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System of Nonlinear Volterra Integro-Differential Equations of Arbitrary Order

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ABSTRACT: In this paper, a new approximate method for solving the system of nonlinear Volterra integro-differential equations of arbitrary (integer and fractional) order is introduced. For this purpose, the generalized fractional order of the Chebyshev orthogonal functions (GFCFs) based on the classical Chebyshev polynomials of the first kind has been introduced that can be used to obtain the solution of the integro-differential equations (IDEs). Also, we construct the fractional derivative operational matrix of order α in the Caputo's definition for GFCFs. This method reduces a system of IDEs by collocation method into a system of algebraic equations. Some examples to illustrate the simplicity and the effectiveness of the propose method have been presented.

Key Words: Fractional order of the Chebyshev functions, Operational matrix, Volterra integro-differential equations, System of Nonlinear IDE, Collocation method.

Contents

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1. Introduction

Many problems of theoretical physics, epidemic models, Hydrodynamic models and many disciplines lead to nonlinear Volterra IDEs [1,2,3], thus application of numerical methods for solving these equations are attractive. There are some methods to solve system of nonlinear Volterra IDEs, such as, Chebyshev wavelets [4], block-pulse functions [5], Sumudu decomposition method [6], Taylor series method [7], radial basis function networks [8], Differential transforms [9], He's Homotopy perturbation method [10], and operational Tau method [11].

The main goal of this paper is to present a numerical method (GFCF collocation method) for approximating the solution of a system of nonlinear Volterra IDEs of arbitrary (integer and fractional) order as follows:

$$\sum_{j=1}^{M} F_{ij}\left(t, y_j, ..., y_j^{(\gamma_{ij})}\right) + \sum_{j=1}^{M} \int_0^t K_{ij}(t,x)\Phi_{ij}\left(t, y_j(x), ..., y_j^{(\lambda_{ij})}(x)\right) dx = f_i(t), \tag{1.1}$$

with the boundary conditions:

$$y_j^{(k)}(t_0) = y_{j0}^k, \; k = 0, 1, ..., s-1, \; s.t. \; \lceil \max_{1\le i\le M}\{\lambda_{ij}, \gamma_{ij}\}\rceil = s, \; s \in \mathbf{N}, \tag{1.2}$$

for $i, j = 1, 2, ..., M$, and

$$F_{ij}\left(t, y_j, ..., y_j^{(\gamma_{ij})}\right) = \sum_{l=0}^{n_1} p_l(t) \prod_{r=0}^{\gamma_{ij}} \left(y_j^{(l)}(t)\right)^{\gamma_{ijlr}}, \tag{1.3}$$

$$\Phi_{ij}\left(t, y_j, ..., y_j^{(\lambda_{ij})}\right) = \sum_{l=0}^{n_2} q_l(t) \prod_{r=0}^{\lambda_{ij}} \left(y_j^{(l)}(t)\right)^{\lambda_{ijlr}}, \tag{1.4}$$

where $t \in [0, \eta]$, $\eta > 0$, $\lceil * \rceil$ is the smallest integer greater than or equal to $*$, n_1, n_2, γ_{ijlr}, $\lambda_{ijlr} \in \mathbf{N}\bigcup\{0\}$, and $f_i(t)$, $p_l(t)$, $q_l(t) \in L^2([0,\eta))$, $K_{ij}(t,x) \in L^2([0,\eta)^2)$ are known functions, $M \in \mathbf{N}$, and $y_j(t)$, $j = 1, ..., M$ are the unknown functions.

In this study, by substituting the GFCFs with the unknown coefficient that satisfies in the boundary conditions, we convert the nonlinear Volterra IDEs system of arbitrary order into nonlinear algebraic system. Because of the fractional derivatives can be in the IDE, we use the fractional derivative operational matrix of GFCFs to convert nonlinear Volterra IDEs system into nonlinear algebraic system. By solving such system, the approximate solution of exact solution is obtained.

The organization of the paper is expressed as follows: in section 2, some basic definitions and theorems are expressed. In section 3, the GFCFs and their properties are obtained. In section 4, the operational matrix of fractional derivative to GFCFs is obtained. In Section 5, application of the method is explained. Illustrative examples of the proposed method are shown in section 6. Finally, a conclusion is provided.

2. Basical definitions

In this section, some basic definitions and theorems which are useful for our method have been introduced [12,13].

Definition 1. For any real function $f(t)$, $t > 0$, if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C(0, \infty)$, is said to be in space C_μ, $\mu \in \Re$, and it is in the space C_μ^n if and only if $f^n \in C_\mu$, $n \in N$.
Definition 2. The fractional derivative of $f(t)$ in the Caputo sense by the Riemann-Liouville fractional integral operator of order $\alpha > 0$ is defined as [14]

$$D^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} D^m f(s) ds, \ \alpha > 0,$$

for $m - 1 < \alpha \leq m$, $m \in N$, $t > 0$ and $f \in C_{-1}^m$.
Some properties of the operator D^α are as follows. For $f \in C_\mu, \mu \geq -1$, $\alpha, \beta \geq 0$, $\gamma \geq -1$, $N_0 = \{0, 1, 2, ...\}$ and constant C:

$$(i) \ D^\alpha C = 0,$$

$$(ii) \ D^\alpha D^\beta f(t) = D^{\alpha+\beta} f(t),$$

$$(iii) \ D^\alpha t^\gamma = \begin{cases} 0 \ \gamma \in N_0 \ \mathit{and} \ \gamma < \alpha, \\ \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} t^{\gamma-\alpha}, \ \mathit{Otherwise}. \end{cases} \tag{2.1}$$

$$(iv) \ D^\alpha (\sum_{i=1}^n c_i f_i(t)) = \sum_{i=1}^n c_i D^\alpha f_i(t), \ \mathit{where} \ c_i \in R. \tag{2.2}$$

Definition 3. Suppose that $f(t)$, $g(t) \in C(0, \eta]$ and $w(t)$ is a weight function, then we define

$$\| f(t) \|_w^2 = \int_0^\eta f^2(t) w(t) dt,$$

$$\langle f(t), g(t) \rangle_w = \int_0^\eta f(t) g(t) w(t) dt,$$

Theorem 1. (*Generalized Taylor's formula*) Suppose that $f(t) \in C[0, \eta]$ and $D^{k\alpha} f(t) \in C[0, \eta]$, where $k = 0, 1, ..., m$, $0 < \alpha \leq 1$, and $\eta > 0$. Then we have

$$f(t) = \sum_{i=0}^{m-1} \frac{t^{i\alpha}}{\Gamma(i\alpha+1)} D^{i\alpha} f(0^+) + \frac{t^{m\alpha}}{\Gamma(m\alpha+1)} D^{m\alpha} f(\xi), \tag{2.3}$$

with $0 < \xi \leqslant t$, $\forall t \in [0, \eta]$. And thus for $M_\alpha \geqslant |D^{m\alpha} f(\xi)|$:

$$|f(t) - \sum_{i=0}^{m-1} \frac{t^{i\alpha}}{\Gamma(i\alpha+1)} D^{i\alpha} f(0^+)| \leqslant M_\alpha \frac{t^{m\alpha}}{\Gamma(m\alpha+1)}. \tag{2.4}$$

Proof: See Ref. [15].
In case of $\alpha = 1$, the generalized Taylor's formula (2.3) reduces to the classical Taylor's formula.

