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Well-posedness and optimal decay rates for the viscoelastic Kirchhoff equation

A. Guesmia, S. A. Messaoudi, C. M. Webler

abstract: In this paper, we investigate the well-posedness as well as optimal decay rate estimates of the energy associated with a Kirchhoff-Carrier problem in n-dimensional bounded domain under an internal finite memory. The considered class of memory kernels is very wide and allows us to derive new and optimal decay rate estimates then those ones considered previously in the literature for Kirchhoff-type models.

Contents

1. Introduction

The nonlinear vibrations of an elastic string are written in the form of partial integro-differential equations by

$$\rho h \frac{\partial^2 u}{\partial t^2} = \left(p_0 + \frac{Eh}{2L} \int_0^L \left(\frac{\partial u}{\partial x} \right)^2 dx \right) \frac{\partial^2 u}{\partial x^2} + f, \tag{1.1}$$

for $0 < x < L$ and $t \geq 0$, where

$$\begin{cases} u \text{ is the lateral deflection,} \\ x \text{ is the space coordenate variable while } t \text{ denotes the time variable,} \\ E \text{ represents the Young's modulus,} \\ \rho \text{ designates the mass density,} \\ L \text{ indicates the string's lengh,} \\ h \text{ represents the cross section,} \\ p_0 \text{ denotes the axial tension,} \\ f \text{ represents an external force.} \end{cases}$$

The model (1.1) has been introduced by Kirchhoff [15] in the study of the oscillations of stretched strings and plates, so that equation (1.1) is called the wave

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equation of Kirchhoff type until now. It is worth mentioning that, when $p_0 = 0$, the model (1.1) is called degenerate, and when $p_0 > 0$, we denominate it as a non-degenerate model.

There is a large literature regarding the Kirchhoff equation. In the sequel, we would like to mention some important works on this subject. Regarding the well-posedness of problem (1.1), the analytic case is rather known in general dimensions, as, for instance, [8], [9] and [25]. In what concerns solutions for (1.1) lying in Sobolev spaces and, as far as we know, the results presented in the literature are only local in time, as, for example, [1] and [24]. However, when equation (1.1) is supplemented by some type of dissipative mechanism, which allows us, roughly speaking, to derive decay rate estimates for the solutions of the linearized problen of (1.1), it is possible to recover the global solvability in time. Consequently, deriving global solutions in time deeply depends on the decay structure of the solutions to the corresponding linearized problem of (1.1). Therefore, we are led naturally to consider the Kirchhoff equation subject to a dissipative term which guarantees the decay properties of the linearized problem. When the dissipation is given by a frictional mechanism, like $g(\partial_t u)$, there is a large body of works in the literature, see, for instance, [2], [10], [4], [13], [16], [17], [18], [24], [21], [23] and a long list of references therein.

In this paper, we investigate the well-posedness as well as optimal decay rate estimates of the energy associated with the following Kirchhoff-Carrier problem with memory:

$$\begin{cases} u'' - M(||\nabla u(t)||_2^2)\Delta u + \displaystyle\int_0^t g(t-s)\Delta u(s)\,ds = 0 \quad \text{in } \Omega \times \mathbb{R}_+\,, \\ u = 0 \quad \text{on } \Gamma \times \mathbb{R}_+, \\ u(x,0) = u_0(x), \;\; u_t(x,0) = u_1(x) \text{ in } \quad \Omega\,, \end{cases} \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}^*$, with smooth boundary $\partial\Omega := \Gamma$. While there is a great number of papers regarding the Kirchhoff equation subject to a frictional damping, in contrast, there is just a few number of papers concerned with the Kirchhoff equation subject to a dissipation given by a memory term. We are aware solely the paper [22], where stronger conditions were considered on the kernel of the memory term. The assumption given in (1.7), firstly introduced in [20], is much more general and allows us to consider a wide class of kernels, and consequently, get new and optimal decay rate estimates then those ones considered previously in the literature for the linear viscoelastic wave equation. In the present paper, we combine techniques given in [20] with new ingredients inherent to the nonlinear character of the Kirchhoff equation (1.2).

It is worth mentioning some important contributions in connection with viscoelasticity, among them, we would like to mention [3], [5], [6], [7], [11], [12], [14], [20], [26] and references therein.

The following assumptions are made on the function M:

Assumption 1.1.

$$\exists m_0 > 0: \quad M \in C^1(\mathbb{R}_+) \quad \textit{and} \quad M(\lambda) \geq m_0, \quad \forall \lambda \geq 0. \tag{1.3}$$
$$\exists \gamma,\, \delta > 0: \quad M(\lambda) \leq \delta \lambda^{\gamma}, \quad \forall \lambda \geq 0. \tag{1.4}$$
$$\exists \alpha,\, \beta > 0: \quad |M'(\lambda)| \leq \beta \lambda^{\alpha}, \quad \forall \lambda \geq 0. \tag{1.5}$$

We shall assume the following assumptions on the kernel g:

Assumption 1.2. *The function $g : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $g \in C^1(\mathbb{R}_+)$, $g' \leq 0$ and, in addition*

$$g(0) > 0 \quad \textit{and} \quad g_0 := \int_0^{+\infty} g(s)\, ds < m_0. \tag{1.6}$$

Moreover, there exists a differentiable non increasing function $\xi : \mathbb{R}_+ \to \mathbb{R}_+^*$ such that

$$\frac{\xi'}{\xi} \in L^\infty(\mathbb{R}_+), \qquad \int_0^{+\infty} \xi(s)\, ds = +\infty$$

and

$$g'(s) \leq -\xi(s) g(s), \quad \forall s \geq 0. \tag{1.7}$$

Now, we are in a position to state our main result.

Theorem 1.3. *Assume that Assumption 1.1 and Assumption 1.2 are in place. Then, there exists an open unbounded set S in $(H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ which contains $(0,0)$ such that, if $(u_0, u_1) \in S$, and, in addition, the initial data are taken in bounded sets of $H_0^1(\Omega) \times L^2(\Omega)$, problem (1.2) possesses a unique global solution u satisfying*

$$u \in L^\infty(\mathbb{R}_+; H^2(\Omega) \cap H_0^1(\Omega)) \cap W^{1,\infty}(\mathbb{R}_+; H_0^1(\Omega)) \cap W^{2,\infty}(\mathbb{R}_+; L^2(\Omega)). \tag{1.8}$$

Furthermore, we have the following decay estimates for the energy $\widehat{E}$ given in (2.10):

$$\widehat{E}(t) \leq c \widehat{E}(0) e^{-\theta \int_0^t \xi(s)\, ds}, \quad \forall t \geq 0, \tag{1.9}$$

where θ and c are positive constants independent of the initial data.

Our paper is organized as follows: in Section 2, we prove the general stability (1.9). The Section 3 is devoted to the proof the well-posedness (1.8).

