Bol. Soc. Paran. Mat. (3s.) **v. 37** 3 (2019): 9–15.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v37i3.31684

Derivation on Vinberg Rings *

G. Lakshmi Devi and K. Jayalakshmi

ABSTRACT: A nonassociative ring which contains a well-known associative ring or left symmetric ring also known as Vinberg ring is of great interest. A method to construct Vinberg nonassociative ring is given; Vinberg nonassociative ring $\overline{VN_{n,m,s}}$ is shown as simple; all the derivations of nonassociative simple Vinberg $\overline{VN_{0,0,1}}$ algebra defined are determined; and finally in solid algebra it is shown that if θ is a nonzero endomorphism of $\overline{VN_{0,0,1}}$, then θ is an epimorphism.

Key Words: Nonassociative ring, Simple, Vinberg ring, Derivation, Solid algebra.

Contents

1. Preliminaries

Let $(A, *, +)$ be a nonassociative algebra then the antisymmetrized algebra $(A^-, [,], +)$ with the same set A and the Lie bracket $[,]$ is defined as follows: $[x, y] = x * y - y * x$ for any $x, y \in A^-$. Choi proposed an interesting problem [9]: Does the equality $Aut_F(A) = Aut_{Lie}(A^-)$ hold? The answer is no generally. Any derivation of an algebra A is a derivation of the antisymmetrized algebra A^- . He also proposed an interesting problem: Is $Der(A) = Der_{Lie}(A^-)$? If θ is an automorphism of Vinberg ring VN then the $Der(VN)$ is also an automorphism. For a p-torsion free Vinberg algebra, we do not know $Der(A)$ generally. Our method of finding $Der(\overline{VN_{0,0,1}})$ will give a good modification to find $Der(A)$ of an algebra A. The authors have given the description of a 2-torsion free Vinberg (-1,1) ring R in [2]. They have shown that if every nonzero root space of R^- for S is one-dimensional where S is a split abelian Cartan subring of R^- which is nil on R then R is a Lie ring isomorphic to R^-. In this paper we extend the results of [2]

* The project is partially supported by Jawaharlal Nehru Technological University Anantha-pur, Registered No. 13PH0909.
2010 *Mathematics Subject Classification:* 17A30, 17A36.
Submitted April 18, 2016. Published April 06, 2017

to $\overline{VN_{0,0,1}}$ algebra. A nonzero endomorphism of $\overline{VN_{0,0,1}}$ is an epimorphism.
A nonassociative ring R is called a Vinberg ring if it satisfies the identity

$$(x, y, z) = (y, x, z) \tag{1.1}$$

where $(x, y, z) = (xy)z - x(yz)$ for $x, y, z \in R$. Throughout this paper Z and N are the sets of integers and non-negative integers respectively.
Let $(R, +, \cdot)$ be a Vinberg ring and ∂ a derivation of R.Let $F[x_1, ..., x_{m+s}]$ be the polynomial ring on the variables $x_1, ..., x_{m+s}$. Let $g_1, ..., g_n$ be given polynomials in $F[x_1, ..., x_{m+s}]$. For $n, m, s \in N$, we define the F - algebra $F_{n,m,s} = F[e^{\pm g_1}, ..., e^{\pm g_n}, x_1^{\pm 1}, ..., x_m^{\pm 1}, x_{m+1}, ..., x_{m+s}]$ with the standard basis [3]

$$B = \{e^{a_1 g_1} \cdot \cdots \cdot e^{a_n g_n} x_1^{i_1} \cdots x_m^{i_m} x_{m+1}^{i_{m+1}} \cdots x_{m+s}^{i_{m+s}} \mid a_1, ..., a_n, i_1, ..., i_m \in Z, \\ i_{m+1}, ..., i_{m+s} \in N\} \tag{1.2}$$

and with the obvious addition and the multiplication [3, 4, 6, 7]. We define the F-Vector space $VN_{(n,m,s)}$ with the standard basis

$$\{e^{a_1 g_1} \cdot \cdots \cdot e^{a_n g_n} x_1^{i_1} \cdots x_m^{i_m} x_{m+1}^{i_{m+1}} \cdots x_{m+s}^{i_{m+s}} \partial_w \mid a_1, ..., a_n, i_1, ..., i_m \in Z, \\ i_{m+1}, ..., i_{m+s} \in N, 1 \leq w \leq m+s\} \tag{1.3}$$

