)}u_{,xx}=-1$, is as:

$$\phi_{(0)}^{(0)}u^{(0)}+\phi_{(1)}^{(0)}u_{,x}^{(0)}+\phi_{(2)}^{(0)}u_{,xx}^{(0)}=-1.$$

Therefore by (4.1) we have:

$$-\frac{1}{2}u_{,xx}^{(0)}=-1.$$

(4.5)

Solving Eq. (4.1) with conditions $u^{(0)}(0)=0$ and $u_{,x}^{(0)}(0)=1$, we have

$$u^{(0)}=x^2+x.$$

For the next step:

$$\phi_{(0)}^{(1)}=\phi_{(0),E_1}*E_1'=\frac{1}{2}E_{,xxE_1}*E_1'=\frac{1}{2}\delta_{,xx}*E_1'=\frac{1}{2}(E_1')_{,xx}=\frac{1}{2}(x^2-\frac{1}{3})_{,xx}=1,$$

and

$$\phi_{(1)}^{(1)}=\phi_{(1),E_1}*E_1'=(\frac{1}{2}E_{,xE_1}-\frac{1}{2}E_{,xxE_1})*E_1'=\frac{1}{2}\delta_{,x}*E_1'-\frac{1}{2}\delta_{,xx}*E_1'=$$
$$=\frac{1}{2}(E_1')_{,x}-\frac{1}{2}(E_1')_{,xx}=x-1,$$

and

$$\phi^{(1)}_{(2)}=\phi_{(2),E_1}*E_1'=(-\frac{3}{2}E)_{,E_1}*E_1'=-\frac{3}{2}\delta*E_1'=-\frac{3}{2}E_1'=-\frac{3}{2}(x^2-\frac{1}{3})=-\frac{3}{2}x^2+\frac{1}{2}.$$

So by the last equality of Eq. (3.12) we have:

$$u^{(0)}+(x-1)u^{(0)}_{,x}+(-\frac{3}{2}x^2+\frac{1}{2})u^{(0)}_{,xx}-\frac{1}{2}u^{(1)}_{,xx}=0.$$

Therefore

$$-\frac{1}{2}u^{(1)}_{,xx}=0. \tag{4.6}$$

Eq. (4.1) differs from Eq. (4.1) of step.1 by the right hand side part only. We solve Eq. (4.1) by homogeneous conditions $u^{(1)}(0)=0$ and $u^{(1)}_{,x}(0)=0$, then

$$u^{(1)}=0.$$

For the next steps $u^{(j)}=0, \quad j=2,3,\cdots$. Therefore the FPM solution is

$$u_{FPM}=u^{(0)}+u^{(1)}+\frac{1}{2!}u^{(2)}=x^2+x.$$

It is observed that the FPM gives the exact solution only by the first term of Frechet series.

4.2. Example 2

Consider

$$-xu_{,x}+(1-x^2)u_{,xx}=0, \tag{4.7}$$

with $u(0)=0$ and $u_{,x}(0)=1$. Considering (4.2) as $\phi_{(1)}u_{,x}+\phi_{(2)}u_{,xx}=0$ and $E(x)=1-x^2$, we have

$$\phi_{(1)}=-x=\frac{1}{2}E_{,x}, \quad \phi_{(2)}=1-x^2=E,$$

Also

$$\langle E\rangle=\int_0^1(1-x^2)dx=\frac{2}{3}, \quad E'=E-\langle E\rangle=\frac{1}{3}-x^2,$$

and

$$\phi^{(0)}_{(1)}=\phi_{(1)}\big|_{E=\langle E\rangle}=\frac{1}{2}(\langle E\rangle)_{,x}=0, \quad \phi^{(0)}_{(2)}=\phi_{(2)}\big|_{E=\langle E\rangle}=\langle E\rangle=\frac{2}{3}. \tag{4.8}$$