2. General stability

In what follows, let us consider the Hilbert space $L^2(\Omega)$ endowed with the inner product

$$(u,v)_{L^2(\Omega)} = \int_\Omega u(x)v(x)\,dx$$

and the corresponding norm

$$||u||_2^2 = \int_\Omega |u(x)|^2 dx,$$

and the Banach space $L^p(\Omega)$, for $p \geq 1$, endowed by the norm

$$||u||_p^p = \int_\Omega |u(x)|^p\,dx.$$

Let $-\Delta$ be the operator defined by the triple $\left\{H_0^1(\Omega), L^2(\Omega), ((\cdot,\cdot))_{H_0^1(\Omega)}\right\}$, where

$$((u,v))_{H_0^1(\Omega)} = \int_\Omega \nabla u \nabla v\,dx, \quad \forall u,v \in H_0^1(\Omega)$$

and

$$D(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega).$$

We recall that the Spectral Theorem for self-adjoint operators guarantees the existence of a complete orthonormal system (ω_ν) of $L^2(\Omega)$ given by the eigenfunctions of $-\Delta$. If (λ_ν) are the corresponding eigenvalues of $-\Delta$, then

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_\nu \leq \cdots \quad \text{and } \lambda_\nu \to +\infty \text{ when } \nu \to +\infty.$$

Moreover,

$$\left(\frac{\omega_\nu}{\sqrt{\lambda_\nu}}\right) \text{ is a complete orthonormal system in } H_0^1(\Omega)$$

and

$$\left(\frac{\omega_\nu}{\lambda_\nu}\right) \text{ is a complete orthonormal system in } H^2(\Omega) \cap H_0^1(\Omega).$$

We denote by V_m the subspace of $H^2(\Omega) \cap H_0^1(\Omega)$ generated by the first m vectors $w_1, \cdots, w_m$, namely, $V_m = [w_1, \cdots, w_m]$ and

$$u_m(t) = \sum_{j=1}^m \gamma_{jm}(t)\omega_j, \tag{2.1}$$

where u_m is the solution of the approximate Cauchy problem

$$\begin{cases} (u''_m(t), w_j)_{L^2(\Omega)} + M(||\nabla u_m(t)||_2^2)(\nabla u_m(t), \nabla w_j)_{L^2(\Omega)} \\ \qquad\qquad - \displaystyle\int_0^t g(t-s)(\nabla u_m(s), \nabla w_j)_{L^2(\Omega)}\, ds = 0, \;\; j = 1, \cdots, m, \\ u_{0m} = \displaystyle\sum_{j=1}^{m} \gamma_{jm}(0) w_j \to u_0 \text{ in } H^2(\Omega) \cap H_0^1(\Omega), \\ u_{1m} = \displaystyle\sum_{j=1}^{m} \gamma'_{jm}(0) w_j \to u_1 \text{ in } H_0^1(\Omega). \end{cases} \tag{2.2}$$

By standard methods in differential equations, we can prove the existence of a solution to (2.2) on some interval $[0, t_m)$. Then, this solution can be extended to the interval $\mathbb{R}_+$ by using of the first estimate below.

The first estimate. Multiplying the first equation in (2.2) by $\gamma'_{jm}(t)$, $j = 1, \cdots, m$, and summing the resulting expressions, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}||u'_m(t)||_2^2 \quad &+ \quad \frac{1}{2} M(||\nabla u_m(t)||_2^2) \frac{d}{dt} ||\nabla u_m(t)||_2^2 \\ &- \quad \int_0^t g(t-s)(\nabla u_m(s), \nabla u'_m(t))_{L^2(\Omega)}\, ds = 0. \end{aligned} \tag{2.3}$$

Defining

$$\widehat{M}(\lambda) = \int_0^\lambda M(s)\, ds, \tag{2.4}$$

and since

$$\begin{aligned} \frac{d}{dt}\widehat{M}(||\nabla u(t)||_2^2) &= \frac{d}{dt}\int_0^{||\nabla u(t)||_2^2} M(s)\, ds \\ &= M(||\nabla u(t)||_2^2)\frac{d}{dt}||\nabla u(t)||_2^2, \end{aligned}$$

then we deduce, taking (2.3) and the last identity into account,

$$\frac{1}{2}\frac{d}{dt}||u'_m(t)||_2^2 + \frac{1}{2}\frac{d}{dt}\widehat{M}(||\nabla u_m(t)||_2^2) - \int_0^t g(t-s)(\nabla u_m(s), \nabla u'_m(t))_{L^2(\Omega)}\, ds = 0. \tag{2.5}$$

Using the binary notation

$$(g\Box u)(t) = \int_0^t g(t-s)|u(t) - u(s)|^2\, ds,$$

we infer

$$\begin{aligned}
\frac{d}{dt}\int_\Omega (g\Box\nabla u)(t)dx &= \int_\Omega\int_0^t g'(t-s)|\nabla u(t)-\nabla u(s)|^2\,ds\,dx \\
&\quad +\int_\Omega\int_0^t g(t-s)\frac{d}{dt}|\nabla u(t)-\nabla u(s)|^2\,ds\,dx \\
&= \int_\Omega (g'\Box\nabla u)(t)\,dx \\
&\quad +2\int_\Omega\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\nabla u'(t)\,ds\,dx \\
&= \int_\Omega (g'\Box\nabla u)(t)\,dx + 2\int_\Omega\int_0^t g(t-s)\nabla u(t)\nabla u'(t)\,ds\,dx \\
&\quad -2\int_\Omega\int_0^t g(t-s)\nabla u(s)\nabla u'(t)\,ds\,dx,
\end{aligned}$$

which implies that, for u_m instead of u,

$$\begin{aligned}
-\int_0^t g(t-s)(\nabla u_m(s),\nabla u_m'(t))_{L^2(\Omega)}\,ds &= \frac{1}{2}\frac{d}{dt}\int_\Omega (g\Box u_m)(t)dx - \frac{1}{2}\int_\Omega (g'\Box\nabla u_m)(t)\,dx \\
&\quad -\frac{1}{2}\left(\int_0^t g(s)\,ds\right)\frac{d}{dt}||\nabla u_m(t)||_2^2. \qquad (2.6)
\end{aligned}$$

Then substituting (2.6) in (2.5) yields

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}||u_m'(t)||_2^2 + \frac{1}{2}\frac{d}{dt}\widehat{M}(||\nabla u_m(t)||_2^2) + \frac{1}{2}\frac{d}{dt}\int_\Omega (g\Box u_m)(t)dx - \frac{1}{2}\left(\int_0^t g(s)\,ds\right)\frac{d}{dt}||\nabla u_m(t)||_2^2 \\
= \frac{1}{2}\int_\Omega (g'\Box\nabla u_m)(t)\,dx,
\end{aligned}$$

and using

$$\frac{1}{2}\frac{d}{dt}\left[\left(\int_0^t g(s)\,ds\right)||\nabla u_m(t)||_2^2\right] = \frac{1}{2}g(t)||\nabla u_m(t)||_2^2 + \frac{1}{2}\left(\int_0^t g(s)\,ds\right)\frac{d}{dt}||\nabla u_m(t)||_2^2,$$

we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[||u_m'(t)||_2^2 + \widehat{M}(||\nabla u_m(t)||_2^2) + \int_\Omega (g\Box\nabla u_m)(t)dx - \left(\int_0^t g(s)\,ds\right)||\nabla u_m(t)||_2^2\right] \\
&= \frac{1}{2}\int_\Omega (g'\Box\nabla u_m)(t)\,dx - \frac{1}{2}g(t)||\nabla u_m(t)||_2^2. \qquad (2.7)
\end{aligned}$$

On the other hand, the hypothesis (1.3) implies that

$$\widehat{M}\left(||\nabla u_m(t)||_2^2\right) = \int_0^{||\nabla u_m(t)||_2^2} M(s)\,ds \geq m_0||\nabla u_m(t)||_2^2,$$

consequently, taking (1.6) into account,

$$\widehat{M}\left(||\nabla u_m(t)||_2^2\right) - \left(\int_0^t g(s)\,ds\right)||\nabla u_m(t)||_2^2 \geq (m_0 - g_0)||\nabla u_m(t)||_2^2. \tag{2.8}$$

Combining (2.7) and (2.8), and observing that $g > 0$ and $g' \leq 0$, we deduce

$$\begin{aligned}
&\frac{1}{2}||u_m'(t)||_2^2 + \frac{1}{2}(m_0 - g_0)||\nabla u_m(t)||_2^2 + \int_\Omega (g\Box\nabla u_m)(t)dx \\
&\leq \frac{1}{2}||u_{1m}||_2^2 + \frac{1}{2}\widehat{M}(||\nabla u_{0m}||_2^2) \\
&\leq L_1\left(||u_1||_2^2, ||\nabla u_0||_2^2\right), \quad \forall t \geq 0,\ \forall m \in \mathbb{N},
\end{aligned} \tag{2.9}$$

where L_1 does not depend neither on $m \in \mathbb{N}$ nor on $t \geq 0$. This implies that the approximated solution u_m exists globally in the topologies given in (2.9).

