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Entropy Solutions For Nonlinear Parabolic Inequalities Involving Measure Data In Musielak-Orlicz-Sobolev Spaces

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ABSTRACT: In this paper, we study an existence result of entropy solutions for some nonlinear parabolic problems in the Musielak-Orlicz-Sobolev spaces.

Key Words: Musielak-Orlicz-Sobolev spaces, parabolic equations, entropy solutions, truncations.

Contents

1. Introduction

Let Ω a bounded open subset of $\mathbb{R}^N$ and let Q be the cylinder $\Omega\times(0,T)$ with some given $T>0$.
We consider the strongly nonlinear parabolic problem

$$(\mathcal{P})\begin{cases}\frac{\partial u}{\partial t}+A(u)+g(x,t,u,\nabla u)=f-\operatorname{div}(F) & \text{in } Q,\\ u\equiv 0 & \text{on } \partial Q=\partial\Omega\times[0,T]\\ u(\cdot,0)=u_0 & \text{on } \Omega,\end{cases}$$

where $A:D(A)\subset W_0^{1,x}L_\varphi(Q)\longrightarrow W^{-1,x}L_\psi(Q)$ (see section 2) defined by $A(u)=-\operatorname{div}(a(x,t,u,\nabla u))$ is an operator of Leray-Lions type, where a is a Carathéodory function such that

$$|a(x,t,s,\xi)|\le\beta\Big(h_1(x,t)+\psi_x^{-1}\gamma(x,\nu|s|)+\psi_x^{-1}\varphi(x,\nu|\xi|)\Big)$$

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$$\Big(a(x,t,s,\xi) - a(x,t,s,\xi')\Big)(\xi - \xi') > 0$$

$$a(x,t,s,\xi).\xi \geq \alpha\varphi(x,|\xi|)$$

with $h_1 \in L^1(Q)$, $\beta, \nu, \alpha > 0$ and γ a Musielak function such that $\gamma \ll \varphi$.
Let g be a Carathéodory function such that

$$|g(x,t,s,\xi)| \leq b(|s|)\Big(h_2(x,t) + \varphi(x,|\xi|)\Big),$$

$$g(x,t,s,\xi)s \geq 0,$$

is satisfied, where b a positive function in $L^1(\mathbb{R}^+)$ and $h_2 \in L^1(Q)$,
and $f \in L^1(Q)$ and $F \in (E_\psi(Q))^N$.
Under these assumptions, the above problem does not admit, in general, a weak solution since the field $a(x,t,u,\nabla u)$ does not belong to $(L^1_{\rm loc}(Q))^N$ in general. To overcome this difficulty we use in this paper the framework of entropy solutions. This notion was introduced by Bénilan and al. [4] for the study of nonlinear elliptic problems.
In the classical Sobolev spaces, the authors in [9, 17] proved the existence of solutions for the problem $(\mathcal{P})$ in the case where $F \equiv 0$, in [7] the authors had proved the existence of solutions for the problem $(\mathcal{P})$ in the elliptic case.
In the setting of Orlicz spaces, the solvability of $(\mathcal{P})$ was proved by Donaldson [10] and Robert [18], and by Elmahi [12] and Elmahi-Meskine [13]. In Musielak framework, recently M. L. Ahmed Oubeid, A. Benkirane and M. Sidi El Vally in [2] had studied the problem $(\mathcal{P})$ in the Inhomogeneous case and the data belongs to $L^1(Q)$, in the elliptic case the authors in [1] proved the existence of weak solutions for the problem $(\mathcal{P})$ where the data assume to be measure and $g \equiv 0$.
It is our purpose in this paper to prove the existence of entropy solutions for problem $(\mathcal{P})$ in the setting of Musielak Orlicz spaces for general Musielak function φ with a nonlinearity $g(x,t,u,\nabla u)$ having natural growth with respect to the gradient.
Our result generalizes that of [13, 1, 2] to the case of inhomogeneous Musielak Orlicz Sobolev spaces.
The plan of the paper is as follows. Section 2 presents the mathematical preliminaries. Section 3 we make precise all the assumptions on a, g, f and u_0. Section 4 is devoted to some technical lemmas with be used in this paper. Section 5 we establish some compactness and approximation results. Final section is consecrate to define the entropy solution of $(\mathcal{P})$ and to prove existence of such a solution.

2. Preliminary

In this section we list briefly some definitions and facts about Musielak-Orlicz-Sobolev spaces. Standard reference is [16]. We also include the definition of inhomogeneous Musielak-Orlicz-Sobolev spaces and some preliminaries Lemmas to be used later.

2.1. Musielak-Orlicz-Sobolev spaces :

Let Ω be an open set in $\mathbb{R}^N$ and let φ be a real-valued function defined in $\Omega \times \mathbb{R}_+$, and satisfying the following conditions :

a) $\varphi(x,\cdot)$ is an N-function $\Big($convex, increasing, continuous, $\varphi(x,0)=0, \varphi(x,t)>0$, $\forall t>0$, $\sup_{x\in\Omega}\frac{\varphi(x,t)}{t} \longrightarrow 0$ as $t \longrightarrow 0$, $\inf_{x\in\Omega}\frac{\varphi(x,t)}{t} \longrightarrow \infty$ as $t \longrightarrow \infty\Big)$.

b) $\varphi(\cdot,t)$ is a measurable function.