Theorem 2. Suppose that $\{P_i(t)\}$ be a sequence of orthogonal polynomials, $w(t)$ is weight function for $\{P_i(t)\}$, and $q(t)$ is a polynomial of degree at most $n - 1$, then for $p_n(t) \in \{P_i(t)\}$ we have: $\langle p_n(t), q(t) \rangle_w = 0$.
Proof: See the section 2.3 in Ref. [16].

3. Generalized Fractional order of the Chebyshev Functions

In this section, the generalized fractional order of the Chebyshev functions of the first kind have been defined, and then some properties and convergence of them for our method have been introduced.

3.1. The Chebyshev functions

The Chebyshev polynomials have been used in numerical analysis, frequently, including polynomial approximation, Gauss-quadrature integration, integral and differential equations and spectral methods. Some of Chebyshev polynomials properties are: orthogonality, recursive, real zeros, complete for the space of polynomials, etc. For these reasons, many researchers have employed these polynomials in their research [17,18,19,20,21,22,23].

The number of researchers by using some transformations extended Chebyshev polynomials to semi-infinite or infinite domain for example by using $x = \frac{t-L}{t+L}, L > 0$ the rational Chebyshev functions are introduced [24,25,26,27,28,29,30].

In this study, by transformation $x = 1 - 2(\frac{t}{\eta})^{\alpha};\ \alpha, \eta > 0$ on the Chebyshev polynomials of the first kind, the generalized fractional order of the Chebyshev orthogonal functions (GFCF) in interval $[0, \eta]$ have been introduced, that we can use them to solve nonlinear Volterra IDEs.

3.2. The GFCFs definition

The efficient methods have been used by many researchers to solve the differential equations (DE) is based on series expansion of the form $\sum_{i=0}^{n} c_i t^i$, such as Adomian's decomposition method [31] and Homotopy perturbation method [32]. But the exact solution of many DEs can't be estimated by polynomials basis, for a simple example: the ODE of $4yy'' = 3t,\ y(0) = y'(0) = 0$, that the exact solution is $y(t) = t^{\frac{3}{2}}$, therefore we have defined a new basis for Spectral methods to solve them as follows:

$$\Phi_n(t) = \sum_{i=0}^{n} c_i t^{i\alpha}.$$

Now by transformation $z = 1 - 2(\frac{t}{\eta})^{\alpha};\ \alpha,\ \eta > 0$ on classical Chebyshev polynomials of the first kind, we defined the GFCFs in interval $[0, \eta]$, that be denoted by ${}_{\eta}FT_n^{\alpha}(t) = T_n(1 - 2(\frac{t}{\eta})^{\alpha})$.

The ${}_{\eta}FT_n^{\alpha}(t)$ can be obtained using recursive relation as follows $(n = 1, 2, \cdots)$:

$$\begin{aligned} {}_{\eta}FT_0^{\alpha}(t) &= 1\ ,\ {}_{\eta}FT_1^{\alpha}(t) = 1 - 2(\frac{t}{\eta})^{\alpha}, \\ {}_{\eta}FT_{n+1}^{\alpha}(t) &= (2 - 4(\frac{t}{\eta})^{\alpha})\ {}_{\eta}FT_n^{\alpha}(t) -\ {}_{\eta}FT_{n-1}^{\alpha}(t), \end{aligned}$$

The analytical form of ${}_{\eta}FT_n^{\alpha}(t)$ of degree $n\alpha$ is given by

$$ {}_{\eta}FT_n^{\alpha}(t) = \sum_{k=0}^{n} \beta_{n,k,\eta,\alpha}.t^{\alpha k},\ \ t \in [0, \eta], \tag{3.1}$$

where

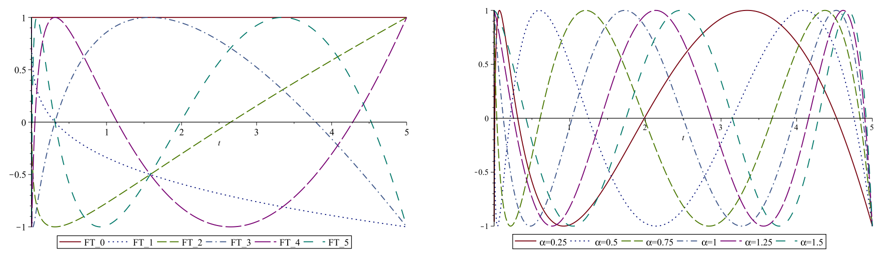
$$\beta_{n,k,\eta,\alpha} = (-1)^k \frac{n2^{2k}(n+k-1)!}{(n-k)!(2k)!\eta^{\alpha k}} \; and \; \beta_{0,k,\eta,\alpha} = 1.$$

Note that ${}_\eta FT_n^\alpha(0) = 1$ and ${}_\eta FT_n^\alpha(\eta) = (-1)^n$.

The GFCFs are orthogonal with respect to the weight function $w(t) = \frac{t^{\frac{\alpha}{2}-1}}{\sqrt{\eta^\alpha - t^\alpha}}$ in the interval $[0, \eta]$:

$$\int_0^\eta {}_\eta FT_n^\alpha(t) \; {}_\eta FT_m^\alpha(t) w(t) dt = \frac{\pi}{2\alpha} c_n \delta_{mn}. \tag{3.2}$$

where δ_{mn} is Kronecker delta, $c_0 = 2$, and $c_n = 1$ for $n \geq 1$. The Eq. (3.2) is provable using the properties of orthogonality in the Chebyshev polynomials. Figs. 1 show the graphs of GFCFs for various values of n and α and $\eta = 5$.



(a) Graph of the GFCFs with $\alpha = 0.25$ and various values of n

(b) Graph of the GFCFs with $n = 5$ and various values of α

Figure 1: Graphs of the GFCFs for various values of n and α.

3.3. Approximation of functions

Any function $y(t)$, $t \in [0, \eta]$, can be expanded as the follows [16]:

$$y(t) = \sum_{n=0}^{\infty} a_n \; {}_\eta FT_n^\alpha(t),$$

where the coefficients a_n are obtained by inner product:

$$\langle y(t), \; {}_\eta FT_n^\alpha(t) \rangle_w = \langle \sum_{n=0}^{\infty} a_n \; {}_\eta FT_n^\alpha(t), \; {}_\eta FT_n^\alpha(t) \rangle_w,$$

and using the property of orthogonality in the GFCFs:

$$a_n = \frac{2\alpha}{\pi c_n} \int_0^\eta {}_\eta FT_n^\alpha(t) y(t) w(t) dt, \; n = 0, 1, 2, \cdots.$$

In practice, we have to use first m-terms GFCFs and approximate $y(t)$:

$$y(t) \simeq y_m(t) = \sum_{n=0}^{m-1} a_n \; {}_\eta FT_n^\alpha(t) = A^T \Phi(t), \tag{3.3}$$

with

$$A = [a_0, a_1, ..., a_{m-1}]^T, \quad (3.4)$$
$$\Phi(t) = [{}_\eta FT_0^\alpha(t),\ {}_\eta FT_1^\alpha(t), ...,\ {}_\eta FT_{m-1}^\alpha(t)]^T. \quad (3.5)$$

3.4. Convergence of method

The following theorem shows that by increasing m, the approximation solution $f_m(t)$ is convergent to $f(t)$ exponentially.