Defining the energy $\widehat{E}$ associated to problem (1.2) by

$$\begin{aligned}
\widehat{E}(t) := \quad & \frac{1}{2}||u'(t)||_2^2 + \frac{1}{2}\widehat{M}(||\nabla u(t)||_2^2) - \frac{1}{2}\left(\int_0^t g(s)\,ds\right)||\nabla u(t)||_2^2 \\
&+\frac{1}{2}\int_\Omega (g\Box\nabla u)(t)dx,
\end{aligned} \tag{2.10}$$

then, in view of (2.7), it is non increasing function. In addition, as a consequence of (2.7), the following identity of the energy holds:

$$\widehat{E}(t_2) - \widehat{E}(t_1) = \frac{1}{2}\int_{t_1}^{t_2}\int_\Omega \left(g'\Box\nabla u - g(t)|\nabla u|^2\right)\,dxdt \leq 0, \quad \forall t_2 \geq t_1 \geq 0. \tag{2.11}$$

Energy decay estimate. Define

$$(g\circ v)(t) = \int_0^t g(t-s)||v(t) - v(s)||_2^2\,ds$$

and

$$(g\diamond v)(t) = \int_0^t g(t-s)(v(t) - v(s))\,ds.$$

Lemma 2.1. *Let $\psi \in L^1(\mathbb{R}_+, \mathbb{R}_+)$ and $u \in L^2(\mathbb{R}_+; L^2(\Omega))$. Then*

$$||(\psi\diamond u)(t)||_2^2 \leq ||\psi||_{L^1(\mathbb{R}_+)}(\psi\circ u)(t).$$

Proof. Applying Hölder inequality and Fubini theorem, we have

$$\begin{aligned} ||(\psi \diamond u)(t)||_2^2 &= \int_\Omega \left(\int_0^t \sqrt{\psi(t-s)}\sqrt{\psi(t-s)}(u(t)-u(s))\,ds \right)^2 dx \\ &\leq \left(\int_0^t \psi(\zeta)\,d\zeta \right) \int_0^t \psi(t-s) \int_\Omega (u(t)-u(s))^2\,dx\,ds. \end{aligned}$$

□

From now on, for short notation, we shall drop the parameter "m" in u_m. We have the following useful lemma:

Lemma 2.2. *Let u be a solution to the approximated problem (2.2) corresponding to initial data taken in bounded sets of $H_0^1(\Omega)\times L^2(\Omega)$. Then, we have the following decay rate estimate:*

$$\widehat{E}(t) \leq c\widehat{E}(0)e^{-\theta\int_0^t \xi(s)\,ds}, \quad t \geq 0,$$

for some positive constants c and θ which do not depend on $m \in \mathbb{N}$.

Proof. From (2.2), we have,

$$\begin{aligned} (u''(t),w)_{L^2(\Omega)} \quad &+ \quad M(||\nabla u(t)||_2^2)(\nabla u(t),\nabla w)_{L^2(\Omega)} \\ &- \quad \int_0^t g(t-s)(\nabla u(s),\nabla w)_{L^2(\Omega)}\,ds = 0, \quad \forall w \in V_m. \end{aligned} \tag{2.12}$$

Recovering the potential energy.

Substituting $w = u$ in (2.12), multiplying by $\xi(t)$ and integrating over $[0,T]$, we can write

$$\begin{aligned} \int_0^T \xi(t)(u''(t),u(t))_{L^2(\Omega)}\,dt \quad &+ \quad \int_0^T \xi(t)M(||\nabla u(t)||_2^2)||\nabla u(t)||_2^2\,dt \\ &- \quad \int_0^T \xi(t)\int_0^t g(t-s)(\nabla u(s),\nabla u(t))_{L^2(\Omega)}\,ds\,dt = 0. \end{aligned} \tag{2.13}$$

Having in mind that

$$\frac{d}{dt}\xi(t)(u'(t),u(t))_{L^2(\Omega)} = \xi(t)(u''(t),u(t)) + \xi(t)||u'(t)||_2^2 + \xi'(t)(u'(t),u(t))_{L^2(\Omega)},$$

from (2.13) we obtain

$$\begin{aligned} \xi(t)(u'(t),u(t))_{L^2(\Omega)}\big|_0^T \quad &- \quad \int_0^T \xi(t)||u'(t)||_2^2\,dt - \int_0^T \xi'(t)(u'(t),u(t))_{L^2(\Omega)}\,dt \\ &+ \quad \int_0^T \xi(t)M(||\nabla u(t)||_2^2)||\nabla u(t)||_2^2\,dt \qquad (2.14) \\ &- \quad \int_0^T \xi(t)\int_0^t g(t-s)(\nabla u(s),\nabla u(t))_{L^2(\Omega)}\,ds\,dt = 0, \end{aligned}$$

and, using (1.3) from (2.14), we find

$$m_0 \int_0^T \xi(t)||\nabla u(t)||_2^2\,dt \le - \xi(t)(u'(t),u(t))_{L^2(\Omega)}|_0^T + \int_0^T \xi(t)||u'(t)||_2^2\,dt + \int_0^T \xi'(t)(u'(t),u(t))_{L^2(\Omega)}\,dt + \int_0^T \xi(t)\int_0^t g(t-s)(\nabla u(s),\nabla u(t))_{L^2(\Omega)}\,ds\,dt. \tag{2.15}$$

Now, we will estimate separately the last terms on the right hand side of (2.15). We have, using Cauchy-Schwarz and Young's inequalities,

$$\int_0^T \xi(t)\int_0^t g(t-s)(\nabla u(s),\nabla u(t))_{L^2(\Omega)}\,ds\,dt$$

$$\begin{aligned}
&\le \int_0^T \xi(t)\int_0^t g(t-s)||\nabla u(s)||_2||\nabla u(t)||_2\,ds\,dt\\
&\le \int_0^T \xi(t)\int_0^t g(t-s)\left(||\nabla u(s)-\nabla u(t)||_2+||\nabla u(t)||_2\right)||\nabla u(t)||_2\,ds\,dt\\
&= \int_0^T \xi(t)\int_0^t g(t-s)||\nabla u(s)-\nabla u(t)||_2||\nabla u(t)||_2\,ds\,dt\\
&\quad+\int_0^T \xi(t)\int_0^t g(t-s)||\nabla u(t)||_2^2\,ds\,dt\\
&\le (1+\varepsilon)\int_0^T \xi(t)\int_0^t g(t-s)||\nabla u(t)||_2^2\,ds\,dt+\frac{1}{4\varepsilon}\int_0^T \xi(t)(g\circ\nabla u)(t)\,dt;
\end{aligned}$$

that is,

$$\int_0^T \xi(t)\int_0^t g(t-s)(\nabla u(s),\nabla u(t))_{L^2(\Omega)}\,ds\,dt \le (1+\varepsilon)g_0\int_0^T \xi(t)||\nabla u(t)||_2^2\,dt + \frac{1}{4\varepsilon}\int_0^T \xi(t)(g\circ\nabla u)(t)\,dt. \tag{2.16}$$

On the other hand, because $\frac{\xi'}{\xi}$ is bounded, we see that, for any $\epsilon_0>0$,

$$\int_0^T \xi'(t)(u'(t),u(t))_{L^2(\Omega)}\,dt \le c_0\int_0^T \xi(t)\left(\epsilon_0||u'(t)||_2^2+\frac{1}{\epsilon_0}||\nabla u(t)||_2^2\right)dt, \tag{2.17}$$

where $c_0=\frac{1}{2}(1+\lambda_1^{-1/2})||\frac{\xi'}{\xi}||_{L^\infty(\mathbb{R}_+)}$.