In the first step which we use $E = \langle E \rangle$, Eq. (4.2) is

$$\phi^{(0)}_{(1)} u^{(0)}_{,x} + \phi^{(0)}_{(2)} u^{(0)}_{,xx} = 0,$$

therefore by (4.8) we have:

$$\frac{2}{3} u^{(0)}_{,xx} = 0. \tag{4.9}$$

Applying conditions $u^{(0)}(0) = 0$ and $u^{(0)}_{,x}(0) = 1$, yields:

$$u^{(0)} = x. \tag{4.10}$$

For the next step:

$$\phi^{(1)}_{(1)} = \phi_{(1),E_1} * E'_1 = \frac{1}{2} E_{,xE_1} * E'_1 = \frac{1}{2} \delta_{,x} * E'_1 = \frac{1}{2} (E'_1)_{,x} = \frac{1}{2} (\frac{1}{3} - x^2)_{,x} = -x,$$

and

$$\phi^{(1)}_{(2)} = \phi_{(2),E_1} * E'_1 = E_{,E_1} * E'_1 = \delta * E'_1 = E'_1 = \frac{1}{3} - x^2,$$

so by the last equality of (3.12) we have:

$$\frac{2}{3} u^{(1)}_{,xx} - x u^{(0)}_{,x} + (\frac{1}{3} - x^2) u^{(0)}_{,xx} = 0, \tag{4.11}$$

therefore by substituting (4.8) and derivatives of (4.10) in the (4.11):

$$\frac{2}{3} u^{(1)}_{,xx} = x. \tag{4.12}$$

We solve Eq. (4.12) subject to homogeneous conditions $u^{(1)}(0) = 0$ and $u^{(1)}_{,x}(0) = 0$:

$$u^{(1)} = \frac{1}{4} x^3.$$

Inasmuch as $\phi^{(2)}_{(1)} = \phi_{(1),E_1E_2} ** E'_1E'_2 = \frac{1}{2} E_{,xE_1E_2} ** E'_1E'_2 = 0$ and $\phi^{(2)}_{(2)} = \phi_{(2),E_1E_2} * *E'_1E'_2 = E_{E_1E_2} * *E'_1E'_2 = 0$, and according to (3.16):

$$\frac{2}{3} u^{(2)}_{,xx} = \frac{9}{2} x^3 - x. \tag{4.13}$$

As we see, Eq. (4.13) differs from Eq. (4.9) and Eq. (4.12) by the right hand side only. Solving Eq. (4.13) with $u^{(2)}(0) = 0$ and $u^{(2)}_{,x}(0) = 0$, leads to:

$$u^{(2)} = \frac{27}{80} x^5 - \frac{1}{4} x^3.$$

Therefore the FPM solution is

$$u_{FPM} = u^{(0)} + u^{(1)} + \frac{1}{2!} u^{(2)} = x + \frac{1}{4} x^3 + \frac{1}{2!} (\frac{27}{80} x^5 - \frac{1}{4} x^3) = x + \frac{1}{8} x^3 + \frac{27}{160} x^5.$$

We have showed FPM solution, exact solution ($u(x) = \arcsin x$) and absolute error in Table 1. The graph of exact and FPM solution is depicted in Fig. 1.

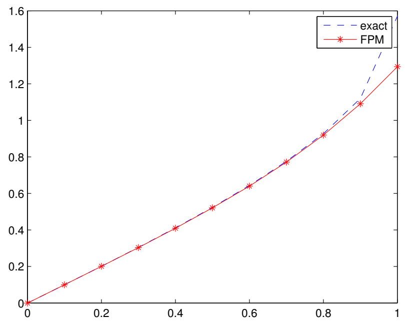

Figure 1: Exact and FPM solutions (Example 2).