Theorem 3. Suppose that $D^{k\alpha}f(t) \in C[0,\eta]$ for $k = 0, 1, ..., m$, and ${}_\eta F_m^\alpha$ is the subspace generated by $\{{}_\eta FT_0^\alpha(t), {}_\eta FT_1^\alpha(t), ..., {}_\eta FT_{m-1}^\alpha(t)\}$. If $f_m = A^T\Phi$ (in Eq. (3.3)) is the best approximation to $f(t)$ from ${}_\eta F_m^\alpha$, then the error bound is presented as follows

$$\| f(t) - f_m(t) \|_w \leq \frac{\eta^{m\alpha} M_\alpha}{2^m \Gamma(m\alpha+1)} \sqrt{\frac{\pi}{\alpha.m!}},$$

where $M_\alpha \geq |D^{m\alpha}f(t)|,\ t \in [0,\eta]$.

Proof. By theorem 1, $y(t) = \sum_{i=0}^{m-1} \frac{t^{i\alpha}}{\Gamma(i\alpha+1)} D^{i\alpha} f(0^+)$ and

$$|f(t) - y(t)| \leq M_\alpha \frac{t^{m\alpha}}{\Gamma(m\alpha+1)}.$$

Since $A^T\Phi(t)$ is the best approximation to $f(t)$ from ${}_\eta F_m^\alpha$, and $y \in {}_\eta F_m^\alpha$, one has

$$\begin{aligned} \| f(t) - f_m(t) \|_w^2 &\leq \| f(t) - y(t) \|_w^2 \\ &\leq \frac{M_\alpha^2}{\Gamma(m\alpha+1)^2} \int_0^\eta \frac{t^{\frac{\alpha}{2}+2m\alpha-1}}{\sqrt{\eta^\alpha - t^\alpha}} dt \\ &= \frac{M_\alpha^2}{\Gamma(m\alpha+1)^2} \frac{\eta^{2m\alpha}\pi}{\alpha 2^{2m} m!}. \end{aligned}$$

Now by taking the square roots, the theorem can be proved. $*$

Theorem 4. The generalized fractional order of the Chebyshev function ${}_\eta FT_n^\alpha(t)$, has precisely n real zeros on interval $(0,\eta)$ in the form

$$t_k = \eta \left(\frac{1 - \cos(\frac{(2k-1)\pi}{2n})}{2} \right)^{\frac{1}{\alpha}},\ k = 1, 2, ..., n.$$

Moreover, $\frac{d}{dt} {}_\eta FT_n^\alpha(t)$ has precisely $n-1$ real zeros on interval $(0,\eta)$ in the following points:

$$t'_k = \eta \left(\frac{1 - \cos(\frac{k\pi}{n})}{2} \right)^{\frac{1}{\alpha}},\ k = 1, 2, ..., n-1.$$

Proof. The Chebyshev polynomial $T_n(x)$ has n real zeros [33,34]:

$$x_k = \cos(\frac{(2k-1)\pi}{2n}),\ k = 1, 2, ..., n.$$

Therefore $T_n(x)$ can be written as

$$T_n(x) = (x - x_1)(x - x_2)...(x - x_n).$$

Using transformation $x = 1 - 2(\frac{t}{\eta})^\alpha$ yields to

$${}_\eta FT_n^\alpha(t) = ((1 - 2(\frac{t}{\eta})^\alpha) - x_1)((1 - 2(\frac{t}{\eta})^\alpha) - x_2)...((1 - 2(\frac{t}{\eta})^\alpha) - x_n),$$

so, the real zeros of ${}_\eta FT_n^\alpha(t)$ are $t_k = \eta(\frac{1-x_k}{2})^{\frac{1}{\alpha}}$.
Also, we know that, the real zeros of $\frac{d}{dt}T_n(t)$ occurs in the following points [34]:

$$x'_k = \cos(\frac{k\pi}{n}),\ k = 1, 2, ..., n - 1.$$

Same as in the previous, the absolute extremes of ${}_\eta FT_n^\alpha(t)$ are $t'_k = \eta(\frac{1-x'_k}{2})^{\frac{1}{\alpha}}$. $*$

4. The fractional derivative operational matrix of GFCFs

In the next theorem, the operational matrix of the Caputo fractional derivative of order $\alpha > 0$ for GFCFs is generalized, which can be expressed by:

$$D^\alpha \Phi(t) = D^{(\alpha)}\Phi(t). \tag{4.1}$$

Theorem 5. Let $\Phi(t)$ be GFCFs vector in Eq. (3.5), and $D^{(\alpha)}$ is an $m \times m$ operational matrix of the Caputo fractional derivatives of order $\alpha > 0$, then:

$$D_{i,j}^{(\alpha)} = \begin{cases} \frac{2}{\sqrt{\pi}c_j}\sum_{k=1}^{i}\sum_{s=0}^{j}\beta_{i,k,\eta,\alpha}\beta_{j,s,\eta,\alpha}\frac{\Gamma(\alpha k+1)\Gamma(s+k-\frac{1}{2})\eta^{\alpha(k+s-1)}}{\Gamma(\alpha k-\alpha+1)\Gamma(s+k)}, & i > j \\ 0 \ otherwise \end{cases}$$

for $i, j = 0, 1, ..., m - 1$.
Proof. Using Eq. (4.1)

$$\begin{bmatrix} D_{0,0} & \cdots & D_{0,j} & \cdots & D_{0,m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ D_{i,0} & \cdots & D_{i,j} & \cdots & D_{i,m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ D_{m-1,0} & \cdots & D_{m-1,j} & \cdots & D_{m-1,m-1} \end{bmatrix} \begin{bmatrix} \Phi_0 \\ \vdots \\ \Phi_j \\ \vdots \\ \Phi_{m-1} \end{bmatrix} = \begin{bmatrix} D^\alpha\Phi_0 \\ \vdots \\ D^\alpha\Phi_i \\ \vdots \\ D^\alpha\Phi_{m-1} \end{bmatrix},$$

and orthogonally property of GFCFs, the Eqs.(2.1) and (3.1), for $i, j = 0, 1, ..., m-1$:

$$D_{i,j}^{(\alpha)} = \frac{2\alpha}{\pi c_j}\int_0^\eta D^\alpha({}_\eta FT_i^\alpha(t))({}_\eta FT_j^\alpha(t))w(t)dt.$$

Since $D^{\alpha}FT_0^{\alpha}(t)=0$, therefore $D_{0,j}^{(\alpha)}=\int_0^{\eta}D^{\alpha}FT_0^{\alpha}(t)FT_j^{\alpha}(t)w(t)dt=0$.
And if $i\leq j$ then $deg(D^{\alpha}({}_{\eta}FT_i^{\alpha}(t)))<deg({}_{\eta}FT_j^{\alpha}(t))$, therefore by theorem 2, $D_{i,j}^{(\alpha)}=0$ for any $i\leq j$. Now for $i>j$ we have:

$$\begin{aligned} D_{i,j}^{(\alpha)} &= \frac{2\alpha}{\pi c_j}\int_0^{\eta}\sum_{k=1}^{i}\beta_{i,k,\eta,\alpha}\frac{\Gamma(\alpha k+1)t^{\alpha k-\alpha}}{\Gamma(\alpha k-\alpha+1)}\sum_{s=0}^{j}\beta_{j,s,\eta,\alpha}t^{\alpha s}\frac{t^{\frac{\alpha}{2}-1}}{\sqrt{\eta^{\alpha}-t^{\alpha}}}dt \\ &= \frac{2\alpha}{\pi c_j}\sum_{k=1}^{i}\sum_{s=0}^{j}\beta_{i,k,\eta,\alpha}\beta_{j,s,\eta,\alpha}\frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k-\alpha+1)}\int_0^{\eta}\frac{t^{\alpha(k+s-\frac{1}{2})-1}}{\sqrt{\eta^{\alpha}-t^{\alpha}}}dt. \end{aligned}$$

Now, by integration of above equation, the theorem can be proved.$*$

Remark: The fractional derivative operational matrix of GFCFs is an lower-triangular matrix and for $\alpha=1$ is same as shifted Chebyshev polynomials [35].