From (2.15), (2.16) and (2.17) we arrive at

$$m_0 \int_0^T \xi(t)||\nabla u(t)||_2^2\,dt \le -\xi(t)(u'(t),u(t))_{L^2(\Omega)}|_0^T + (1+\epsilon_0 c_0)\int_0^T \xi(t)||u'(t)||_2^2\,dt \tag{2.18}$$
$$+\left((1+\varepsilon)g_0+\frac{c_0}{\epsilon_0}\right)\int_0^T \xi(t)||\nabla u(t)||_2^2\,dt + \frac{1}{4\varepsilon}\int_0^T \xi(t)(g\circ\nabla u)(t)\,dt.$$

Recovering the kinectic energy. Substituting $w = g\diamond u \in V_m$ in (2.12) and multiplying by $\xi(t)$, it results that

$$\begin{aligned}
&\int_0^T \xi(t)(u''(t),(g\diamond u)(t))_{L^2(\Omega)}\,dt \\
&+\int_0^T \xi(t)M\left(||\nabla u(t)||_2^2\right)(\nabla u(t),(g\diamond\nabla u)(t))_{L^2(\Omega)}\,dt \\
&-\int_0^T \xi(t)\int_0^t g(t-s)(\nabla u(s),(g\diamond\nabla u)(t))_{L^2(\Omega)}\,ds\,dt = 0.
\end{aligned}\tag{2.19}$$

But

$$\begin{aligned}
\frac{d}{dt}\xi(t)(u'(t),(g\diamond u)(t))_{L^2(\Omega)} &= \xi(t)(u''(t),(g\diamond u)(t))_{L^2(\Omega)} \\
&+\xi(t)(u'(t),(g'\diamond u)(t))_{L^2(\Omega)} \\
&+\xi(t)\left(u'(t),\int_0^t g(t-s)u'(t)\,ds\right)_{L^2(\Omega)} \\
&+\xi'(t)(u'(t),(g\diamond u)(t))_{L^2(\Omega)}.
\end{aligned}$$

Integrating the last identity over $(0,T)$, we obtain,

$$\begin{aligned}
\int_0^T \xi(t)(u''(t),(g\diamond u)(t))_{L^2(\Omega)}\,dt &= \xi(t)(u'(t),(g\diamond u)(t))_{L^2(\Omega)}|_0^T \\
&-\int_0^T \xi'(t)(u'(t),(g\diamond u)(t))_{L^2(\Omega)}\,dt \\
&-\int_0^T \xi(t)(u'(t),(g'\diamond u)(t))_{L^2(\Omega)}\,dt \\
&-\int_0^T \xi(t)\left(\int_0^t g(s)\,ds\right)||u'(t)||_2^2\,dt.
\end{aligned}\tag{2.20}$$

Substituting (2.20) in (2.19), we conclude

$$\begin{aligned}
&\int_0^T \xi(t)\left(\int_0^t g(s)\,ds\right)||u'(t)||_2^2\,dt \qquad (2.21)\\
= \;& \xi(t)(u'(t),(g\diamond u)(t))_{L^2(\Omega)}|_0^T - \int_0^T \xi(t)(u'(t),(g'\diamond u)(t))\\
&+\int_0^T \xi(t)M\left(||\nabla u(t)||_2^2\right)(\nabla u(t),(g\diamond\nabla u)(t))_{L^2(\Omega)}\,dt\\
&-\int_0^T \xi(t)\int_0^t g(t-s)\,(\nabla u(s),(g\diamond\nabla u)(t))_{L^2(\Omega)}\,ds\,dt\\
&-\int_0^T \xi'(t)(u'(t),(g\diamond u)(t))_{L^2(\Omega)}\,dt.
\end{aligned}$$

Let $t_0 > 0$ such that $g(t_0)t_0 > 0$. This is possible in vertue of Assumption 1.2. Then one has

$$\int_0^t g(s)\,ds \geq g(t_0)t_0 > 0, \quad \forall t \geq t_0. \qquad (2.22)$$

Combining (2.21) and (2.22) yields

$$\begin{aligned}
&g(t_0)t_0\int_{t_0}^T \xi(t)||u'(t)||_2^2\,dt \qquad (2.23)\\
\leq\;& \xi(t)(u'(t),(g\diamond u)(t))_{L^2(\Omega)}|_0^T - \int_0^T \xi(t)(u'(t),(g'\diamond u)(t))_{L^2(\Omega)}\,dt\\
&+\int_0^T \xi(t)M\left(||\nabla u(t)||_2^2\right)(\nabla u(t),(g\diamond\nabla u)(t))_{L^2(\Omega)}\,dt\\
&-\int_0^T \xi(t)\int_0^t g(t-s)\,(\nabla u(s),(g\diamond\nabla u)(t))_{L^2(\Omega)}\,ds\,dt\\
&-\int_0^T \xi'(t)(u'(t),(g\diamond u)(t))_{L^2(\Omega)}\,dt, \quad \forall T\geq t_0.
\end{aligned}$$

On the other hand, it is convenient to observe that

$$\begin{aligned}
&\int_0^T \xi(t)\left(\int_0^t g(t-s)\nabla u(t)\,ds,(g\diamond\nabla u)(t)\right)_{L^2(\Omega)}dt \qquad (2.24)\\
=\;&\int_0^T \xi(t)\left((g\diamond\nabla u)(t),(g\diamond\nabla u)(t)\right)_{L^2(\Omega)}\,dt\\
&+\int_0^T \xi(t)\left(\int_0^t g(t-s)\nabla u(s)\,ds,(g\diamond\nabla u)(t)\right)_{L^2(\Omega)}dt.
\end{aligned}$$

Combining (2.23) and (2.24) we infer, for all $T \geq t_0$,

$$\begin{aligned} g(t_0)t_0 \int_{t_0}^T \xi(t)||u'(t)||_2^2\,dt \quad \leq \quad & \xi(t)(u'(t),(g\diamond u)(t))_{L^2(\Omega)}|_0^T \\ & + \int_0^T \xi(t)||(g\diamond\nabla u)(t)||_2^2\,dt \\ & - \int_0^T \xi(t)(u'(t),(g'\diamond u)(t))_{L^2(\Omega)}\,dt \\ & + \int_0^T \xi(t)M\left(||\nabla u(t)||_2^2\right)(\nabla u(t),(g\diamond\nabla u)(t))_{L^2(\Omega)}\,dt \\ & - \int_0^T \xi(t)\left(\int_0^t g(t-s)\nabla u(t)\,ds,(g\diamond\nabla u)(t)\right)_{L^2(\Omega)} dt \\ & - \int_0^T \xi'(t)(u'(t),(g\diamond u)(t))_{L^2(\Omega)}\,dt. \end{aligned} \tag{2.25}$$

Next, we shall analyse the terms on the right hand side of (2.25).