where ∂_w is the usual partial derivative with respect to x_w . We define the multiplication $*$ on $VN_{n,m,s}$ as

$$f\partial_w * h\partial_u = f\partial_w(h)\partial_u \tag{1.4}$$

for $f\partial_w$ and $h\partial_u \in VN_{n,m,s}$. Thus we can define the Vinberg-type nonassociative ring $\overline{VN_{n,m,s}}$ with the multiplication in (1.4) and with the set $VN_{(n,m,s)}$. The nonassociative ring $\overline{VN_{n,m,s}}(s \geq 2)$ is not a Vinberg ring as it does not satisfy (1.1). But $\overline{VN_{1,0,1}}$ is a Vinberg ring. For any element $l = e^{a_1 g_1} \cdots e^{a_n g_n} x_1^{i_1} \cdots x_{m+s}^{i_{m+s}} \partial_w$ $(1 \leq w \leq m+s)$, let us call $i_1, ..., i_{m+s}$ the powers of l. An ideal in a nonassociative ring is a two sided ideal of it. In this paper, we prove that the ring $\overline{VN_{n,m,s}}$ is simple. The ring $\overline{VN_{n,m,s}}$ is not a Jordan ring. The right annihilators of $\overline{VN_{n,m,s}}$ is the sub ring $T_s = \{\sum_{t=1}^{s} c_t d_t \mid c_t \in F\}$, and the left annihilator of $\overline{VN_{n,m,s}}$ is $\{0\}$. We can see that the center of $\overline{VN_{n,m,s}}$ is $\{0\}$ since for any $l \in \overline{VN_{n,m,s}}$, there is $l_1 \in \overline{VN_{n,m,s}}$ such that $[l, l_1] = l*l_1 - l_1*l \neq 0$. In $\overline{VN_{n,m,s}}$, $\{x_t\partial_t + c_t\partial_t | 1 \leq t \leq m+s, c_t \in F\}$ is a set of orthogonal idempotents in $\overline{VN_{n,m,s}}$, and $\{\sum_{v=1}^{m+s} x_v\partial_v + \sum_{v=1}^{m+s} c_v\partial_v : c_v \in F\}$ is the set of right units of $\overline{VN_{n,m,s}}$, where $n \leq m+s$. A nonassociative ring VN is called power-associative if the subring $F[a]$generated by any element a of VN is associative (see [8]). From $(a^n\partial * a^n\partial) * a^n\partial = a^n\partial * (a^n\partial * a^n\partial)$, we know that the algebra $\overline{VN_{n,m,s}}$ is not power associative.