5. Application of the GFCF collocation method

In this section, we apply the GFCFs collocation method to solve the system of nonlinear Volterra IDEs of arbitrary order.

For satisfying the boundary conditions the Eq. (1.2), we satisfy the boundary conditions as follows, for $j=1,2,...,M$:

$$\widehat{y_{jm}}(t)=\sum_{k=0}^{s-1}y_{j0}^{k}\ t^{k}+t^{s}\ y_{jm}(t), \tag{5.1}$$

where $y_{jm}(t)$ is defined in Eq. (3.3).

To apply the collocation method, we construct the residual functions by substituting $\widehat{y_{jm}}(t)$ in Eq. (5.1) for $y_j(t)$ in the system of nonlinear Volterra IDEs of arbitrary order the Eq. (1.1), for $i=1,2,...,M$:

$$\begin{aligned} Res_i(t) &= \sum_{j=1}^{M}F_{ij}\left(t,\widehat{y_{jm}},...,\widehat{y_{jm}}^{(\gamma_{ij})}\right) \\ &+ \sum_{j=1}^{M}\int_0^{t}K_{ij}(t,x)\Phi_{ij}\left(t,\widehat{y_{jm}},...,\widehat{y_{jm}}^{(\lambda_{ij})}\right)dx-f_i(t) \end{aligned} \tag{5.2}$$

The equations for obtaining the coefficient $\{a_{jn}\}_{j=1,n=0}^{M,m-1}$ arise from equalizing $Res_i(t)$ to zero on m collocation points:

$$Res_i(t_k)=0,\ \ k=0,1,...,m-1. \tag{5.3}$$

In this study, the roots of the GFCFs in the interval $[0,\eta]$ (Theorem 4) are used as collocation points. By solving the obtained set of $2m\times 2m$ equations, we have the approximating functions $\widehat{y_{jm}}(t),\ j=1,2,...,M$.

Note that if we have a fractional derivative in system, we can not use the Eq. (5.1) for the unknown function, instead, we use Eqs. (3.3) and (4.1), and then the boundary conditions will applied.

And also consider that all of the computations have been done by Maple 2015 on a laptop with CPU Core i7, Windows 8.1 64bit, and 8GB of RAM.

6. Illustrative examples

In this section, by using the present method we solve some well-known examples to show efficiently and applicability GFCFs method based on Spectral method. These examples have been studied by other researchers. We apply the present method to solve the system of nonlinear Volterra IDEs of arbitrary order, and their outputs are compared with the corresponding analytical solution.

6.1. Nonlinear system of Volterra IDEs of integer order

We applied the present method to solve some system of Volterra IDEs of integer order.

Example 1. Consider nonlinear system of Volterra IDEs with boundary conditions $u(0) = 0,\ u'(0) = 1,\ u''(0) = 0,\ v(0) = 1,\ v'(0) = 0$ and $v''(0) = -1$, as follows [4]

$$\begin{cases} u'''(t) + u'(t) + \int_0^t (u''^2(x) + v''^2(x))dx = t \\ v'''(t) - \int_0^t (u''(x)v(x))dx = \sin(t) + \frac{1}{2}\sin^2(t) \end{cases} \tag{6.1}$$

The exact solutions of this problem are $u(t) = \sin(t)$ and $v(t) = \cos(t)$.
By applying the technique described in last section, we satisfy the boundary conditions as follows:

$$\begin{aligned} \widehat{y_{1m}}(t) &= t + t^3\ y_{1m}(t), \\ \widehat{y_{2m}}(t) &= 1 - \frac{t^2}{2} + t^3\ y_{2m}(t). \end{aligned}$$

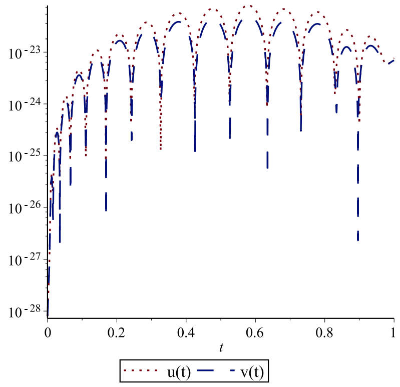
We construct the residual functions as follows:

$$Res_1(t) = \widehat{y_{1m}}'''(t) + \widehat{y_{1m}}'(t) + \int_0^t (\widehat{y_{1m}}''^2(x) + \widehat{y_{2m}}''^2(x))dx - t$$

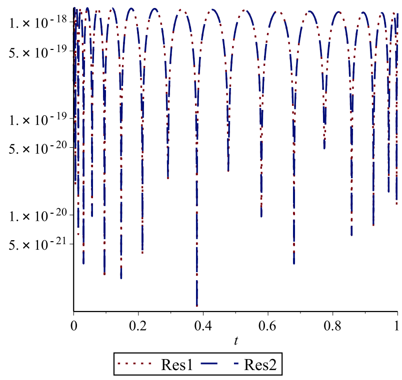
$$Res_2(t) = \widehat{y_{2m}}'''(t) - \int_0^t (\widehat{y_{1m}}''(x)\widehat{y_{2m}}(x))dx - \sin(t) - \frac{1}{2}\sin^2(t)$$

Therefore, to obtain the coeffcient $\{a_{jn}\}_{j=1,n=0}^{2,m-1}$; $Res_i(t),\ i = 1, 2$ is equalized to zero at m collocation point. By solving this set of nonlinear algebraic equations, we can find the approximating function $\widehat{y_{jm}}(t)$.

Figure 2 shows the logarithmic graph of the absolute and the residual errors of the approximate solutions and the analytic solutions for $m = 20$, and $\alpha = 0.50$.



(a) Graphs of the absolute errors



(b) Graphs of the residual errors

Figure 2: Log graphs of the absolute and the residual errors for example 1.

Example 2. Consider the following linear system of Fredholm IDEs with boundary conditions $u(0)=1,\ u'(0)=0,\ v(0)=-1$ and $v'(0)=2$, as follows [4]

$$\begin{cases} u''(t)+v'(t)+\int_0^1 2tx(u(x)-3v(x))dx = 3t^2+\frac{3}{10}t+8 \\ v''(t)+u'(t)+\int_0^1 (6t+3x^2)(u(x)-2v(x))dx = 21t+\frac{4}{5} \end{cases} \tag{6.2}$$