Estimate for $I_1 := \xi(t)(u'(t),(g\diamond u)(t))_{L^2(\Omega)}|_0^T$. We have,

$$I_1 = \xi(T)\left(u'(T),\int_0^T g(T-s)(u(T)-u(s))ds\right)_{L^2(\Omega)}. \tag{2.26}$$

Thus, having in mind lemma 2.1, the definition of the energy in (2.10) and that ξ is non increasing, we deduce

$$\begin{aligned} |I_1| \quad & = \xi(T)|\int_0^T g(T-s)(u'(T),u(T)-u(s))_{L^2(\Omega)}\,ds| \qquad (2.27) \\ & \leq \xi(0)\int_0^T g(T-s)||u'(T)||_2\,||u(T)-u(s)||_2\,ds \\ & \leq \xi(0)\int_0^T g(T-s)\left(\frac{1}{2}||u'(T)||_2^2+\frac{1}{2}||u(T)-u(s)||_2^2\right)ds \\ & \leq \frac{1}{2}\xi(0)g_0||u'(T)||_2^2+\frac{\lambda_1^{-1/2}\xi(0)}{2}\int_0^T g(T-s)||\nabla u(T)-\nabla u(s)||_2^2\,ds \\ & = \frac{1}{2}\xi(0)g_0||u'(T)||_2^2+\frac{\lambda_1^{-1/2}\xi(0)}{2}(g\circ\nabla u)(T) \\ & \leq \xi(0)\left(g_0+\lambda_1^{-1/2}\right)\widehat{E}(T). \end{aligned}$$

Therefore

$$|I_1| \leq C\widehat{E}(T), \tag{2.28}$$

for some $C > 0$, which, from now on, will represent various constants do not depend on T and $m \in \mathbb{N}$, which is crucial in the proof.

Estimate for $I_2 := -\int_0^T \xi(t)(u'(t),(g' \diamond u)(t))_{L^2(\Omega)}\,dt$. Employing lemma 2.1 and the property $\xi(t) \le \xi(0)$, one has

$$
\begin{aligned}
|I_2| &\le \int_0^T \xi(t)||u'(t)||_2\,||(g' \diamond u)(t)||_2\,dt \qquad (2.29)\\
&\le \varepsilon \int_0^T \xi(t)||u'(t)||_2^2\,dt + \frac{1}{4\varepsilon}\int_0^T \xi(t)||(g' \diamond u)(t)||_2^2\,dt\\
&\le \varepsilon \int_0^T \xi(t)||u'(t)||_2^2\,dt + \frac{\xi(0)}{4\varepsilon}||g'||_{L^1(\mathbb{R}_+)}\int_0^T (|g'| \circ u)(t)dt\\
&\le \varepsilon \int_0^T \xi(t)||u'(t)||^2\,dt - \frac{\xi(0)\lambda_1^{-1/2}}{4\varepsilon}||g'||_{L^1(\mathbb{R}_+)}\int_0^T (g' \circ \nabla u)(t)\,dt,
\end{aligned}
$$

where ε is an arbitrary positive constant.

Similarly, because $\frac{\xi'}{\xi}$ is bounded, we have

$$
\begin{aligned}
|-\int_0^T \xi'(t)(u'(t),(g \diamond u)(t))_{L^2(\Omega)}\,dt| &\le \varepsilon \int_0^T \xi(t)||u'(t)||^2\,dt \qquad (2.30)\\
&+\frac{\lambda_1^{-1/2}g_0}{4\varepsilon}||\frac{\xi'}{\xi}||_{L^\infty(\mathbb{R}_+)}\int_0^T \xi(t)(g \circ \nabla u)(t)dt,
\end{aligned}
$$

where ε is an arbitrary positive constant.

Estimate for $I_3 := \int_0^T \xi(t)M\left(||\nabla u(t)||_2^2\right)(\nabla u(t),(g \diamond \nabla u)(t))_{L^2(\Omega)}\,dt$. Let us define:

$$
E(t) := \frac{1}{2}||u'(t)||_2^2 + \frac{1}{2}||\nabla u(t)||_2^2, \qquad (2.31)
$$

the mechanical energy associated to problem (1.2). First, we observe that

$$
\widehat{E}(t) \ge \frac{1}{2}\left(||u'(t)||_2^2 + (m_0 - g_0)||\nabla u(t)||_2^2\right),
$$

which implies that

$$
\widehat{E}(0) \ge \widehat{E}(t) \ge \alpha_0 E(t), \quad \forall t \ge 0,
$$

where $\alpha_0 = min\{1, m_0 - g_0\}$ ($\alpha_0 > 0$ in vertue of (1.6)). Thus, we get

$$
E(t) \le \alpha_0^{-1}\widehat{E}(t) \le \alpha_0^{-1}\widehat{E}(0), \quad \forall t \ge 0. \qquad (2.32)
$$

Using the assumption (1.4) and taking (2.32) into account, we deduce, similarly to the estimate (2.29),

$$\begin{aligned} |I_3| &\leq \delta\alpha_0^{-\gamma}(2\widehat{E}(0))^\gamma \int_0^T \xi(t)\|\nabla u(t)\|_2 \,\|(g\diamond\nabla u)(t)\|_2\,dt \qquad (2.33)\\ &\leq \varepsilon\int_0^T \xi(t)\|\nabla u(t)\|_2^2\,dt + \frac{\delta^2\alpha_0^{-2\gamma}(2\widehat{E}(0))^{2\gamma}}{4\varepsilon}\int_0^T \xi(t)\|(g\diamond\nabla u)(t)\|_2^2\,dt\\ &\leq \varepsilon\int_0^T \xi(t)\|\nabla u(t)\|_2^2\,dt + \frac{\delta^2\alpha_0^{-2\gamma}g_0(2\widehat{E}(0))^{2\gamma}}{4\varepsilon}\int_0^T \xi(t)(g\circ\nabla u)(t)dt, \end{aligned}$$

where ε is an arbitrary positive constant.

Estimate for $I_4 := \int_0^T \xi(t)\|(g\diamond\nabla u)(t)\|_2^2$. Lemma 2.1 implies that

$$|I_4| \leq g_0\int_0^T \xi(t)(g\circ\nabla u)(t)\,dt. \qquad (2.34)$$

Estimate for $I_5 := -\int_0^T \xi(t)\left(\int_0^t g(t-s)\nabla u(t)\,ds, (g\diamond\nabla u)(t)\right)_{L^2(\Omega)} dt$. One has, using again lemma 2.1,

$$\begin{aligned} |I_5| &\leq g_0\int_0^T \xi(t)\|\nabla u(t)\|_2\,\|(g\diamond\nabla u)(t)\|_2\,dt \qquad (2.35)\\ &\leq \varepsilon\int_0^T \xi(t)\|\nabla u(t)\|_2^2\,dt + \frac{g_0^2}{4\varepsilon}\int_0^T \xi(t)\|(g\diamond\nabla u)(t)\|_2^2\,dt\\ &\leq \varepsilon\int_0^T \xi(t)\|\nabla u(t)\|_2^2\,dt + \frac{g_0^3}{4\varepsilon}\int_0^T \xi(t)(g\circ\nabla u)(t)dt. \end{aligned}$$