A function φ, which satisfies the conditions $a)$ and $b)$ is called Musielak-Orlicz function.
For a Musielak-orlicz function φ we put $\varphi_x(t)=\varphi(x,t)$ and we associate its non-negative reciprocal function φ_x^{-1}, with respect to t that is

$$\varphi_x^{-1}(\varphi(x,t)) = \varphi(x,\varphi_x^{-1}(t)) = t.$$

The Musielak-orlicz function φ is said to satisfy the Δ_2-condition if for some $k>0$ and a non negative function h integrable in Ω, we have

$$\varphi(x,2t) \le k\varphi(x,t) + h(x) \text{ for all } x\in\Omega \text{ and } t \ge 0. \tag{2.1}$$

When (2.1) holds only for $t \ge t_0 > 0$; then φ said to satisfy Δ_2 near infinity.
Let φ and γ be two Musielak-orlicz functions, we say that φ dominate γ, and we write $\gamma \prec \varphi$, near infinity (resp. globally) if there exist two positive constants c and t_0 such that for almost all $x\in\Omega$

$$\gamma(x,t) \le \varphi(x,ct) \text{ for all } t \ge t_0, \quad (\text{ resp. for all } t\ge 0 \text{ i.e. } t_0=0).$$

We say that γ grows essentially less rapidly than φ at 0 (resp. near infinity), and we write $\gamma \prec\prec \varphi$, If for every positive constant c we have

$$\lim_{t\longrightarrow 0}\left(\sup_{x\in\Omega}\frac{\gamma(x,ct)}{\varphi(x,t)}\right)=0, \quad (\text{resp. } \lim_{t\longrightarrow \infty}\left(\sup_{x\in\Omega}\frac{\gamma(x,ct)}{\varphi(x,t)}\right)=0).$$

Remark 2.1. [6] If $\gamma \prec\prec \varphi$ near infinity, then $\forall\varepsilon>0$ there exist $k(\varepsilon)>0$ such that for almost all $x\in\Omega$ we have

$$\gamma(x,t) \le k(\varepsilon)\varphi(x,\varepsilon t), \quad \text{for all } t \ge 0. \tag{2.2}$$

We define the functional

$$\rho_{\varphi,\Omega}(u) = \int_\Omega \varphi(x,|u(x)|)dx.$$

where $u:\Omega \longrightarrow \mathbb{R}$ a Lebesgue measurable function. In the following, the measurability of a function $u:\Omega\longrightarrow\mathbb{R}$ means the Lebesgue measurability.
The set

$$K_\varphi(\Omega) = \Big\{u:\Omega\longrightarrow\mathbb{R} \text{ measurable : } \rho_{\varphi,\Omega}(u) < +\infty\Big\}.$$

is called the generalized Orlicz class.
The Musielak-Orlicz space (or the generalized Orlicz spaces) $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is, $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$.
Equivalently

$$L_\varphi(\Omega) = \left\{ u : \Omega \longrightarrow \mathbb{R} \text{ measurable} : \; \rho_{\varphi,\Omega}\Big(\frac{u}{\lambda}\Big) < +\infty, \text{ for some } \lambda > 0 \right\}.$$

Let

$$\psi(x,s) = \sup_{t\geq 0}\{st - \varphi(x,t)\}.$$

that is, ψ is the Musielak-Orlicz function complementary to φ in the sens of Young with respect to the variable s.
We define in the space $L_\varphi(\Omega)$ the following two norms

$$\|u\|_{\varphi,\Omega} = \inf\left\{ \lambda > 0 / \int_\Omega \varphi\Big(x, \frac{|u(x)|}{\lambda}\Big) dx \leq 1 \right\}.$$

which is called the Luxemburg norm and the so called Orlicz norm by :

$$|||u|||_{\varphi,\Omega} = \sup_{\|v\|_\psi \leq 1} \int_\Omega |u(x)v(x)|dx.$$

where ψ is the Musielak Orlicz function complementary to φ. These two norms are equivalent [16].
The closure in $L_\varphi(\Omega)$ of the bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_\varphi(\Omega)$. A Musielak function φ is called locally integrable on Ω if $\rho_\varphi(t\chi_E) < \infty$ for all $t > 0$ and all measurable $E \subset \Omega$ with meas$(E) < \infty$. Note that local integrability in the previous definition differs from the one used in $L^1_{\text{loc}}(\Omega)$, where we assume integrability over compact subsets.

Lemma 2.1. *[15] Let φ a Musielak function which is locally integrable. Then $E_\varphi(\Omega)$ is separable.*

We say that sequence of functions $u_n \in L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant $\lambda > 0$ such that

$$\lim_{n\to\infty} \rho_{\varphi,\Omega}\Big(\frac{u_n - u}{\lambda}\Big) = 0.$$

For any fixed nonnegative integer m we define

$$W^m L_\varphi(\Omega) = \left\{ u \in L_\varphi(\Omega) : \forall |\alpha| \leq m, \; D^\alpha u \in L_\varphi(\Omega) \right\}.$$

and

$$W^m E_\varphi(\Omega) = \left\{ u \in E_\varphi(\Omega) : \forall |\alpha| \leq m, \; D^\alpha u \in E_\varphi(\Omega) \right\}.$$

where $\alpha = (\alpha_1, ..., \alpha_n)$ with nonnegative integers α_i, $|\alpha| = |\alpha_1| + ... + |\alpha_n|$ and $D^\alpha u$ denote the distributional derivatives. The space $W^m L_\varphi(\Omega)$ is called the Musielak Orlicz Sobolev space.
Let

$$\overline{\rho}_{\varphi,\Omega}(u) = \sum_{|\alpha|\leq m} \rho_{\varphi,\Omega}\Big(D^\alpha u\Big) \text{ and } \|u\|^m_{\varphi,\Omega} = \inf\Big\{\lambda > 0 : \overline{\rho}_{\varphi,\Omega}\Big(\frac{u}{\lambda}\Big) \leq 1\Big\}$$

for $u \in W^m L_\varphi(\Omega)$, these functionals are a convex modular and a norm on $W^m L_\varphi(\Omega)$, respectively, and the pair $\Big(W^m L_\varphi(\Omega), \|\|^m_{\varphi,\Omega}\Big)$ is a Banach space if φ satisfies the following condition [16] :

$$\text{there exist a constant } c > 0 \text{ such that } \inf_{x\in\Omega} \varphi(x,1) \geq c. \tag{2.3}$$