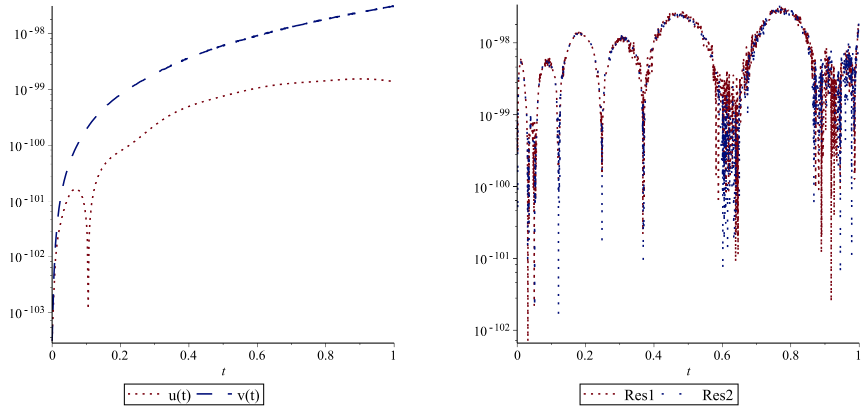
The exact solutions of this problem are $u(t)=3t^2+1$ and $v(t)=t^3+2t-1$.
We satisfy the boundary conditions:

$$\begin{aligned} \widehat{y_{1m}}(t) &= 1+t^2\ y_{1m}(t), \\ \widehat{y_{2m}}(t) &= -1+2t+t^2\ y_{2m}(t), \end{aligned}$$

and construct the residual functions as follows:

$$Res_1(t) = \widehat{y_{1m}}''(t)+\widehat{y_{2m}}'(t)+\int_0^1 2tx(\widehat{y_{1m}}(x)-3\widehat{y_{2m}}(x))dx-3t^2-\frac{3}{10}t-8$$
$$Res_2(t) = \widehat{y_{2m}}''(t)+\widehat{y_{1m}}'(t)+\int_0^1 (6t+3x^2)(\widehat{y_{1m}}(x)-2\widehat{y_{2m}}(x))dx-21t-\frac{4}{5}$$

Figure 3 shows the logarithmic graph of the absolute and the residual errors of the approximate solutions and the analytic solutions for $m=15$, and $\alpha=0.50$.



(a) Graphs of the absolute errors (b) Graphs of the residual errors

Figure 3: Log graphs of the absolute and the residual errors for example 2.

Example 3. Consider the nonlinear Volterra IDEs problem with boundary conditions $u(0) = 0$ and $v(0) = 1$, as follows [5]

$$\begin{cases} u'(t) + \frac{1}{2}v'^2(t) - \int_0^t((t-x)v(x) + v(x)u(x))dx = 1 \\ v'(t) - \int_0^t((t-x)u(x) - v^2(x) + u^2(x))dx = 2t \end{cases} \tag{6.3}$$

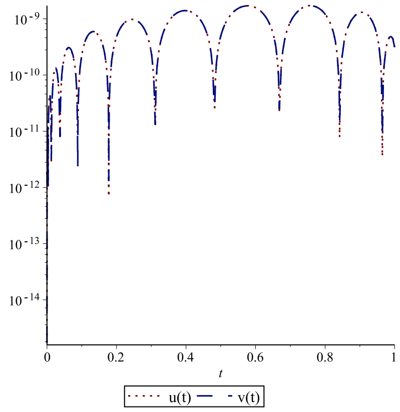
The exact solutions of this problem are $u(t) = \sinh(t)$ and $v(t) = \cosh(t)$.
We satisfy the boundary conditions:

$$\begin{aligned} \widehat{y_{1m}}(t) &= t\ y_{1m}(t), \\ \widehat{y_{2m}}(t) &= 1 + t\ y_{2m}(t), \end{aligned}$$

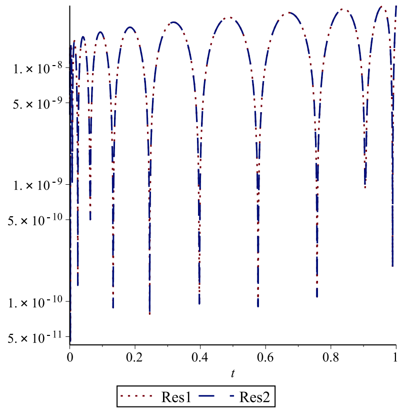
and construct the residual functions as follows:

$$Res_1(t) = \widehat{y_{1m}}'(t) + \frac{1}{2}\widehat{y_{2m}}'^2(t) - \int_0^t ((t-x)\widehat{y_{2m}}(x) + \widehat{y_{2m}}(x)\widehat{y_{1m}}(x))dx - 1$$
$$Res_2(t) = \widehat{y_{2m}}'(t) - \int_0^t ((t-x)\widehat{y_{1m}}(x) - \widehat{y_{2m}}^2(x) + \widehat{y_{1m}}^2(x))dx - 2t$$

Figure 4 shows the logarithmic graph of the absolute and the residual errors of the approximate solutions and the analytic solutions for $m = 15$, and $\alpha = 0.25$.



(a) Graphs of the absolute errors



(b) Graphs of the residual errors

Figure 4: Log graphs of the absolute and the residual errors for example 3.

Example 4. Consider the nonlinear Volterra IDEs problem with boundary conditions $u(0) = 1,\ u'(0) = 2,\ v(0) = -1$ and $v'(0) = 0$, as follows [5]

$$\begin{cases} u''(t) + \frac{1}{2}v'^2(t) - \frac{1}{2}\int_0^t (u^2(x) + v^2(x))dx = 1 - \frac{t^3}{3} \\ v''(t) + tu(t) - \frac{1}{4}\int_0^t (u^2(x) - v^2(x))dx = -1 + t^2 \end{cases} \tag{6.4}$$

The exact solutions of this problem are $u(t) = t + e^t$ and $v(t) = t - e^t$.
We satisfy the boundary conditions:

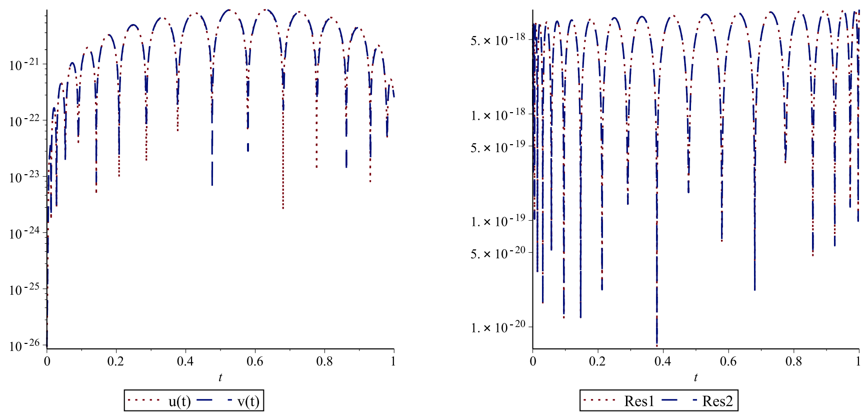
$$\begin{aligned} \widehat{y_{1m}}(t) &= 1 + 2t + t^2\ y_{1m}(t), \\ \widehat{y_{2m}}(t) &= -1 + t^2\ y_{2m}(t), \end{aligned}$$

and construct the residual functions as follows:

$$Res_1(t) = \widehat{y_{1m}}''(t) + \frac{1}{2}\widehat{y_{2m}}'^2(t) - \frac{1}{2}\int_0^t (\widehat{y_{1m}}^2(x) + \widehat{y_{2m}}^2(x))dx - 1 + \frac{t^3}{3}$$

$$Res_2(t) = \widehat{y_{2m}}''(t) + t\widehat{y_{1m}}(t) - \frac{1}{4}\int_0^t (\widehat{y_{1m}}^2(x) - \widehat{y_{2m}}^2(x))dx + 1 - t^2$$

Figure 5 shows the logarithmic graph of the absolute and the residual errors of the approximate solutions and the analytic solutions for $m = 20$, and $\alpha = 0.50$.



(a) Graphs of the absolute errors (b) Graphs of the residual errors

Figure 5: Log graphs of the absolute and the residual errors for example 4.