Combining (2.25), (2.28), (2.29), (2.30), (2.33), (2.34) and (2.35), we conclude, for all $T \geq t_0$,

$$\begin{aligned} g(t_0)t_0\int_{t_0}^T \xi(t)\|u'(t)\|_2^2\,dt &\leq 2\varepsilon\int_0^T \xi(t)\|u'(t)\|^2\,dt + 2\varepsilon\int_0^T \xi(t)\|\nabla u(t)\|_2^2\,dt \qquad (2.36)\\ &\quad + C\widehat{E}(T) + C\int_0^T (\xi(t)(g\circ\nabla u)(t) - (g'\circ\nabla u)(t))\,dt. \end{aligned}$$

Multiplying (2.18) by a constant $\beta_1 > 0$, adding (2.36) and having in mind that, according to the properity $\xi(t) \leq \xi(0)$ and (2.32),

$$g(t_0)t_0\int_0^t \xi(t)\|u'(t)\|_2^2\,dt \leq g(t_0)t_0\xi(0)\int_0^{t_0} \|u'(t)\|_2^2\,dt \leq C\widehat{E}(0), \quad \forall t\in[0,t_0]$$

and

$$|-\xi(t)(u'(t),u(t))_{L^2(\Omega)}|_0^T| \leq C\widehat{E}(0), \quad \forall t \geq 0,$$

we can write

$$\begin{aligned}
&(g(t_0)t_0 - 2\varepsilon - \beta_1(1+\epsilon_0 c_0))\int_0^T \xi(t)||u'(t)||_2^2\,dt \\
&+\left(\beta_1\left(m_0 - g_0 - \frac{c_0}{\epsilon_0}\right) - \varepsilon(\beta_1 g_0 + 2)\right)\int_0^T \xi(t)||\nabla u(t)||_2^2\,dt \\
&\leq C\widehat{E}(0) + C\int_0^T (\xi(t)(g\circ\nabla u)(t) - (g'\circ\nabla u)(t))dt, \quad \forall T \geq t_0.
\end{aligned} \tag{2.37}$$

Choosing $\epsilon_0 > \frac{c_0}{m_0 - g_0}$, $0 < \beta_1 < \frac{g(t_0)t_0}{1+\epsilon_0 c_0}$ and $0 < \varepsilon < \min\left\{\frac{1}{2}(g(t_0)t_0 - \beta_1(1+\epsilon_0 c_0)), \frac{\beta_1(m_0 - g_0 - \frac{c_0}{\epsilon_0})}{\beta_1 g_0 + 2}\right\}$. Hence, from (2.37), we deduce

$$\begin{aligned}
&\int_0^T \xi(t)||\xi(t)u'(t)||_2^2\,dt + \int_0^T \xi(t)||\nabla u(t)||_2^2\,dt \\
&\leq C\widehat{E}(0) + C\int_0^T (\xi(t)(g\circ\nabla u)(t) - (g'\circ\nabla u)(t))\,dt, \quad \forall T \geq t_0.
\end{aligned} \tag{2.38}$$

Taking (2.31) and (2.38) into consideration, it results that

$$\int_0^T \xi(t)E(t)\,dt \leq C\widehat{E}(0) + C\int_0^T (\xi(t)(g\circ\nabla u)(t) - (g'\circ\nabla u)(t))\,dt, \quad \forall T \geq t_0. \tag{2.39}$$

Recalling that $\widehat{M}(\lambda) = \int_0^\lambda M(s)\,ds$, from (1.3) and (1.4), we infer

$$m_0\lambda \leq \widehat{M}(\lambda) \leq \frac{\delta}{\gamma+1}\lambda^{\gamma+1}, \quad \forall \lambda \geq 0. \tag{2.40}$$

Considering (2.40) and using (2.32), we can write

$$\begin{aligned}
\widehat{E}(t) &= \frac{1}{2}\left(||u'(t)||_2^2 + \widehat{M}(||\nabla u(t)||_2^2) + (g\circ\nabla u)(t) - \left(\int_0^t g(s)\,ds\right)||\nabla u(t)||_2^2\right) \\
&\leq \frac{1}{2}g\circ\nabla u + \frac{1}{2}\left(||u'(t)||_2^2 + \frac{\delta}{\gamma+1}||\nabla u(t)||_2^{2\gamma}||\nabla u(t)||_2^2\right) \\
&\leq \frac{1}{2}(g\circ\nabla u)(t) + \frac{1}{2}\left(||u'(t)||_2^2 + \frac{\delta}{\gamma+1}\left(\frac{2}{\alpha_0}\widehat{E}(0)\right)^\gamma ||\nabla u(t)||_2^2\right).
\end{aligned} \tag{2.41}$$

We are assuming, by assumption, that the initial data are taken in bounded sets of $H_0^1(\Omega)\times L^2(\Omega)$. Consequently, let $L > 0$ (not depending neither on $m \in \mathbb{N}$

nor on $t \in \mathbb{R}_+$) such that $E(t) \leq L$. This implies that there exists $d > 0$ such that $\widehat{E}(0) < d$. Then, from (2.41), we conclude

$$\widehat{E}(t) \leq \frac{1}{2}(g \circ \nabla u)(t) + B_0 E(t), \quad \forall t \geq 0, \tag{2.42}$$

where $B_0 = \max\left\{1, \dfrac{\delta(2d)^\gamma}{(\gamma+1)\alpha_0^\gamma}\right\}$, and, therefore

$$\int_0^T \xi(t)\widehat{E}(t)\,dt \leq \frac{1}{2}\int_0^T \xi(t)(g \circ \nabla u)(t)\,dt + B_0 \int_0^T \xi(t)E(t)\,dt, \quad \forall T \geq 0. \tag{2.43}$$

Combining (2.39) and (2.43) we deduce, for all $T \geq t_0$,

$$\int_0^T \xi(t)\widehat{E}(t)\,dt \leq C\widehat{E}(0) + C\int_0^T (\xi(t)(g \circ \nabla u)(t) - (g' \circ \nabla u)(t))\,dt. \tag{2.44}$$

Since, according to (2.7),

$$\widehat{E}'(t) \leq \frac{1}{2}(g' \circ \nabla u)(t), \quad \forall t \geq 0,$$

it implies that

$$(-g' \circ \nabla u)(t) \leq -2\widehat{E}'(t), \quad \forall t \geq 0,$$

and consequently, from (2.44), we have

$$\int_0^T \xi(t)\widehat{E}(t)\,dt \leq C\widehat{E}(0) + C\int_0^T \xi(t)(g \circ \nabla u)(t)\,dt - C\int_0^T \widehat{E}'(t)\,dt, \quad \forall T \geq t_0,$$

namely,

$$\int_0^T \xi(t)\widehat{E}(t)\,dt \leq C\widehat{E}(0) + C\int_0^T \xi(t)(g \circ \nabla u)(t)\,dt, \quad \forall T \geq t_0. \tag{2.45}$$

Once we are assuming (1.7) and because ξ is non increasing, we see that

$$\xi(t)(g \circ \nabla u)(t) \leq ((\xi g) \circ \nabla u)(t) \leq -(g' \circ \nabla u)(t) \leq -2\widehat{E}'(t),$$

then, we deduce from (2.45) that

$$\int_0^T \xi(t)\widehat{E}(t)\,dt \leq C\widehat{E}(0) - C\int_0^T \widehat{E}'(t), \quad \forall T \geq t_0, \tag{2.46}$$

which leads us

$$\int_0^T \xi(t)\widehat{E}(t)\,dt \leq C\widehat{E}(0), \quad \forall T \geq t_0. \tag{2.47}$$

For $0 \le T \le t_0$, one has, using (2.11) and the fact that $\xi(t) \le \xi(0)$,

$$\int_0^T \xi(t)\widehat{E}(t)\,dt \le T\xi(0)\widehat{E}(0) \le t_0\xi(0)\widehat{E}(0),$$

which gives us (2.47), for all $T > 0$.