2. Main results

Theorem 2.1. *The algebra* $\overline{VN_{n,m,s}}$ *is simple.*

Proof: First we show that the ideal $\langle \partial_w \rangle$ generated by ∂_w, where $1 \le w \le m+s$, is $\overline{VN_{n,m,s}}$. For any basis element $e^{a_1 g_1} \cdots e^{a_n g_n} x_1^{i_1} x_k^{i_k} x_{k+1}^{i_{k+1}} \cdots x_{m+s}^{i_{m+s}} \partial_u$ of $\overline{VN_{n,m,s}}$ with $a_k \neq 0$, we have $\partial_k * \frac{1}{a_k} e^{a_1 g_1} \cdots e^{a_n g_n} x_1^{i_1} \cdots \widehat{x_k^{i_k}} x_{k+1}^{i_{k+1}} \cdots x_{m+s}^{i_{m+s}} \partial_u = e^{a_1 g_1} \cdot \cdot \cdot e^{a_n g_n} x_1^{i_1} \cdots \widehat{x_k^{i_k}} x_{k+1}^{i_{k+1}} \cdots x_{m+s}^{i_{m+s}} \partial_u \in \langle \partial_w \rangle$ for $a_1, ..., a_{k-1}, a_{k+1}, ...a_n, i_1, ..., i_m \in Z$ and $i_{m+1}, , ..., i_{m+s} \in N$, where $\widehat{x_k^{i_k}}$ means that the term $x_k^{i_k}$ is omitted. For any $e^{a_1 g_1} \cdots e^{a_n g_n} x_1^{i_1} \cdots \widehat{x_k^{i_k}} x_{k+1}^{i_{k+1}} \cdots x_{m+s}^{i_{m+s}} \partial_u \in \langle \partial_w \rangle$ with $a_k \neq 0$, we have $x_k^{i_k} \partial_k * \frac{1}{a_k} e^{a_1 g_1} \cdot \cdot \cdot e^{a_n g_n} x_1^{i_1} \cdots \widehat{x_k^{i_k}} x_{k+1}^{i_{k+1}} \cdots x_{m+s}^{i_{m+s}} \partial_u = e^{a_1 g_1} \cdots e^{a_n g_n} x_1^{i_1} \cdots x_k^{i_k} x_{k+1}^{i_{k+1}} \cdots x_{m+s}^{i_{m+s}} \partial_u$. This implies that $e^{a_1 g_1} \cdots e^{a_n g_n} x_1^{i_1} \cdots x_k^{i_k} x_{k+1}^{i_{k+1}} \cdots x_{m+s}^{i_{m+s}} \partial_u \in \langle \partial_w \rangle$ holds for any $i_k \in Z$ or $i_k \in N$. Therefore, we have proved that $\langle \partial_w \rangle = \overline{VN_{n,m,s}}$. Let I be a non - zero ideal of $\overline{VN_{n,m,s}}$. Let us prove the theorem by induction on the number of distinct homogeneous components of any non - zero element l in I. Assume that l has only one $(0, ..., 0)$ - homogeneous component. We may assume that l has positive powers from $l_2 = l_1 * l \in I$ by taking an appropriate element $l_1 \in \overline{VN_{n,m,s}}$. We can get the element

$$\partial_{q_1} * \cdots \partial_{q_1} * (\cdots * (\partial_{q_t} * (\cdots (*(\partial_{q_t} * l_2) \cdots) = c\partial_{q_k} \tag{2.1}$$

by taking appropriate $q_1, ..., q_t, 1 \le q_1, ..., q_t \le m+s$, and applying $\partial_{q_1}, ..., \partial_{q_t}$ in (2.1) with appropriate times, where c is a non- zero scalar. This implies that $\overline{VN_{n,m,s}} = \langle \partial_w \rangle \subset I$. Therefore, we have the theorem. Assume that l is in the $(a_1, ..., a_n)$ - homogeneous component, then $0 \neq e^{-a_1 g_1} \cdots e^{-a_n g_n} \partial_t * l \in VN_{(0,...,0)}$ by taking an appropriate $t, 1 \le t \le m+s$, where atleast one of $a_1, ..., a_n$ is not zero. In this case, we have the theorem already. We may assume that l has n homogeneous components by induction. Let us assume that l has the $(0, ..., 0, a_w, ..., a_n)$- homogeneous component such that $a_w \neq 0$.By taking $l_1 = e^{-a_w g_w} \cdots e^{-a_s g_s} x_1^{i_1} \cdots x_{m+s}^{i_{m+s}} \partial_t$, where $i_1, ..., i_{m+s}$ are sufficiently large positive integers so that $l_1 * l \in I$ has positive powers. By taking an appropriate $\partial_k, 1 \le k \le m+s$, we have $0 \neq \partial_k * (\cdots * (\partial_k * (l_1 * l) \cdots) \in I$ with appropriate times so that $\partial_k * (\cdots * (\partial_k * (l_1 * l) \cdots) \neq 0$ has atmost $n-1$ homogeneous components. Therefore, we have the theorem by induction.

□

3. Derivations of $\overline{VN_{0,0,1}}$

The right annihilator of l in $\overline{VN_{n,m,s}}$ is the set $\{l_1 \in \overline{VN_{n,m,s}} | l * l_1 = 0\}$ and similarly the left annihilator is the set $\{l_2 \in \overline{VN_{n,m,s}} | l_2 * l = 0\}$. An additive F - linear map D of $\overline{VN_{n,m,s}}$ is a derivation if $D(l_1 * l_2) = D(l_1) * l_2 + l_1 * D(l_2)$ holds for any $l_1, l_2 \in \overline{VN_{n,m,s}}$ [1].