Example 5. Consider the nonlinear Volterra IDEs problem with boundary conditions $u(0) = 1,\ u'(0) = 1,\ v(0) = 1$ and $v'(0) = 2$, as follows [6]

$$\begin{cases} u''(t) - \int_0^t e^{t-x}(u^2(x) + v^2(x))dx = \frac{7}{3}e^t - e^{2t} - \frac{1}{3}e^{4t} \\ v''(t) - \int_0^t e^{t-x}(u^2(x) - v^2(x))dx = \frac{2}{3}e^t + 3e^{2t} + \frac{1}{3}e^{4t} \end{cases} \tag{6.5}$$

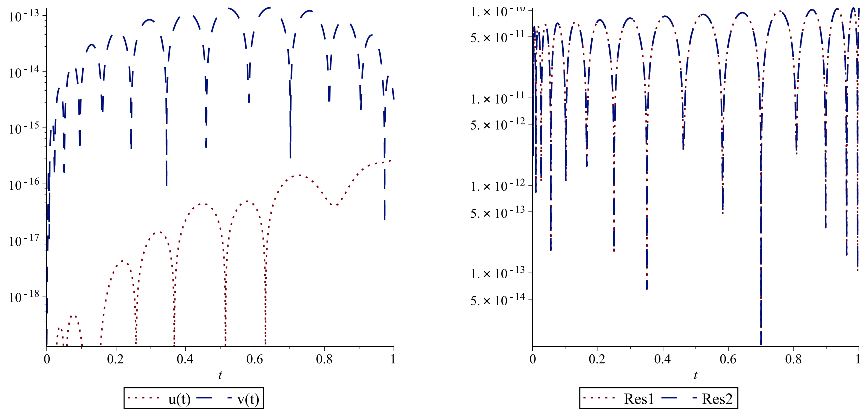
The exact solutions of this problem are $u(t) = e^t$ and $v(t) = e^{2t}$.
We satisfy the boundary conditions:

$$\begin{aligned} \widehat{y_{1m}}(t) &= 1 + t + t^2\ y_{1m}(t), \\ \widehat{y_{2m}}(t) &= 1 + 2t + t^2\ y_{2m}(t), \end{aligned}$$

and construct the residual functions as follows:

$$Res_1(t) = \widehat{y_{1m}}''(t) - \int_0^t e^{t-x}(\widehat{y_{1m}}^2(x) + \widehat{y_{2m}}^2(x))dx - (\frac{7}{3}e^t - e^{2t} - \frac{1}{3}e^{4t})$$
$$Res_2(t) = \widehat{y_{2m}}''(t) - \int_0^t e^{t-x}(\widehat{y_{1m}}^2(x) - \widehat{y_{2m}}^2(x))dx - (\frac{2}{3}e^t + 3e^{2t} + \frac{1}{3}e^{4t})$$

Figure 6 shows the logarithmic graph of the absolute and the residual errors of the approximate solutions and the analytic solutions for $m = 17$, and $\alpha = 0.50$.



(a) Graphs of the absolute errors (b) Graphs of the residual errors

Figure 6: Log graphs of the absolute and the residual errors for example 5.

Example 6. Consider the nonlinear Volterra IDEs problem with boundary conditions $u(0) = u''(0) = 0,\ u'(0) = 1,\ v(0) = 1,\ v'(0) = 0$ and $v''(0) = -1$, as follows [7]

$$\begin{cases} u'''(t) + u'(t) + \int_0^t e^{t+x}(u''^2(x) + v''^2(x))dx = e^t(e^t - 1) \\ v'''(t) - \int_0^t u''(x)v(x)dx = \sin(t)\left(1 + \frac{1}{2}\sin(t)\right) \end{cases} \tag{6.6}$$

The exact solutions of this problem are $u(t) = \sin(t)$ and $v(t) = \cos(t)$.
We satisfy the boundary conditions:

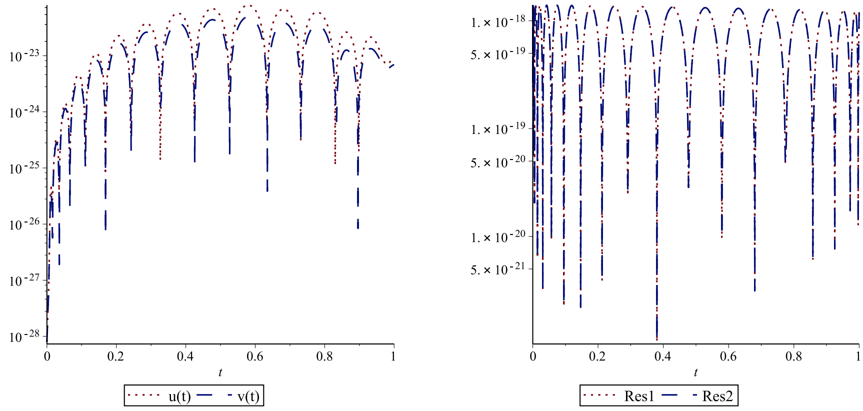
$$\begin{aligned} \widehat{y_{1m}}(t) &= t + t^3\, y_{1m}(t), \\ \widehat{y_{2m}}(t) &= 1 - \frac{t^2}{2} + t^3\, y_{2m}(t), \end{aligned}$$

and construct the residual functions as follows:

$$Res_1(t) = \widehat{y_{1m}}'''(t) + \widehat{y_{1m}}'(t) + \int_0^t e^{t+x}(\widehat{y_{1m}}''^2(x) + \widehat{y_{2m}}''^2(x))dx - e^t(e^t - 1)$$

$$Res_2(t) = \widehat{y_{2m}}'''(t) - \int_0^t \widehat{y_{1m}}''(x)\widehat{y_{2m}}(x)dx - \sin(t)\left(1 + \frac{1}{2}\sin(t)\right)$$

Figure 7 shows the logarithmic graph of the absolute and the residual errors of the approximate solutions and the analytic solutions for $m = 20$, and $\alpha = 0.50$.



(a) Graphs of the absolute errors

(b) Graphs of the residual errors

Figure 7: Log graphs of the absolute and the residual errors for example 6.

Example 7. Consider the nonlinear Volterra IDEs problem with boundary conditions $u(0) = 0,\ u'(0) = 1,\ v(0) = 0$ and $v'(0) = 0$, as follows [7]

$$\begin{cases} u''(t) + \frac{t}{2}v'(t) + v^2(t) - \int_0^t \left((t-x)v(x) + u(x)v(x)\right) dx = f_1(t) \\ u''(t) + u^2(t) - \int_0^t \left((t-x)u(x) - v^2(x) + u^2(x)\right) dx = f_2(t) \end{cases} \tag{6.7}$$

where

$$\begin{cases} f_1(t) = \frac{7}{6}t^6 - \frac{49}{20}t^5 + \frac{4}{3}t^4 + \frac{3}{2}t^3 - t^2 - 2 \\ f_2(t) = \frac{1}{7}t^7 - \frac{1}{3}t^6 + \frac{19}{12}t^4 - \frac{5}{2}t^3 + t^2 + 6t - 2 \end{cases}$$

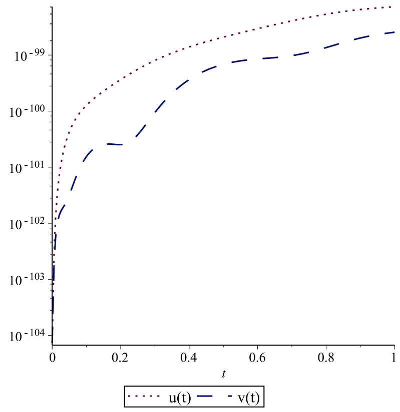
The exact solutions of this problem are $u(t) = t - t^2$ and $v(t) = t^3 - t^2$.
We satisfy the boundary conditions:

$$\begin{aligned} \widehat{y_{1m}}(t) &= t + t^2\ y_{1m}(t), \\ \widehat{y_{2m}}(t) &= t^2\ y_{2m}(t), \end{aligned}$$