Let $\widehat{\xi}(t) = \int_0^t \xi(s)\,ds$ and $F(t) = \widehat{E}(\widehat{\xi}^{-1}(t))$. Thanks to Assumption 1.2, $\widehat{\xi}$ defines a bijection from $\mathbb{R}_+$ to $\mathbb{R}_+$, F is non increasing and $F(0) = \widehat{E}(0)$, and then (2.47) implies that

$$\int_0^T F(t)\,dt \le CF(0), \quad \forall T \ge 0.$$

Consequently, by applying Theorem 9.1 in [16], we find that ther exist positive constants c and θ not depending on $\hat{E}(0)$ such that

$$F(t) \le cF(0)e^{-\theta t}, \quad \forall t \ge 0.$$

□

By the definition of F, this last inequality implies the general stability (1.9), which finishes the proof.

3. Well-posedness

Lemma 3.1 ($H^2(\Omega)$ a priori bounds)**.** *Suppose that u is a local solution on $[0,T[$ such that*

$$\sup_{t\in[0,T[} \{||\nabla u'(t)||_2, ||\Delta u(t)||_2\} < K,$$

for some $K > 0$ and $T > 0$. Then, the following estimate holds:

$$\begin{aligned} ||\nabla u'(t)||_2^2 + ||\Delta u(t)||_2^2 &\le CK^3(\widehat{E}(0))^{\frac{2\alpha+1}{2}} \int_0^t e^{-\frac{\theta(2\alpha+1)}{2}\int_0^s \xi(\tau)\,d\tau}\,ds \\ &\quad +\alpha_0^{-1}\left(||\nabla u_1||_2^2 + M(||\nabla u_0||_2^2)||\Delta u_0||_2^2\right) \\ &:= G(t, I_0, I_1, K) \quad \text{on } [0,T[, \end{aligned} \tag{3.1}$$

with $I_0 = \widehat{E}(0)$ and $I_1 = ||\nabla u_1||_2^2 + M(||\nabla u_0||_2^2)||\Delta u_0||_2^2$.

Proof. Taking $w = -\Delta u' \in V_m$ in the approximate problem (2.2) yields

$$\begin{aligned} &\tfrac{1}{2}\tfrac{d}{dt}\left[||\nabla u'(t)||_2^2 + M(||\nabla u(t)||_2^2)||\Delta u(t)||_2^2\right] - \int_0^t g(t-s)(\Delta u(s), \Delta u'(s))_{L^2(\Omega)}\,ds \\ &\quad = M'(||\nabla u(t)||_2^2)(\nabla u'(t), \nabla u(t))_{L^2(\Omega)}||\Delta u(t)||_2^2. \end{aligned} \tag{3.2}$$

Considering similar computations as done before, from (3.2), we infer

$$\frac{1}{2}\frac{d}{dt}\left[||\nabla u'(t)||_2^2+M(||\nabla u(t)||_2^2)||\Delta u(t)||_2^2-\left(\int_0^t g(s)ds\right)||\Delta u(t)||_2^2+(g\circ\Delta u)(t)\right]$$
$$=\frac{1}{2}(g'\circ\Delta u)(t)\,dx-\frac{1}{2}g(t)||\Delta u(t)||_2^2+M'(||\nabla u||_2^2)(\nabla u'(t),\nabla u(t))_{L^2(\Omega)}||\Delta u(t)||_2^2. \tag{3.3}$$

Integrating (3.3) over $(0,t)$, $t>0$, we deduce

$$\begin{aligned}&||\nabla u'(t)||_2^2+M\left(||\nabla u(t)||_2^2\right)||\Delta u(t)||_2^2-\left(\int_0^t g(s)ds\right)||\Delta u(t)||_2^2+(g\circ\Delta u)(t)\\&-\left(||\nabla u_1||_2^2+M\left(||\nabla u_0||_2^2\right)||\Delta u_0||_2^2\right)\\&\le 2\int_0^t M'\left(||\nabla u(s)||_2^2\right)\left(\nabla u'(s),\nabla u(s)\right)_{L^2(\Omega)}||\Delta u(s)||_2^2\,ds.\end{aligned} \tag{3.4}$$

On the other hand, we have, in vertue of (1.3) of and (1.6),

$$\begin{aligned}&||\nabla u'(t)||_2^2+M\left(||\nabla u(t)||_2^2\right)||\Delta u(t)||_2^2-\left(\int_0^t g(s)ds\right)||\Delta u(t)||_2^2+(g\circ\Delta u)(t)\,dx\\&\ge||\nabla u'(t)||_2^2+(m_0-g_0)||\Delta u(t)||_2^2\\&\ge\alpha_0\left(||\nabla u'(t)||_2^2+||\Delta u(t)||_2^2\right).\end{aligned} \tag{3.5}$$

Combining (3.4) and (3.5), and taking (1.5) and (2.32) into account, we obtain

$$\begin{aligned}&\alpha_0\left(||\nabla u'(t)||_2^2+||\Delta u(t)||_2^2\right)-\left(||\nabla u_1||_2^2+M(||\nabla u_0||_2^2)||\Delta u_0||_2^2\right)\\&\le 2\int_0^t |M'(||\nabla u(s)||_2^2)|||\nabla u'(s)||_2||\nabla u(s)||_2||\Delta u(s)||_2^2\,ds\\&\le 2\beta K^3\int_0^t ||\nabla u(s)||_2^{2\alpha+1}\,ds\\&\le 2^{\frac{2\alpha+3}{2}}\beta K^3\int_0^t (E(s))^{\frac{2\alpha+1}{2}}\,ds\\&\le 2^{\frac{2\alpha+3}{2}}\alpha_0^{-\frac{2\alpha+1}{2}}\beta K^3\int_0^t (\widehat{E}(s))^{\frac{2\alpha+1}{2}}\,ds.\end{aligned} \tag{3.6}$$

Inequality (3.6) combined with Lemma 2.2 yields

$$||\nabla u'(t)||_2^2+||\Delta u(t)||_2^2 \tag{3.7}$$

$$\begin{aligned}&\le\alpha_0^{-1}\left(||\nabla u_1||_2^2+M(||\nabla u_0||_2^2)||\Delta u_0||_2^2\right)\\&+CK^3(\widehat{E}(0))^{\frac{2\alpha+1}{2}}\textstyle\int_0^t e^{-\frac{(2\alpha+1)\theta}{2}\int_0^s\xi(\tau)\,d\tau}\,ds,\end{aligned}$$

which proves the Lemma 3.1. □

Now, we finish the proof of (1.8) when

$$\int_0^{+\infty} e^{-\frac{(2\alpha+1)\theta}{2}\int_0^s \xi(\tau)\,d\tau}\,ds < +\infty. \tag{3.8}$$

Remark 3.4. Condition (3.8) as well as Assumption 1.2 are satisfied, for example, if g converges to zero at infinity faster than $\frac{1}{t^d}$, for any $d > 0$, like

$$g_1(t) = a_1 e^{-b_1(t+1)^{q_1}} \quad \text{and} \quad g_2(t) = a_2 e^{-b_2(\ln(t+e^{q_2-1}))^{q_2}},$$

where a_i, b_i, $q_1 > 0$ and $q_2 > 1$ such that a_i are small enough so that (1.6) holds. For these two particular examples, ξ is given, respectively, by

$$\xi(t) = b_1 q_1 (t+1)^{\min\{0,q_1-1\}} \quad \text{and} \quad \xi(t) = b_2 q_2 (t+e^{q_2-1})^{-1}(\ln(t+e^{q_2-1}))^{q_2-1}.$$

However, when g converges to zero at infinity slower than $\frac{1}{t^d}$, for some $d > 0$, like

$$g_3(t) = a_3(t+1)^{-q_3},$$

where $a_3 > 0$ and $q_3 > 1$, Assumption 1.2 is satisfied with

$$\xi(t) = q_3(t+1)^{-1}$$

provided that a_3 is small enough so that (1.6) holds. But (3.8) is not always satisfied, since (3.8) is equivalent to $\frac{1}{2}(2\alpha+1)\theta q_3 > 1$.