The space $W^m L_\varphi(\Omega)$ will always be identified to a subspace of the product $\prod_{|\alpha|\leq m} L_\varphi(\Omega) = \Pi L_\varphi$, this subspace is $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closed.
We denote by $\mathcal{D}(\Omega)$ the space of infinitely smooth functions with compact support in Ω and by $\mathcal{D}(\overline{\Omega}))$ the restriction of $\mathcal{D}(\mathbb{R}^N)$ on Ω.
Let $W^m_0 L_\varphi(\Omega)$ be the $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closure of $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$.
Let $W^m E_\varphi(\Omega)$ the space of functions u such that u and its distribution derivatives up to order m lie to $E_\varphi(\Omega)$, and $W^m_0 E_\varphi(\Omega)$ is the (norm) closure of $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$.
The following spaces of distributions will also be used :

$$W^{-m} L_\psi(\Omega) = \Big\{f \in D'(\Omega);\ f = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in L_\psi(\Omega)\Big\}.$$

and

$$W^{-m} E_\psi(\Omega) = \Big\{f \in D'(\Omega);\ f = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in E_\psi(\Omega)\Big\}.$$

We say that a sequence of functions $u_n \in W^m L_\varphi(\Omega)$ is modular convergent to $u \in W^m L_\varphi(\Omega)$ if there exists a constant $k > 0$ such that

$$\lim_{n\to\infty} \overline{\rho}_{\varphi,\Omega}\Big(\frac{u_n - u}{k}\Big) = 0.$$

For φ and her complementary function ψ, the following inequality is called the Young inequality [16]:

$$ts \leq \varphi(x,t) + \psi(x,s), \quad \forall t, s \geq 0, x \in \Omega. \tag{2.4}$$

This inequality implies that

$$\||u|\|_{\varphi,\Omega} \leq \rho_{\varphi,\Omega}(u) + 1. \tag{2.5}$$

In $L_\varphi(\Omega)$ we have the relation between the norm and the modular

$$\|u\|_{\varphi,\Omega} \leq \rho_{\varphi,\Omega}(u) \text{ if } \|u\|_{\varphi,\Omega} > 1. \tag{2.6}$$

$$\|u\|_{\varphi,\Omega} \geq \rho_{\varphi,\Omega}(u) \text{ if } \|u\|_{\varphi,\Omega} \leq 1. \tag{2.7}$$

For two complementary Musielak Orlicz functions φ and ψ, let $u \in L_\varphi(\Omega)$ and $v \in L_\psi(\Omega)$, then we have the Hölder inequality [16]

$$\left| \int_\Omega u(x)v(x)dx \right| \leq \|u\|_{\varphi,\Omega} \| |v| \|_{\psi,\Omega}. \tag{2.8}$$

2.2. Inhomogeneous Musielak-Orlicz-Sobolev spaces :

Let Ω a bounded open subset of $\mathbb{R}^N$ and let $Q = \Omega \times]0,T[$ with some given $T > 0$. Let φ be a Musielak function. For each $\alpha \in \mathbb{N}^N$, denote by D_x^α the distributional derivative on Q of order α with respect to the variable $x \in \mathbb{R}^N$. The inhomogeneous Musielak-Orlicz-Sobolev spaces of order 1 are defined as follows.

$$W^{1,x}L_\varphi(Q) = \{u \in L_\varphi(Q) : \forall |\alpha| \leq 1 \; D_x^\alpha u \in L_\varphi(Q)\}$$

et

$$W^{1,x}E_\varphi(Q) = \{u \in E_\varphi(Q) : \forall |\alpha| \leq 1 \; D_x^\alpha u \in E_\varphi(Q)\}$$

The last space is a subspace of the first one, and both are Banach spaces under the norm

$$\|u\| = \sum_{|\alpha| \leq m} \|D_x^\alpha u\|_{\varphi,Q}.$$

We can easily show that they form a complementary system when Ω is a Lipschitz domain [5]. These spaces are considered as subspaces of the product space $\Pi L_\varphi(Q)$ which has $(N+1)$ copies. We shall also consider the weak topologies $\sigma(\Pi L_\varphi, \Pi E_\psi)$ and $\sigma(\Pi L_\varphi, \Pi L_\psi)$. If $u \in W^{1,x}L_\varphi(Q)$ then the function : $t \longmapsto u(t) = u(t,\cdot)$ is defined on $[0,T]$ with values in $W^1L_\varphi(\Omega)$. If, further, $u \in W^{1,x}E_\varphi(Q)$ then this function is $W^1E_\varphi(\Omega)$ valued and is strongly measurable. Furthermore the following imbedding holds:: $W^{1,x}E_\varphi(Q) \subset L^1(0,T; W^1E_\varphi(\Omega))$. The space $W^{1,x}L_\varphi(Q)$ is not in general separable, if $u \in W^{1,x}L_\varphi(Q)$, we can not conclude that the function $u(t)$ is measurable on $[0,T]$.