Remark 3.1. *Let $c \in F$. The map D_1 such that $D_1(cx^i \partial) = cix^{i-1}\partial$ for any basis element $x^i\partial$ can be extended linearly on $\overline{VN_{0,0,1}}$, which is a derivation of $\overline{VN_{0,0,1}}$. Similarly, the F-linear map D_2 on $\overline{VN_{0,0,1}}$ such that $D_2(x^i\partial) = (1-i)x^i\partial$ for any basis element $x^i\partial$ of $\overline{VN_{0,0,1}}$ is a derivation of $\overline{VN_{0,0,1}}$.*

Lemma 3.2. *The left annihilator of ∂ is $\overline{VN_{0,0,1}}$, and the right annihilator of ∂ is $\{c\partial | c \in F\}$.*

Proof: The proof is straightforward by the definitions of the right and left annihilators of ∂ in $\overline{VN_{0,0,1}}$. □

Theorem 3.3. *For any derivation D of $\overline{VN_{0,0,1}}$, $D = c_1D_1 + c_2D_2, c_1, c_2 \in F$, where D_1 and D_2 are the derivations of $\overline{VN_{0,0,1}}$ in Remark 3.1.*

Proof: Let D be any derivation of $\overline{VN_{0,0,1}}$. Then

$$D(\partial * \partial) = D(\partial) * \partial + \partial * D(\partial) = \partial * D(\partial) = 0.$$

By Lemma 3.1, we have

$$D(\partial) = C(0)\partial forsome C(0) \in F. \tag{3.1}$$

By $D(\partial * x\partial) = D(\partial) * x\partial + \partial * D(x\partial) = C(0)\partial = C(0)\partial + \partial * D(x\partial)$, we have

$$D(x\partial) = C(1)\partial forsome C(1) \in F. \tag{3.2}$$

This implies that $D(\partial * x^2\partial) = 2D(x\partial) = 2C(1)\partial$. But,

$$D(\partial) * x^2\partial + \partial * D(x^2\partial) = C(0)\partial * x^2\partial + \partial * D(x^2\partial) = 2C(0)x\partial + \partial * D(x^2\partial).$$

This implies that $\partial * D(x^2\partial) = -2C(0)x\partial + 2C(1)\partial$. Then $D(x^2\partial) = -C(0)x^2\partial + 2C(1)x\partial + C(2,0)\partial$ for some $C(2,0) \in F$. We have

$$D(x\partial * x^2\partial) = 2D(x^2\partial) = -2C(0)x^2\partial) + 4C(1)x\partial + C(2,0)\partial. \tag{3.3}$$

Also, we have

$$D(x\partial) * x^2\partial + x\partial * D(x^2\partial) = 2C(1)x\partial + x\partial * (-C(0)x^2\partial) + 2C(1)x\partial + C(2,0)\partial).$$

Thus

$$D(x\partial) * x^2\partial + x\partial * D(x^2\partial) = -2C(0)x^2\partial + 4C(1)x\partial. \tag{3.4}$$

By (3.3) and (3.4), we have $C(2,0) = 0$. Let us assume that
$D(x^n\partial) = C(0)(1-n)x^n\partial + C(1)nx^{n-1}\partial$ for some fixed $n \in N$, by induction.
Thus we have

$$D(\partial * x^{n+1}\partial) = (n+1)D(x^n\partial) = (n+1)C(0)(1-n)x^n\partial + (n+1)C(1)nx^{n-1}\partial.$$

But we have $D(\partial) * x^{n+1}\partial + \partial * D(x^{n+1}\partial) = C(0)(n+1)x^n\partial + \partial * D(x^{n+1}\partial)$.
This implies that
$\partial * D(x^{n+1}\partial) = -C(0)(n+1)x^n\partial + C(0)(n+1)(1-n)x^n\partial + C(1)n(n+1)x^{n-1}\partial$
$= -nC(0)(n+1)x^n\partial + C(1)n(n+1)x^{n-1}\partial$.
Hence,

$$D(x^{n+1}\partial) = -nC(0)x^{n+1}\partial + C(1)(n+1)x^n\partial + C(n,0)\partial, C(n,0) \in F.$$