Assume that Assumption 1.1, Assumption 1.2 and (3.8) hold, let $K > 0$ and set

$$S_K := \{(u_0, u_1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega),\ G(t, I_0, I_1, K) < K^2,\ \forall t \geq 0\}, \tag{3.9}$$

and

$$S = \bigcup_{K>0} S_K. \tag{3.10}$$

Recalling Lemma 2.2, one can assert that u (the approximate solution constructed by Galerkin method) and u' exist globally in $\mathbb{R}_+$. Suppose that $(u_0, u_1) \in S_K$ for some $K > 0$. Thus, we would like to prove that

$$||\Delta u(t)||_2 < K \quad \text{and} \quad ||\nabla u'(t)||_2 < K, \quad \forall t \geq 0. \tag{3.11}$$

In order to prove (3.11), we argue by contradiction. So, assume that (3.11) does not hold. Then, there exists some $T > 0$ such that

$$||\Delta u(t)||_2 < K \quad \text{and} \quad ||\nabla u'(t)||_2 < K, \quad \forall t \in [0, T[\tag{3.12}$$

and

$$||\Delta u(T)||_2 = K \quad \text{or} \quad ||\nabla u'(T)||_2 = K. \tag{3.13}$$

Repeating the proof of Lemma 3.1, we see from (3.12) and (3.13) that (3.1) remains valid, for $0 \leq t < T$, so that, taking (3.9) into account, one has

$$||\nabla u'(T)||_2^2 + ||\Delta u(T)||_2^2 \leq G(T, I_0, I_1, K) \leq \lim_{t\to+\infty} G(t, I_0, I_1, K) < K^2, \tag{3.14}$$

which contradicts (3.13). Thus, we have shown (3.11). As a consequence, we can repeat the continuation procedure indefinitely and we can conclude that, if $(u_0, u_1) \in S$, the solution u can be continued globally on $\mathbb{R}_+$ and $(u(t), u'(t)) \in S$, for all $t \geq 0$.

Uniqueness. Let u and v be two solutions to problem (1.2). Then $w = u - v$ satisfies

$$\begin{cases} w'' - M(||\nabla u(t)||_2^2)\Delta w + \displaystyle\int_0^t g(t-s)\Delta w(s)\, ds \\ \quad = \left(M(||\nabla u(t)||_2^2) - M(||\nabla v(t)||_2^2)\right)\Delta v \quad \text{in } \Omega \times \mathbb{R}_+, \\ w = 0 \quad \text{on } \Gamma \times \mathbb{R}_+, \\ w(0) = w'(0) = 0 \quad \text{in } \Omega. \end{cases} \tag{3.15}$$

Taking the inner product in $L^2(\Omega)$ of the first equation of the above system with w', we deduce

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[||w'(t)||_2^2 + M\left(||\nabla u(t)||_2^2\right)||\nabla w(t)||_2^2 - \left(\int_0^t g(s)ds\right)||\nabla w(t)||_2^2 + (g\circ\nabla w)(t)\right] \\ &= \frac{1}{2}(g'\circ\nabla w)(t) - \frac{1}{2}g(t)||\nabla w(t)||_2^2 + M'\left(||\nabla u||_2^2\right)\left(\nabla u'(t), \nabla u(t)\right)_{L^2(\Omega)}||\nabla w(t)||_2^2 \\ &\quad + \left(M\left(||\nabla u(t)||_2^2\right) - M\left(||\nabla v(t)||_2^2\right)\right)\left(\Delta v(t), w'(t)\right)_{L^2(\Omega)}, \end{aligned}$$

which implies

$$\begin{aligned} &\tfrac{1}{2}\tfrac{d}{dt}\left[||w'(t)||_2^2 + M\left(||\nabla u(t)||_2^2\right)||\nabla w(t)||_2^2 - \left(\int_0^t g(s)ds\right)||\nabla w(t)||_2^2 + (g\circ\nabla w)(t)\right] \\ &\leq M'\left(||\nabla u||_2^2\right)\left(\nabla u'(t), \nabla u(t)\right)_{L^2(\Omega)}||\nabla w(t)||_2^2 \\ &\quad + \left(M\left(||\nabla u(t)||_2^2\right) - M\left(||\nabla v(t)||_2^2\right)\right)\left(\Delta v(t), w'(t)\right)_{L^2(\Omega)}. \end{aligned}$$

Making use of the main value theorem, we infer

$$\begin{aligned} \left|M\left(||\nabla u(t)||_2^2\right) - M\left(||\nabla v(t)||_2^2\right)\right| &\leq C\left|||\nabla u(t)||_2^2 - ||\nabla v(t)||_2^2\right| \\ &\leq C\left(||\nabla u(t)||_2 + ||\nabla v(t)||_2\right)||\nabla u(t)||_2 \\ &\quad -||\nabla v(t)||_2| \\ &\leq C||\nabla w(t)||_2. \end{aligned}$$

Thus,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[||w'(t)||_2^2 + M\left(||\nabla u(t)||_2^2\right)||\nabla w(t)||_2^2 - \left(\int_0^t g(s)ds\right)||\nabla w(t)||_2^2 + (g\circ\nabla w)(t)\right] \\ &\leq C\left(||\nabla w(t)||_2^2 + ||\nabla w(t)||_2||w'(t)||_2\right). \end{aligned}$$

Integrating the last inequality over $(0,t)$ and noting that $w(0)=w'(0)=0$ yields

$$\frac{1}{2}\left(||w'(t)||_2^2+(m_0-g_0)||\nabla w(t)||_2^2+(g\circ\nabla w)(t)\right)\leq C\int_0^t\left(||w'(s)||_2^2+||\nabla w(s)||_2^2\right)\,ds.$$

This implies that

$$||w'(t)||_2^2+||\nabla w(t)||_2^2\leq C\int_0^t(||w'(s)||_2^2+||\nabla w(s)||_2^2)\,ds,\quad \forall t\geq 0,$$

which, by Gronwall's inequality, implies $w=0$. This completes the proof of (1.8) in case (3.8).

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A. Guesmia (Corresponding author)
Institut Elie Cartan de Lorraine, UMR 7502, Université de Lorraine,
Bat. A, Ile du Saulcy, 57045 Metz Cedex 01, France.
E-mail address: aissa.guesmia@univ-lorraine.fr

and

S. A. Messaoudi
Department of Mathematics and Statistics, College of Sciences,
King Fahd University of Petroleum and Minerals,
P.O.Box. 5005, Dhahran 31261, Saudi Arabia.
E-mail address: messaoud@kfupm.edu.sa

and

C. M. Webler
Department of Mathematics, State University of Maringá,
87020-900, Maringá, PR, Brazil.
E-mail address: cmwebler@uem.br