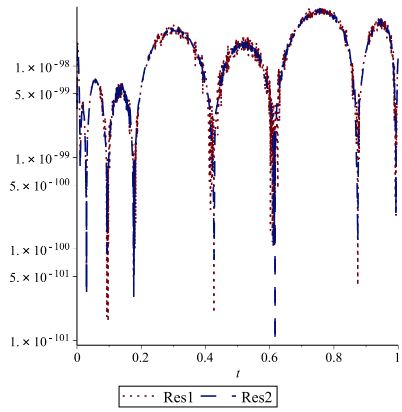
and construct the residual functions as follows:

$$\begin{aligned} Res_1(t) &= \widehat{y_{1m}}''(t) + \frac{t}{2}\widehat{y_{2m}}'(t) + \widehat{y_{2m}}^2(t) \\ &\quad - \int_0^t \left((t-x)\widehat{y_{2m}}(x) + \widehat{y_{1m}}(x)\widehat{y_{2m}}(x)\right) dx - f_1(t) \\ Res_2(t) &= \widehat{y_{1m}}''(t) + \widehat{y_{1m}}^2(t) \\ &\quad - \int_0^t \left((t-x)\widehat{y_{1m}}(x) - \widehat{y_{2m}}^2(x) + \widehat{y_{1m}}^2(x)\right) dx - f_2(t) \end{aligned}$$

Figure 8 shows the logarithmic graph of the absolute and the residual errors of the approximate solutions and the analytic solutions for $m = 10$, and $\alpha = 0.50$.



(a) Graphs of the absolute errors



(b) Graphs of the residual errors

Figure 8: Log graphs of the absolute and the residual errors for example 7.

6.2. Nonlinear system of Volterra IDEs of fractional order

We applied the present method to solve some system of Volterra IDEs of fractional order.

Example 8. Consider the nonlinear Volterra IDEs problem with initial conditions $u(0) = 1,\ v(0) = 1$ and $v'(0) = 1$, as follows

$$\begin{cases} D^{\frac{1}{2}}u(t) - v'(t) + \int_0^t (v(x) + u(x))dx = f_1(t) \\ D^{\frac{1}{4}}u(t) + 2v''(t) + \int_0^t v^2(x)dx = f_2(t) \end{cases} \tag{6.8}$$

where

$$\begin{cases} f_1(t) = \frac{1}{\Gamma(0.50)}\left(\frac{16}{5}t^{\frac{5}{2}} + 2t^{\frac{1}{2}}\right) + \frac{t^4}{4} + \frac{t^2}{2} + t + 1 \\ f_2(t) = \frac{1}{\Gamma(0.75)}\left(\frac{128}{77}t^{\frac{11}{4}} + \frac{4}{3}t^{\frac{3}{4}}\right) + 2e^t + \frac{1}{2}e^{2t} - \frac{1}{2} \end{cases}$$

The exact solutions of this problem are $u(t) = t^3 + t + 1$ and $v(t) = e^t$.
By applying the technique described in last section, we satisfy some boundary conditions as follows:

$$\begin{aligned} \widehat{y_{1m}}(t) &= y_{1m}(t) = A_1^T\Phi(t), \\ \widehat{y_{2m}}(t) &= 1 + t + t^2\ y_{2m}(t), \end{aligned}$$

and construct the residual functions as follows:

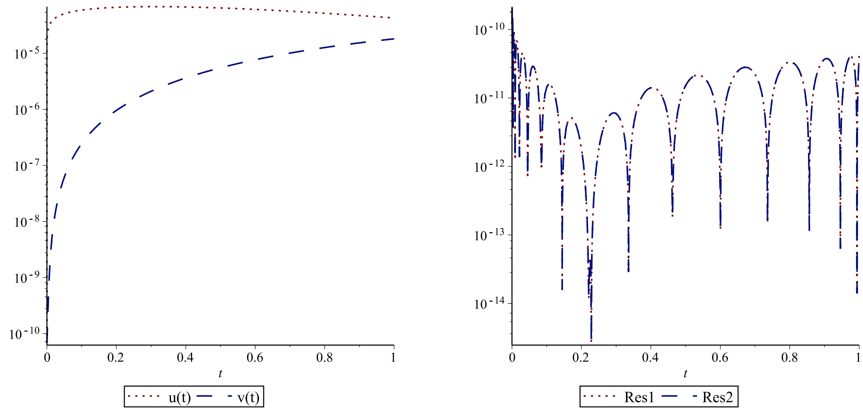
$$Res_1(t) = A_1^T D^{(\frac{1}{2})}\Phi(t) - \widehat{y_{1m}}'(t) + \int_0^t (\widehat{y_{1m}}(x) + \widehat{y_{1m}}(x))dx - f_1(t)$$

$$Res_2(t) = A_1^T D^{(\frac{1}{4})}\Phi(t) + 2\widehat{y_{1m}}''(t) + \int_0^t \widehat{y_{1m}}^2(x)dx - f_2(t)$$

where $D^{(\frac{1}{2})} = D^{(\frac{1}{4})}D^{(\frac{1}{4})}$, $D^{(\frac{1}{4})}$ are the operational matrix of fractional derivative defined in Eq. (4.1) and A_1^T is defined in Eq. (3.3).

Therefore, to obtain the coeffcient $\{a_{jn}\}_{j=1,n=0}^{2,m-1}$; $Res_i(t),\ i = 1,2$ is equalized to zero at m collocation point and replace the one collocation with the boundary condition $\widehat{y_{1m}}(0) - 1 = 0$.
By solving this set of nonlinear algebraic equations, we can find the approximating function $\widehat{y_{jm}}(t)$.

Figure 9 shows the logarithmic graph of the absolute and the residual errors of the approximate solutions and the analytic solutions for $m = 20$, and $\alpha = 0.25$.



(a) Graphs of the absolute error

(b) Graphs of the residual error

Figure 9: Log graphs of the absolute and the residual errors for example 8.

Example 9. Consider the nonlinear Volterra IDEs problem with boundary conditions $u(0) = 1,\ v(0) = 2$ and $v(1) = 1$, as follows

$$\begin{cases} D^{\frac{1}{2}}u(t) + D^{\frac{3}{2}}v(t) + \int_0^t (u(x) - v(x))dx = f_1(t) \\ u'(t) + D^{\frac{1}{2}}v(t) + \int_0^t (u(x) + v(x))dx = f_2(t) \end{cases} \tag{6.9}$$

where

$$\begin{cases} f_1(t) = \frac{1}{\Gamma(0.50)}\left(16t^{\frac{5}{2}} - 10t^{\frac{1}{2}}\right) + \frac{4t^3}{3} - 2t \\ f_2(t) = \frac{1}{\Gamma(0.50)}\left(\frac{128}{35}t^{\frac{7}{2}} - 8t^{\frac{3}{2}} + 2t^{\frac{1}{2}}\right) + \frac{7t^4}{4} + \frac{t^2}{2} + t + 1 \end{cases}$$

The exact solutions of this problem are $u(t) = t^3 + t + 1$ and $v(t) = t^4 - 3t^2 + t + 2$. By applying the technique described in last section, we define:

$$\begin{aligned} \widehat{y_{1m}}(t) &= y_{1m}(t) = A_1^T \Phi(t), \\ \widehat{y_{2m}}(t) &= y_{2m}(t) = A_2^T \Phi(t), \end{aligned}$$