Table 1: Absolute error of Example 2

x_i	FPM	Exact	Absolute error
0	0	0	0
0.1	0.1001	0.1002	$1.0E-04$
0.2	0.2011	0.2014	$3.0E-04$
0.3	0.3038	0.3047	$9.0E-04$
0.4	0.4097	0.4115	$1.8E-03$
0.5	0.5209	0.5236	$2.7E-03$
0.6	0.6401	0.6435	$3.4E-03$
0.7	0.7712	0.7754	$4.2E-03$
0.8	0.9193	0.9273	$8.0E-03$
0.9	1.0908	1.1198	$2.9E-02$
$RMSE = 3.1E-03$			

4.3. Example 3

As the third example, consider

$$u_{,x} + (1+x)u_{,xx} = 0, \tag{4.14}$$

with $u(0) = 0$ and $u_{,x}(1) = \frac{1}{6}$. By considering $E(x) = 1 + x$, we will have:

$$\langle E \rangle = \int_0^1 (1+x)dx = \frac{3}{2}, \quad E'(x) = E(x) - \langle E \rangle = x - \frac{1}{2}.$$

Comparing (4.14) with $\phi_{(1)}u_{,x} + \phi_{(2)}u_{,xx} = 0$ yields $\phi_{(1)} = 1 = E_{,x}$ and $\phi_{(2)} = 1 + x = E$. We obtain $\phi^{(0)}_{(i)}$, $i = 1, 2$ for $E = \langle E \rangle$, as follow

$$\phi^{(0)}_{(1)} = (\langle E \rangle)_{,x} = 0, \quad \phi^{(0)}_{(2)} = \langle E \rangle = \frac{3}{2}. \tag{4.15}$$

For the first step, (4.14) is:

$$\phi_{(1)}^{(0)}u_{,x}^{(0)} + \phi_{(2)}^{(0)}u_{,xx}^{(0)} = 0,$$

by (4.15), we have:

$$\frac{3}{2}u_{,xx}^{(0)} = 0. \tag{4.16}$$

Applying conditions $u(0) = 0$ and $u_{,x}(1) = \frac{1}{6}$, yields

$$u^{(0)} = \frac{1}{6}x.$$

As the second step, we obtain $\phi_{(i)}^{(1)}$, $i = 1, 2$ as follow

$$\phi_{(1)}^{(1)} = \phi_{(1),E_1} * E_1' = E_{,xE_1} * E_1' = \delta_{,x} * E_1' = (E_1')_{,x} = (x - \frac{1}{2})_{,x} = 1,$$

$$\phi_{(2)}^{(1)} = \phi_{(2),E_1} * E_1' = E_{,E_1} * E_1' = \delta * E_1' = E_1' = x - \frac{1}{2}.$$

According to the last equality of Eq. (3.12) we have

$$\frac{3}{2}u_{,xx}^{(1)} + u_{,x}^{(0)} + (x - \frac{1}{2})u_{,xx}^{(0)} = 0,$$

then

$$\frac{3}{2}u_{,xx}^{(1)} = -\frac{1}{6}. \tag{4.17}$$

Now, we consider homogeneous conditions $u(0) = 0$ and $u_{,x}(1) = 0$. Therefore

$$u^{(1)} = -\frac{1}{18}x^2 + \frac{1}{9}x.$$

To obtain $u^{(2)}$, from Eq. (3.16) we have

$$\phi_{(1)}^{(2)}u_{,x}^{(0)} + \phi_{(2)}^{(2)}u_{,xx}^{(0)} + 2\phi_{(1)}^{(1)}u_{,x}^{(1)} + 2\phi_{(2)}^{(1)}u_{,xx}^{(1)} + \phi_{(1)}^{(0)}u_{,x}^{(2)} + \phi_{(2)}^{(0)}u_{,xx}^{(2)} = 0. \tag{4.18}$$