However, the scalar function $t \mapsto \|u(t)\|_{\varphi,\Omega}$ is in $L^1(0,T)$. The space $W_0^{1,x}E_\varphi(Q)$ is defined as the (norm) closure in $W^{1,x}E_\varphi(Q)$ of $\mathcal{D}(Q)$.

We can easily show as in [5] that when Ω a Lipschitz domain then each element u of the closure of $\mathcal{D}(Q)$ with respect of the weak * topology $\sigma(\Pi L_\varphi, \Pi E_\psi)$ is limit, in $W^{1,x}L_\varphi(Q)$, of some subsequence $(u_i) \subset \mathcal{D}(Q)$ for the modular convergence; i.e., there exists $\exists\lambda > 0$ such that for all $|\alpha| \leq 1$,

$$\int_Q \varphi(x, (\frac{D_x^\alpha u_i - D_x^\alpha u}{\lambda}))\, dx\, dt \to 0 \text{ as } i \to \infty,$$

this implies that (u_i) converges to u in $W^{1,x}L_\varphi(Q)$ for the weak topology $\sigma(\Pi L_\varphi, \Pi L_\psi)$.

Consequently

$$\overline{\mathcal{D}(Q)}^{\sigma(\Pi L_\varphi,\Pi E_\psi)} = \overline{\mathcal{D}(Q)}^{\sigma(\Pi L_\varphi,\Pi L_\psi)},$$

this space will be denoted by $W_0^{1,x}L_\psi(Q)$. Furthermore, $W_0^{1,x}E_\varphi(Q) = W_0^{1,x}L_\varphi(Q) \cap \Pi E_\varphi$.

We have the following complementary system

$$\begin{pmatrix} W_0^{1,x}L_\varphi(Q) & F \\ W_0^{1,x}E_\varphi(Q) & F_0 \end{pmatrix},$$

F being the dual space of $W_0^{1,x}E_\varphi(Q)$. It is also, except for an isomorphism, the quotient of ΠL_ψ by the polar set $W_0^{1,x}E_\varphi(Q)^\perp$, and will be denoted by $F = W^{-1,x}L_\psi(Q)$ and it is shown that

$$W^{-1,x}L_\psi(Q) = \Big\{ f = \sum_{|\alpha|\le 1} D_x^\alpha f_\alpha : f_\alpha \in L_\psi(Q) \Big\}.$$

This space will be equipped with the usual quotient norm

$$\|f\| = \inf \sum_{|\alpha|\le 1} \|f_\alpha\|_{\psi,Q}$$

where the inf is taken on all possible decompositions

$$f = \sum_{|\alpha|\le 1} D_x^\alpha f_\alpha, \quad f_\alpha \in L_\psi(Q).$$

The space F_0 is then given by

$$F_0 = \Big\{ f = \sum_{|\alpha|\le 1} D_x^\alpha f_\alpha : f_\alpha \in E_\psi(Q) \Big\}$$

and is denoted by $F_0 = W^{-1,x}E_\psi(Q)$.

3. Essential assumptions

Let Ω be a bounded open subset of $\mathbb{R}^N$ satisfying the segment property and $T > 0$ we denote $Q = \Omega \times [0,T]$, and let φ and γ be two Musielak-Orlicz functions such that $\gamma \prec\prec \varphi$.

Let $A : D(A) \subset W_0^{1,x}L_\varphi(Q) \longrightarrow W^{-1,x}L_\psi(Q)$ be a mapping given by

$$A(u) = -\mathrm{div}(a(x,t,u,\nabla u)),$$

where $a : a(x,t,s,\xi) : \Omega \times [0,t] \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a Carathéodory function satisfying, for a.e $(x,t) \in Q$ and for all $s \in \mathbb{R}$ and all $\xi, \xi' \in \mathbb{R}^N$, $\xi \neq \xi'$:

$$|a(x,t,s,\xi)| \le \beta\Big(h_1(x,t) + \psi_x^{-1}\gamma(x,\nu|s|) + \psi_x^{-1}\varphi(x,\nu|\xi|) \Big) \tag{3.1}$$

$$\Big(a(x,t,s,\xi)-a(x,t,s,\xi')\Big)(\xi-\xi')>0 \tag{3.2}$$

$$a(x,t,s,\xi).\xi \geq \alpha\varphi(x,|\xi|) \tag{3.3}$$

where $c(x,t)$ a positive function, $c(x,t)\in E_{\psi}(Q)$ and positive constants ν,α. Furthermore, let $g(x,t,s,\xi):\Omega\times]0,T[\times\mathbb{R}\times\mathbb{R}^N\longrightarrow\mathbb{R}$ be a Caratheodory function such that for a.e. $(x,t)\in\Omega\times]0,T[$ and for all $s\in\mathbb{R}$, $\xi\in\mathbb{R}^N$, the following conditions

$$|g(x,t,s,\xi)|\leq b(|s|)\Big(h_2(x,t)+\varphi(x,|\xi|)\Big), \tag{3.4}$$

$$g(x,t,s,\xi)s\geq 0, \tag{3.5}$$

are satisfied, where $b:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ is a continuous positive function which belongs to $L^1(\mathbb{R})$ and $h_2(x,t)\in L^1(Q)$.

$$f\in L^1(Q) \quad \text{and} \quad F\in(E_{\psi}(Q))^N. \tag{3.6}$$

$$u_0\in L^1(\Omega). \tag{3.7}$$

4. Some technical Lemmas

Lemma 4.1. *[5]. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and let φ and ψ be two complementary Musielak-Orlicz functions which satisfy the following conditions:*

i) There exist a constant $c>0$ such that $\inf_{x\in\Omega}\varphi(x,1)\geq c$.

ii) There exist a constant $A>0$ such that for all $x,y\in\Omega$ with $|x-y|\leq\frac{1}{2}$ we have

$$\frac{\varphi(x,t)}{\varphi(y,t)}\leq t^{\left(\frac{A}{log(\frac{1}{|x-y|})}\right)}, \quad \forall t\geq 1. \tag{4.1}$$

iii)

$$\textit{If } D\subset\Omega \textit{ is a bounded measurable set, then } \int_D\varphi(x,1)dx<\infty. \tag{4.2}$$

iv) There exist a constant $C>0$ such that $\psi(x,1)\leq C$ a.e in Ω.