Then

$$D(x\partial * x^{n+1}\partial) = (n+1)D(x^{n+1}\partial)$$
$$= -nC(0)(n+1)x^{n+1}\partial + C(1)(n+1)^2x^n\partial + C(n,0)(n+1)\partial.$$

On the other hand, we have

$$C(1)\partial * x^{n+1}\partial + x\partial * (-nC(0)x^{n+1}\partial + C(1)(n+1)x^n\partial + C(n,0)\partial)$$
$$= -nC(0)(n+1)x^{n+1}\partial + C(1)(n+1)^2x^n\partial.$$

This implies that $C(n,0) = 0$. Therefore, we have proved that

$$D(x^n\partial) = C(0)(1-n)x^n\partial + C(1)nx^{n-1}\partial, n \in N.$$

This shows that $D = C(0)D_2 + C(1)D_1$ and completes the proof of the theorem. □

4. Solid Algebras

Let A be an F-algebra. Let $End_F(A)$ be the set of all F-endomorphisms of A, and $Aut_F(A)$ the set of all automorphisms of A. An F-algebra A is solid if every non-zero endomorphism of A is surjective.

Proposition 4.1. *A simple algebra A is solid if and only if $End_F(A) = \{0\} \cup Aut_F(A)$.*

Proof: It is straightforward by the fact that A is a simple algebra and the definition of the solid algebra. □

Lemma 4.2. *For any $\theta \in End_F(\overline{VN_{0,0,1}})$, if $\theta(\partial) = 0$, then θ is the zero map of $\overline{VN_{0,0,1}}$.*

Proof: We have $\theta(\partial * x^n\partial) = n\theta(x^{n-1}\partial) = 0$ for any $n \in N$, which implies that θ is the zero map by induction on the degree of $x^n\partial$. □

Lemma 4.3. *For any non-zero F-endomorphism θ of $\overline{VN_{0,0,1}}$, $\theta(\partial) = c_o\partial$ holds for some fixed $0 \neq c_0 \in F$.*

Proof: We have $\theta(\partial * \partial) = \theta(\partial) * \theta(\partial) = 0$. Since $\theta(\partial) \neq 0$, by Lemma 4.1, we have $\theta(\partial) = c_o\partial$, $0 \neq c_0 \in F$. □

Proposition 4.4. *If θ is a non-zero endomorphism of $\overline{VN_{0,0,1}}$, then θ is an epimorphism.*

Proof: By Lemma 4.3 we have $\theta(\partial) = c_0\partial$ for some non-zero $c_0 \in F$. From

$$\theta(\partial * x\partial) = \theta(\partial),$$

we have $c_0\partial * \theta(x\partial) = c_0\partial$. This implies that $\theta(x\partial) = c_1\partial + x\partial$ for some $c_1 \in F$. By $\theta(\partial * x^2\partial) = 2\theta(x\partial)$, we have $\theta(x^2\partial) = c_r\partial + \frac{2c_1x}{c_0}\partial + \frac{x^2}{c_0}\partial$ for $c_r \in F$. By $\theta(x\partial * x^2\partial) = 2\theta(x^2\partial)$, we have

$$(c_1\partial + x\partial) * (c_r\partial + \frac{2c_1x}{c_0}\partial + \frac{x^2}{c_0}\partial) = 2c_r\partial + \frac{4c_1x}{c_0}\partial + \frac{2x^2}{c_0}\partial. \tag{4.1}$$

By comparing the coefficients of both sides of (4.1), we have $c_r = \frac{c_1^2}{c_0}$. Thus, we have $\theta(x^2\partial) = c_0^{-1}(x + c_1)^2\partial$. Let us assume that $\theta(x^n\partial) = c_0^{1-n}(x + c_1)^n\partial$ for some fixed non-negative integer n inductively.
From $\theta(\partial * x^{n+1}\partial) = (n + 1)\theta(x^n\partial)$, we have

$$\partial * \theta(x^{n+1}\partial) = (n + 1)c_0^{1-n}(x + c_1)^n\partial.$$

This implies that $\theta(x^{n+1}\partial) = c_0^{-n}(x + c_1)^{n+1}\partial + c_u\partial$ for some $c_u \in F$. By