Apparently when $\phi_{(i)} = E_{,x^i}$, then $\phi_{(i),E_1E_2\dots} = 0$. So

$$\phi_{(i)}^{(2)} = \phi_{(i),E_1E_2} * *E_1'E_2' = 0.$$

Therefore, from Eq. (4.18) we have

$$\frac{3}{2}u_{,xx}^{(2)} = \frac{4}{9}x - \frac{1}{3},$$

with homogeneous conditions $u^{(2)}(0) = 0$ and $u_{,x}^{(2)}(1) = 0$, $u^{(2)}$ can be obtained

$$u^{(2)} = \frac{4}{81}x^3 - \frac{1}{9}x^2 + \frac{2}{27}x.$$

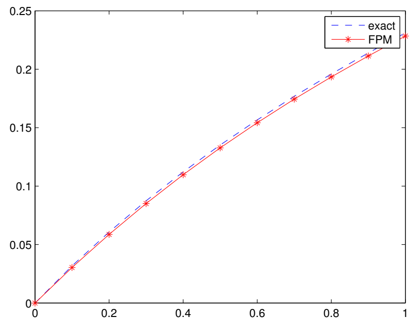

Figure 2: Exact and FPM solutions (Example 3).

The solution of Eq. (4.14) is

$$\begin{aligned} u_{FPM} =& u^{(0)} + u^{(1)} + \frac{1}{2!}u^{(2)} \\ =& \frac{1}{6}x - \frac{1}{18}x^2 + \frac{1}{9}x + \frac{1}{2!}(\frac{4}{81}x^3 - \frac{1}{9}x^2 + \frac{2}{27}x) \\ =& \frac{17}{54}x - \frac{1}{9}x^2 + \frac{2}{81}x^3. \end{aligned}$$

The exact solution ($u(x) = \frac{1}{3}\ln(1 + x)$) and the FPM solution are shown in Fig. 2. The accuracy of the solution is clearly seen by $RMSE = 6.9957E - 04$ with x_i from Table.2.

Table 2: Absolute error of Example 3

x_i	FPM	Exact	Absolute error
0	0	0	0
0.1	0.0304	0.0318	$1.4E-03$
0.2	0.0587	0.0608	$2.1E-03$
0.3	0.0851	0.0875	$2.3E-03$
0.4	0.1097	0.1122	$2.4E-03$
0.5	0.1327	0.1352	$2.4E-03$
0.6	0.1542	0.1567	$2.4E-03$
0.7	0.1744	0.1769	$2.5E-03$
0.8	0.1934	0.1959	$2.5E-03$
0.9	0.2113	0.2140	$2.6E-03$
$RMSE = 6.9957E-04$			

4.4. Example 4

As the last example, we consider a nonlinear differential equation

$$uu_{,xx} - u_{,x}^2 = -2, \tag{4.19}$$

with $u(0) = 1$ and $u_{,x}(0) = 1$. By rewriting Eq. (4.19) as

$$\tilde{u}u_{,xx} - \tilde{u}_{,x}u_{,x} = -2.$$

We can consider $\phi_{(0)} = 0$, $\phi_{(1)} = -\tilde{u}_{,x}$ and $\phi_{(2)} = \tilde{u}$. First of all, we use two terms of Taylor expansion for $u(x)$

$$u(x) \simeq \tilde{u}(x) = u(0) + xu_{,x}(0) = 1 + x.$$

If we consider $E(x) = \tilde{u} = 1 + x$, then:

$$\phi_{(1)} = -E_{,x}, \qquad \phi_{(2)} = E.$$

Also

$$\langle E \rangle = \int_0^1 (1 + x)dx = \frac{3}{2},$$

and

$$E'(x) = E(x) - \langle E \rangle = x - \frac{1}{2}.$$

Now, we obtain $\phi_{(i)}^{(0)}$, $i = 1, 2$ for $E = \langle E \rangle$ as follow

$$\phi_{(1)}^{(0)} = -E_{,x}\big|_{E=\langle E \rangle} = 0, \quad \phi_{(2)}^{(0)} = E\big|_{E=\langle E \rangle} = \frac{3}{2}.$$