Under this assumptions, $\mathcal{D}(\Omega)$ is dense in $L_{\varphi}(\Omega)$ with respect to the modular topology, $\mathcal{D}(\Omega)$ is dense in $W_0^1L_{\varphi}(\Omega)$ for the modular convergence and $\mathcal{D}(\overline{\Omega})$ is dense in $W^1L_{\varphi}(\Omega)$ the modular convergence.

Consequently, the action of a distribution S in $W^{-1}L_{\psi}(\Omega)$ on an element u of $W_0^1L_{\varphi}(\Omega)$ is well defined. It will be denoted by $<S,u>$.

Lemma 4.2. *[6]. Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let φ be a Musielak- Orlicz function and let $u \in W_0^1 L_\varphi(\Omega)$. Then $F(u) \in W_0^1 L_\varphi(\Omega)$. Moreover, if the set D of discontinuity points of F' is finite, we have*

$$\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} \text{ a.e in } \{x \in \Omega : u(x) \in D\}. \\ 0 \quad \text{a.e in } \{x \in \Omega : u(x) \notin D\}. \end{cases}$$

Lemma 4.3. *Let $(f_n), f \in L^1(\Omega)$ such that*

i) $f_n \geq 0$ *a.e in* Ω.

ii) $f_n \longrightarrow f$ *a.e in* Ω.

iii) $\int_\Omega f_n(x)dx \longrightarrow \int_\Omega f(x)dx$.

then $f_n \longrightarrow f$ strongly in $L^1(\Omega)$.

Lemma 4.4 (Jensen inequality)**.** *[19]. Let $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ a convex function and $g : \Omega \longrightarrow \mathbb{R}$ is function measurable, then*

$$\varphi\left(\int_\Omega g \, d\mu\right) \leq \int_\Omega \varphi \; o \; g \, d\mu.$$

Lemma 4.5 (Poincaré inequality)**.** *[11].Let φ a Musielak Orlicz function which satisfies the assumptions of lemma 4.1, suppose that $\varphi(x,t)$ decreases with respect of one of coordinate of x.*
Then, that exists a constant $c > 0$ depends only of Ω such that

$$\int_\Omega \varphi(x, |u(x)|)dx \leq \int_\Omega \varphi(x, c|\nabla u(x)|)dx, \quad \forall u \in W_0^1 L_\varphi(\Omega). \tag{4.3}$$

Proof Since $\varphi(x,t)$ decreases with respect to one of coordinates of x , there exists $i_0 \in \{1, \ldots, N\}$ such that the function $\sigma \longrightarrow \varphi(x_1, ..., x_{i_0-1}, \sigma, x_{i_0+1}, ..., x_N, t)$ is decreasing for every $x_1, ..., x_{i_0-1}, x_{i_0+1}, ..., x_N \in \mathbb{R}$ and $\forall t > 0$.
To prove our result, it suffices to show that

$$\int_\Omega \varphi(x, |u(x)|)dx \leq \int_\Omega \varphi\Big(x, 2d\Big|\frac{\partial u}{\partial x_{i_0}}(x)\Big|\Big)dx, \quad \forall u \in W_0^1 L_\varphi(\Omega). \tag{4.4}$$

with $d = \max\Big(\text{diam}(\Omega), 1\Big)$ and $\text{diam}(\Omega)$ is the diameter of Ω.
First, suppose that $u \in \mathcal{D}(\Omega)$, then so by the Jensen integral inequality we obtain

$$\begin{aligned}
&\varphi(x, |u(x_1, ..., x_N)|) \\
&\leq \varphi\Big(x, \int_{-\infty}^{x_{i_0}} \Big|\frac{\partial u}{\partial x_{i_0}}\Big|(x_1, ..., x_{i_0-1}, \sigma, x_{i_0+1}, ..., x_N)d\sigma\Big), \\
&\leq \frac{1}{d}\int_{-\infty}^{+\infty} \varphi\Big(x, d\Big|\frac{\partial u}{\partial x_{i_0}}\Big|(x_1, ..., x_{i_0-1}, \sigma, x_{i_0+1}, ..., x_N)\Big)d\sigma \\
&\leq \frac{1}{d}\int_{-\infty}^{+\infty} f(\sigma)d\sigma,
\end{aligned}$$

where $f(\sigma)=\varphi\Big(x_1,...,x_{i_0-1},\sigma,x_{i_0+1},...,x_N,d\Big|\frac{\partial u}{\partial x_{i_0}}\Big|(x_1,...,x_{i_0-1},\sigma,x_{i_0+1},...,x_N)\Big)$.