$$\theta(x\partial * x^{n+1}\partial) = (n + 1)\theta(x^{n+1}\partial), \tag{4.2}$$

we have $(x+c_1)\partial*(c_0^{-n}(x+c_1)^{n+1}\partial+c_u\partial) = c_0^{-n}(n+1)(x+c_1)^{n+1}\partial+(n+1)c_u\partial$. By comparing the coefficients of both sides of (4.2), we have $c_u = 0$. Thus,$\theta(x^m\partial) = c_0^{1-m}(x + c_1)^m\partial$ holds for any $m \in F$ inductively.
Therefore, any $l \in \overline{VN_{0,0,1}}$ can be written as

$$l = c_t^{"}c_0^{1-t}(x + c_1)^t\partial + \cdots + c_0^{"}c_0^{-1}\partial = c_t^{"}\theta(x^t\partial) + \cdots + c_0^{"}\theta(\partial),$$

Where $c_t^{"}, ..., c_0^{"} \in F$. This implies that θ is surjective. The following corollary is the version of Jacobian conjecture on $\overline{VN_{0,0,1}}$. □

Corollary 4.5. *For any non-zero endomorphism θ of $\overline{VN_{0,0,1}}$, θ is an automorphism of $\overline{VN_{0,0,1}}$.*

Proof: By Lemma 4.3, $\theta(\partial) = c_0\partial$ for some non-zero $c_0 \in F$. Since $\overline{VN_{0,0,1}}$ is simple, θ is one to one. By Proposition 4.4, θ is onto. □

Corollary 4.6. *$End(\overline{VN_{0,0,1}}) = Aut\ (\overline{VN_{0,0,1}}) \bigcup \{0\}$,where 0 is the zero map of $\overline{VN_{0,0,1}}$.*

Proof: It is straightforward by Corollary 4.5. □

By Corollary 4.6, we know that $\overline{VN_{0,0,1}}$ is solid.

Proposition 4.7. *For any $\theta \in Aut(\overline{VN_{n,m,s}})$, we have $\theta(T_{s_1}) = T_{s_1}$.*

Proof: Since T_{s_1} is the unique maximal right annihilator of $\overline{VN_{n,m,s}}$, $\theta(T_{s_1}) = T_{s_1}$ holds for any $\theta \in Aut(\overline{VN_{n,m,s}})$. □

Acknowledgements
The authors express their sincere thanks to the referees for suggesting corrections and bringing the paper in present form.

References

1. J. Dixmier, *Enveloping Algebras, North Holland, Amsterdam, 1977.*
2. *K. Jayalakshmi, and G. Lakshmi Devi, On Vinberg (-1, 1) rings, Asian-European Journal of Mathematics, Vol.9, No.2 (2016) (9pages).*
3. *V.G. Kac, Description of filtered Lie Algebra with which graded Lie Algebras of Cartan type are associated, Izv.Akad.Nauk SSSR, Ser.Mat,38,1974, 832-834.*
4. *Ki-Bong Nam, Generalized W and H type Lie Algebras, Algebra Colloquium, 1999, 329- 340.*
5. *H. Mohammad, Ahmadi, Ki-Bong Nam and Jonathan Pakinathan, Lie admissible non- associative algebras, Algebra Colloquium, Vol.12, No.1, World Scientific, March, 2005, 113-120.*
6. *D. Passman, Simple Lie Algebras of Witt-Type, Journal of Algebra, 206, 1998, 682-692.*
7. *A. N. Rudakov, Groups of Automorphisms of Infinite-Dimensional Simple Lie Algebras, Math. USSR- Izvestijs, 3, 1969, 707-722.*
8. *R. D. Schafer, Introduction to Non-associative Algebras, Dover, 1995.*
9. *Seul Hee Choi, Derivations of a restricted Weyl type Algebra I*.*

G. Lakshmi Devi,
Department of Mathematics,
Government College(Autonomous,)
Ananthapuramu, Andhra Pradesh, India.
E-mail address: glakshmi229@gmail.com

and

K. Jayalakshmi,
Department of Mathematics,
JNTUA College Of Engineering (Anathapuramu),
Andhra Pradesh, India.
E-mail address: kjay.maths@jntua.ac.in