In first step we have

$$\phi_{(1)}^{(0)}u_{,x}^{(0)} + \phi_{(2)}^{(0)}u_{,xx}^{(0)} = -2,$$

so

$$\frac{3}{2}u_{,xx}^{(0)} = -2.$$

By using $u^{(0)}(0) = 1$ and $u_{,x}^{(0)}(0) = 1$ we have

$$u^{(0)} = -\frac{2}{3}x^2 + x + 1.$$

Also $\phi_{(i)}^{(1)}$, $i = 1, 2$ are obtained as

$$\phi_{(1)}^{(1)} = -E'_{,x} = -1, \quad \phi_{(2)}^{(1)} = E' = x - \frac{1}{2}.$$

So by Eq. (3.12) we have

$$\frac{3}{2}u_{,xx}^{(1)} = \frac{1}{3},$$

with homogeneous conditions $u^{(1)}(0) = 0$ and $u^{(1)}_{,x}(0) = 0$, $u^{(1)}$ is obtained

$$u^{(1)} = \frac{1}{9}x^2.$$

Inasmuch as $\phi^{(2)}_{(1)} = \phi_{(1),E_1E_2} * *E'_1E'_2 = -E_{,xE_1E_2} * *E'_1E'_2 = 0$ and $\phi^{(2)}_{(2)} = \phi_{(2),E_1E_2} * *E'_1E'_2 = E_{E_1E_2} * *E'_1E'_2 = 0$, Eq. (3.16) is reduced to

$$\frac{3}{2}u^{(2)}_{,xx} = \frac{2}{9}.$$

In this step we use homogeneous conditions too, so

$$u^{(2)} = \frac{2}{27}x^2.$$

Therefore u_{FPM} is obtained

$$u_{FPM} = u^{(0)} + u^{(1)} + \frac{1}{2!}u^{(2)} = -\frac{2}{3}x^2 + x + 1 + \frac{1}{9}x^2 + \frac{1}{2!}(\frac{2}{27}x^2) = 1 + x - \frac{14}{27}x^2.$$

The exact solution ($u(x) = \sin x + \cos x$) and the FPM solutions are depicted in

Table 3: Absolute error of Example 4

x_i	FPM	Exact	relative error
0	1.0000	1.0000	0
0.1	1.0948	1.0948	0
0.2	1.1793	1.1787	$4.0E-04$
0.3	1.2533	1.2509	$2.0E-03$
0.4	1.3170	1.3105	$5.0E-03$
0.5	1.3704	1.3570	$9.8E-03$
0.6	1.4133	1.3900	$1.68E-02$
0.7	1.4459	1.4091	$2.62E-02$
0.8	1.4681	1.4141	$3.82E-02$
0.9	1.4800	1.4049	$5.34E-02$
$RMSE = 7.4E-03$			

Fig. 3. The relative errors of them are shown in Table. 3.

Remark 4.1. *We have verified all the examples with the aid of MATLAB R2013a.*

5. Conclusion

In this article we have studied the functional perturbation method (FPM) which is an effective tool for analytical solution of linear problems and can be used for some nonlinear problems too. We expand differential equations functionally, yielding some ODEs which have constant coefficients and differ only in their right hand side. The right hand side functions that exist in each step, correct the inconsistencies of all previous approximations. The initial condition is fulfilled by the

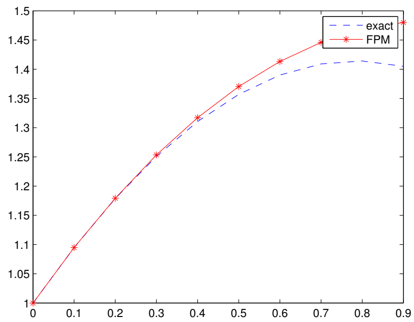

Figure 3: Exact solution and FPM (Example 4).

zero-order approximation only. Higher-order approximations are considered with homogeneous conditions. We have successfully applied the proposed approach to solve four equations. First, the idea of FPM is applied for linear equations, then we generate the idea to a nonlinear differential equation. In the nonlinear case, the unknown u is replaced by two terms of Taylor expansion $\tilde{u}$. For polynomial case, the exact solution is obtained by only the first term of expansion. The results have shown that the new described idea produces acceptable results.

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