By integrating with respect to x, we get

$$\begin{aligned}&\int_\Omega \varphi(x,|u(x_1,...,x_N)|)dx\\ &\leq \int_\Omega \frac{1}{d}\int_{-\infty}^{+\infty} f(\sigma)d\sigma dx,\end{aligned}$$

since $\varphi\Big(x_1,...,x_{i_0-1},\sigma,x_{i_0+1},...,x_N,d\Big|\frac{\partial u}{\partial x_{i_0}}\Big|(x_1,...,x_{i_0-1},\sigma,x_{i_0+1},...,x_N)\Big)$ independent of x_{i_0}, we can get it out of the integral to respect of x_{i_0} and by the fact that σ is arbitrary, then by Fubini's Theorem we get

$$\int_\Omega \varphi(x,|u(x)|)dx \leq \int_\Omega \varphi\Big(x,d\Big|\frac{\partial u}{\partial x_{i_0}}\Big|(x)\Big)dx,\quad \forall u\in \mathcal{D}(\Omega). \tag{4.5}$$

For $u\in W_0^1L_\varphi(\Omega)$ according to Lemma 4.1, we have the existence of $u_n\in\mathcal{D}(\Omega)$ and $\lambda>0$ such that

$$\overline{\varrho}_{\varphi,\Omega}\Big(\frac{u_n-u}{\lambda}\Big)=0,\qquad \text{as } n\longrightarrow+\infty,$$

hence

$$\begin{cases}\int_\Omega \varphi\Big(x,\frac{|u_n-u|}{\lambda}\Big)dx\longrightarrow 0, \qquad \text{as } n\longrightarrow+\infty,\\ \int_\Omega \varphi\Big(x,\frac{|\nabla u_n-\nabla u|}{\lambda}\Big)dx\longrightarrow 0, \qquad \text{as } n\longrightarrow+\infty,\\ u_n\longrightarrow u \quad \text{a.e in } \Omega, \quad (\text{ for a subsequence still denote } u_n).\end{cases}$$

Then, we have

$$\begin{aligned}\int_\Omega \varphi\Big(x,\frac{|u(x)|}{2d\lambda}\Big)dx &\leq \liminf_{n\longrightarrow+\infty}\int_\Omega \varphi\Big(x,\frac{|u_n(x)|}{2d\lambda}\Big)dx\\ &\leq \liminf_{n\longrightarrow+\infty}\int_\Omega \varphi\Big(x,\frac{1}{2\lambda}\Big|\frac{\partial u_n}{\partial x_{i_0}}(x)\Big|\Big)dx\\ &= \liminf_{n\longrightarrow+\infty}\int_\Omega \varphi\Big(x,\frac{1}{2\lambda}\Big|\frac{\partial u_n}{\partial x_{i_0}}(x)-\frac{\partial u}{\partial x_{i_0}}(x)+\frac{\partial u}{\partial x_{i_0}}(x)\Big|\Big)dx\\ &\leq \frac{1}{2}\liminf_{n\longrightarrow+\infty}\int_\Omega \varphi\Big(x,\frac{1}{\lambda}\Big|\frac{\partial u_n}{\partial x_{i_0}}(x)-\frac{\partial u}{\partial x_{i_0}}(x)\Big|\Big)dx\\ &+ \frac{1}{2}\int_\Omega \varphi\Big(x,\frac{1}{\lambda}\Big|\frac{\partial u}{\partial x_{i_0}}(x)\Big|\Big)dx\\ &\leq \int_\Omega \varphi\Big(x,\frac{1}{\lambda}\Big|\frac{\partial u}{\partial x_{i_0}}(x)\Big|\Big)dx.\end{aligned}$$

Hence

$$\int_\Omega \varphi\Big(x,|u(x)|\Big)dx\leq \int_\Omega \varphi\Big(x,2d\Big|\frac{\partial u}{\partial x_{i_0}}(x)\Big|\Big)dx,\quad \forall u\in W_0^1L_\varphi(\Omega).$$

□

Lemma 4.6 (The Nemytskii Operator)**.** *Let Ω be an open subset of $\mathbb{R}^N$ with finite measure and let φ and ψ be two Musielak Orlicz functions. Let $f : \Omega \times \mathbb{R}^p \longrightarrow \mathbb{R}^q$ be a Carathodory function such that for a.e. $x \in \Omega$ and all $s \in \mathbb{R}^p$:*

$$|f(x, s)| \leq c(x) + k_1 \psi_x^{-1} \varphi(x, k_2|s|). \tag{4.6}$$

where k_1 and k_2 are real positives constants and $c(.) \in E_\psi(\Omega)$.
Then the Nemytskii Operator N_f defined by $N_f(u)(x) = f(x, u(x))$ is continuous from

$$\mathcal{P}\Big(E_\varphi(\Omega), \frac{1}{k_2}\Big)^p = \prod \Big\{u \in L_\varphi(\Omega) : d(u, E_\varphi(\Omega)) < \frac{1}{k_2}\Big\}.$$

into $(L_\psi(\Omega))^q$ for the modular convergence.
Furthermore if $c(\cdot) \in E_\gamma(\Omega)$ and $\gamma \prec\prec \psi$ then N_f is strongly continuous from $\mathcal{P}\Big(E_\varphi(\Omega), \frac{1}{k_2}\Big)^p$ to $(E_\gamma(\Omega))^q$