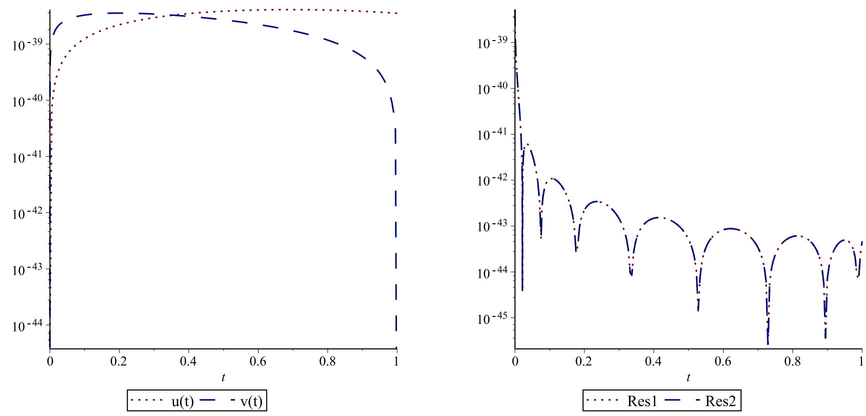
and construct the residual functions as follows:

$$\begin{aligned} Res_1(t) &= A_1^T D^{(\frac{1}{2})}\Phi(t) + A_2^T D^{(\frac{3}{2})}\Phi(t) + \int_0^t (\widehat{y_{1m}}(x) - \widehat{y_{2m}}(x))dx - f_1(t) \\ Res_2(t) &= \widehat{y_{1m}}'(t) + A_2^T D^{(\frac{1}{2})}\Phi(t) + \int_0^t (\widehat{y_{1m}}(x) + \widehat{y_{2m}}(x))dx - f_2(t) \end{aligned}$$

where $D^{(\frac{3}{2})} = D^{(\frac{1}{2})}D^{(\frac{1}{2})}D^{(\frac{1}{2})}$, $D^{(\frac{1}{2})}$ are the operational matrix of fractional derivative defined in Eq. (4.1) and A_1^T, A_2^T are defined in Eq. (3.3).

Therefore, to obtain the coeffcient $\{a_{jn}\}_{j=1,n=0}^{2,m-1}$; $Res_i(t)$, $i = 1, 2$ is equalized to zero at m collocation point and replace the three collocations with the boundary conditions $\widehat{y_{1m}}(0) - 1 = 0$, $\widehat{y_{2m}}(0) - 2 = 0$ and $\widehat{y_{2m}}(1) - 1 = 0$.
By solving this set of nonlinear algebraic equations, we can find the approximating function $\widehat{y_{jm}}(t)$.

Figure 10 shows the logarithmic graph of the absolute and the residual errors of the approximate solutions and the analytic solutions for $m = 10$, and $\alpha = 0.50$.



(a) Graphs of the absolute errors

(b) Graphs of the residual errors

Figure 10: Log graphs of the absolute and the residual errors for example 9.

Example 10. Consider the nonlinear Volterra IDEs problem with boundary conditions $D^{\frac{1}{2}}u(0) = 0$, $v(0) = 1$ and $v'(0) = -1$, as follows

$$\begin{cases} D^{\frac{1}{2}}u(t) + v'(t) + \int_0^t (u(x)v(x))dx = f_1(t) \\ D^{\frac{1}{2}}u(t) + v''(t) + \int_0^t (u(x) - v(x))dx = f_2(t) \end{cases} \tag{6.10}$$

where

$$\begin{cases} f_1(t) = t - e^{-t} + \sum_{k=1}^{\infty} \frac{1}{\Gamma(k+\frac{1}{2})} t^{k-\frac{1}{2}} \\ f_2(t) = e^t + 2e^{-t} - 2 + \sum_{k=1}^{\infty} \frac{1}{\Gamma(k+\frac{1}{2})} t^{k-\frac{1}{2}} \end{cases}$$

The exact solutions of this problem are $u(t) = e^t$ and $v(t) = e^{-t}$.
By applying the technique described in last section, we satisfy some boundary conditions as follows:

$$\begin{aligned} \widehat{y_{1m}}(t) &= y_{1m}(t) = A_1^T \Phi(t), \\ \widehat{y_{2m}}(t) &= 1 - t + t^2\ y_{2m}(t). \end{aligned}$$

We construct the residual functions as follows:

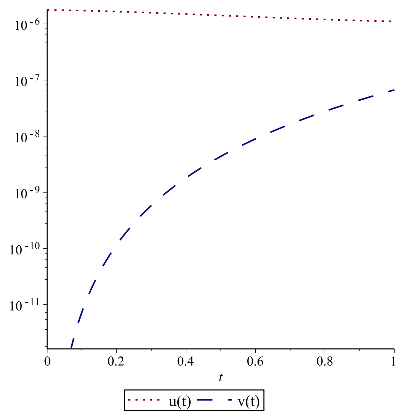
$$Res_1(t) = A_1^T D^{(\frac{1}{2})} \Phi(t) + \widehat{y_{2m}}'(t) + \int_0^t \widehat{y_{1m}}(x)\widehat{y_{2m}}(x)dx - f_1(t)$$

$$Res_2(t) = A_1^T D^{(\frac{1}{2})} \Phi(t) + \widehat{y_{2m}}''(t) + \int_0^t (\widehat{y_{1m}}(x) - \widehat{y_{2m}}(x))dx - f_2(t)$$

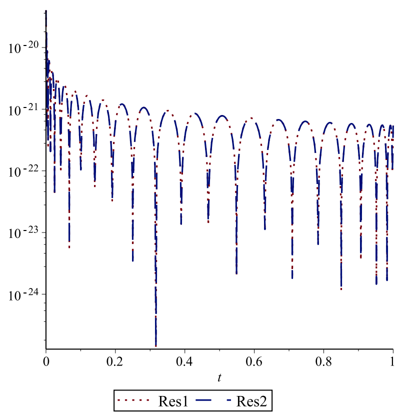
where $D^{(\frac{1}{2})}$ is the operational matrix of fractional derivative defined in Eq. (4.1) and A_1^T is defined in Eq. (3.3).

Therefore, to obtain the coeffcient $\{a_{jn}\}_{j=1,n=0}^{2,m-1}$; $Res_i(t),\ i = 1, 2$ is equalized to zero at m collocation point and replace the one collocation with the boundary condition $A_1^T D^{(\frac{1}{2})} \Phi(0) = 0$.
By solving this set of nonlinear algebraic equations, we can find the approximating function $\widehat{y_{jm}}(t)$.

Figure 11 shows the logarithmic graph of the absolute and the residual errors of the approximate solutions and the analytic solutions for $m = 25$, and $\alpha = 0.50$.



(a) Graphs of the absolute errors



(b) Graphs of the residual errors

Figure 11: Log graphs of the absolute and the residual errors for example 10.

7. Conclusion

In this paper, the generalized fractional order of the Chebyshev functions (GFCFs) of the first kind have been introduced as a new basis for Spectral methods and this basis can be used to develop a framework or theory in Spectral methods. Next an operational matrix of fractional derivative for these orthogonal functions is obtained. These functions and matrix can be used to solve the system of nonlinear Volterra integro-differential equations of arbitrary order. The comparison of the approximate solutions and the exact solutions shows that the proposed method is more efficient tool and more practical for solving linear and non-linear systems of integro-differential equations and plots confirm.

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