5. Approximation and trace results

In this section, Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ with the segment property and I is a subinterval of $\mathbb{R}$ (both possibly unbounded) and $Q = \Omega \times I$. It is easy to see that Q also satisfies Lipschitz domain. We say that $u_n \longrightarrow u$ in $W^{-1,x}L_\psi(Q) + L^2(Q)$ for the modular convergence if we can write

$$u_n = \sum_{|\alpha| \leq 1} D_x^\alpha u_n^\alpha + u_n^0 \text{ and } u = \sum_{|\alpha| \leq 1} D_x^\alpha u^\alpha + u^0,$$

with $u_n^\alpha \longrightarrow u^\alpha$ in $L_\psi(Q)$ for the modular convergence for all $|\alpha| \leq 1$, and $u_n^0 \longrightarrow u^0$ strongly in $L^2(Q)$. We shall prove the following approximation theorem, which plays a fundamental role when the existence of solutions for parabolic problems is proved. [2] Let φ be an Musielak-Orlicz function satisfies the assumption (4.1). If $u \in W^{1,x}L_\varphi(Q)$ (respectively $u \in W_0^{1,x}L_\varphi(Q)$) and $\frac{\partial u}{\partial t} \in W^{-1,x}L_\psi(Q) + L^1(Q)$, then there exists a sequence $(v_j) \in \mathcal{D}(\overline{Q})$ (respectively $\mathcal{D}(\overline{I}, \mathcal{D}(\Omega))$) such that $v_j \longrightarrow u$ in $W^{1,x}L_\varphi(Q)$ and $\frac{\partial v_j}{\partial t} \longrightarrow \frac{\partial u}{\partial t}$ in $W^{-1,x}L_\psi(Q) + L^1(Q)$ for the modular convergence.

Lemma 5.1. *[2] Let $a < b \in \mathbb{R}$ and let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$. Then*

$$\Big\{u \in W_0^{1,x}L_\varphi(\Omega\times]a, b[) \;:\; \frac{\partial u}{\partial t} \in W^{-1,x}L_\psi(\Omega\times]a, b[) + L^1(\Omega\times]a, b[)\Big\}$$

is a subset of $\mathcal{C}(]a, b[, L^1(\Omega))$.

In order to deal with the time derivative, we introduce a time mollification of a function $u \in W_0^{1,x}L_\varphi(Q)$.
Thus we define, for all $\mu > 0$ and all $(x, t) \in Q$

$$u_\mu(x, t) = \int_{-\infty}^{t} \tilde{u}(x, \sigma) \exp(\mu(\sigma - t)) d\sigma \tag{5.1}$$

where $\tilde{u}(x,t) = u(x,t)\chi_{[0,T]}(t)$.
Throughout the paper the index ì always indicates this mollification.

Lemma 5.2. *.[2] If $u \in L_\varphi(Q)$ then u_μ is measurable in Q and $\frac{\partial u_\mu}{\partial t} = \mu(u - u_\mu)$ and if $u \in K_\varphi(Q)$ then*

$$\int_Q \varphi(x, u_\mu)dxdt \leq \int_Q \varphi(x,u)dxdt.$$

Lemma 5.3. *1. If $u \in L_\varphi(Q)$ then $u_\mu \longrightarrow u$ for the modular convergence in $L_\varphi(Q)$ as $\mu \longrightarrow \infty$.*

2. If $u \in W_0^{1,x}L_\varphi(Q)$ then $u_\mu \longrightarrow u$ for the modular convergence in $W_0^{1,x}L_\varphi(Q)$ as $\mu \longrightarrow \infty$.

Proof

1. Let $(v_k)_k \subset \mathcal{D}(Q)$ such that $v_k \longrightarrow u$ in $L_\varphi(Q)$ for the modular convergence. Let $\lambda > 0$ large enough such that

$$\frac{u}{\lambda} \in K_\varphi(Q), \quad \int_Q \varphi\Big(x, \frac{|v_k - u|}{\lambda}\Big)dxdt \longrightarrow 0 \text{ as } k \longrightarrow +\infty.$$

On the one hand, for a.e $(x,t) \in Q$, we have

$$\Big|(v_k)_\mu(x,t) - v_k(x,t)\Big| = \frac{1}{\mu}\Big|\frac{\partial v_k}{\partial t}(x,t)\Big| \leq \Big\|\frac{\partial v_k}{\partial t}\Big\|_{L^\infty(Q)}$$

On the other hand, one has

$$\begin{aligned}
\int_Q \varphi\Big(x, \frac{|u_\mu - u|}{3\lambda}\Big)dxdt \quad &\leq \quad \frac{1}{3}\int_Q \varphi\Big(x, \frac{|u_\mu - (v_k)_\mu|}{\lambda}\Big)dxdt \\
&+ \quad \frac{1}{3}\int_Q \varphi\Big(x, \frac{|(v_k)_\mu - v_k|}{\lambda}\Big)dxdt \\
&+ \quad \frac{1}{3}\int_Q \varphi\Big(x, \frac{|v_k - u|}{\lambda}\Big)dxdt \\
&\leq \quad \frac{1}{3}\int_Q \varphi\Big(x, \frac{|(u - v_k)_\mu|}{\lambda}\Big)dxdt \\
&+ \quad \frac{1}{3}\int_Q \varphi\Big(x, \frac{|(v_k)_\mu - v_k|}{\lambda}\Big)dxdt \\
&+ \quad \frac{1}{3}\int_Q \varphi\Big(x, \frac{|v_k - u|}{\lambda}\Big)dxdt.
\end{aligned}$$

This implies that

$$\begin{aligned}
\int_Q \varphi\Big(x, \frac{|u_\mu - u|}{3\lambda}\Big)dxdt \leq \qquad & \tfrac{2}{3}\textstyle\int_Q \varphi\Big(x, \frac{|v_k - u|}{\lambda}\Big)dxdt \\
&+ \textstyle\int_Q \varphi\Big(x, \frac{1}{\lambda\mu}\Big\|\frac{\partial v_k}{\partial t}\Big\|_{L^\infty(Q)}\Big)dxdt.
\end{aligned}$$

Let $\varepsilon > 0$ there exists $k_0 > 0$ such that $\forall k > k_0$, we have

$$\int_Q \varphi\Big(x, \frac{|v_k - u|}{\lambda}\Big) dxdt < \varepsilon$$

and there exists $\mu_0 > 0$ such that $\forall \mu > \mu_0$ and for all $k > k_0$

$$\frac{1}{\lambda\mu}\Big\|\frac{\partial v_k}{\partial t}\Big\|_{L^\infty(Q)} \leq 1$$

Then, we get

$$\int_Q \varphi\Big(x, \frac{|u_\mu - u|}{3\lambda}\Big) dxdt \leq \varepsilon + \frac{1}{\lambda\mu}\Big\|\frac{\partial v_k}{\partial t}\Big\|_{L^\infty(Q)} T \int_\Omega \varphi(x,1) dxdt$$

Finely, by using (iii) of Lemma 4.1 and by letting $\mu \longrightarrow +\infty$, there exits $\mu_1 > 0$ such that

$$\int_Q \varphi\Big(x, \frac{|u_\mu - u|}{3\lambda}\Big) dxdt \leq \varepsilon, \quad \text{for all } \mu > \mu_1.$$

2. Since for all indice α such that $|\alpha| \leq 1$, we have $D_x^\alpha(u_\mu) = (D_x^\alpha u)_\mu$, consequently, the first part above applied on each $D_x^\alpha u$, gives the result.

□

Remark 5.1. If $u \in E_\varphi(Q)$, we can choose λ arbitrary small since $\mathcal{D}(Q)$ is (norm) dense in $E_\varphi(Q)$.
Thus, for all $\lambda > 0$, we have

$$\int_Q \varphi\Big(x, \frac{|u_\mu - u|}{\lambda}\Big) dxdt \quad \text{as } \mu \longrightarrow +\infty.$$

and $u_\mu \longrightarrow u$ strongly in $E_\varphi(Q)$.Idem for $W^{1,x}E_\varphi(Q)$.

Lemma 5.4. *If $u_n \longrightarrow u$ in $W_0^{1,x}L_\varphi(Q)$ strongly (resp., for the modular convergence), then $(u_n)_\mu \longrightarrow u_\mu$ strongly (resp., for the modular convergence).*

Proof For all $\lambda > 0$ (resp., for some $\lambda > 0$),

$$\int_Q \varphi\Big(x, \frac{|D_x^\alpha((u_n))_\mu - D_x^\alpha(u)_\mu|}{\lambda}\Big) dxdt \longrightarrow \int_Q \varphi\Big(x, \frac{|D_x^\alpha u_n - D_x^\alpha u|}{\lambda}\Big) dxdt \longrightarrow 0,$$

as $n \longrightarrow +\infty$. Then $(u_n)_\mu \longrightarrow u_\mu$ in $W^{1,x}L_\varphi(Q)$ strongly (resp., for the modular convergence). □

6. Compactness Results

For each $h>0$, define the usual translated $\tau_h f$ of the function f by $\tau_h f(t) = f(t+h)$.
If f is defined on $[0,T]$ then $\tau_h f$ is defined on $[-h, T-h]$.
First of all, recall the following compactness results proved by the authors in [2].

Lemma 6.1. *Let φ be a Musielak function. Let Y be a Banach space such that the following continuous imbedding holds $L^1(\Omega)\subset Y$. Then for all $\varepsilon>0$ and all $\lambda>0$, there is $C_\varepsilon>0$ such that for all $u\in W_0^{1,x}L_\varphi(Q)$ with $\frac{|\nabla u|}{\lambda}\in K_\varphi(Q)$, we have*

$$\|u\|_1 \leq \varepsilon\lambda\bigg(\int_Q \varphi\Big(x,\frac{|\nabla u|}{\lambda}\Big)dxdt + T\bigg) + C_\varepsilon\|u\|_{L^1(0,T,Y)}.$$

Proof Since $W_0^1L_\varphi(\Omega)\subset L^1(\Omega)$ with compact imbedding, then for all $\varepsilon>0$, there is $C_\varepsilon>0$ such that for all $v\in W_0^1L_\varphi(\Omega)$

$$\|v\|_{L^1(\Omega)} \leq \varepsilon\|\nabla u\|_{L_\varphi(\Omega)} + C_\varepsilon\|v\|_Y. \tag{6.1}$$

Indeed, if the above assertion holds false, there is $\varepsilon_0>0$ and $v_n\in W_0^1L_\varphi(\Omega)$ such that

$$\|v_n\|_{L^1(\Omega)} \geq \varepsilon_0\|\nabla v_n\|_{L_\varphi(\Omega)} + n\|v_n\|_Y.$$

This gives, by setting $w_n = \frac{v_n}{\|\nabla v_n\|_{L_\varphi(\Omega)}}$,

$$\|w_n\|_{L^1(\Omega)} \geq \varepsilon_0 + n\|w_n\|_Y, \quad \|\nabla w_n\|_{L_\varphi(\Omega)} = 1.$$

Since $(w_n)_n$ is bounded in $W_0^1L_\varphi(\Omega)$ then for a subsequence

$$w_n \rightharpoonup w \text{ in } W_0^1L_\varphi(\Omega) \text{ for } \sigma(\Pi L_\varphi, \Pi E_\psi) \text{ and strongly in } L^1(\Omega).$$

Thus, $\|w_n\|_{L^1(\Omega)}$ is bounded and $\|w_n\|_Y\to 0$ as $n\to+\infty$.
We conclude $w_n\to 0$ in Y and that $w=0$ implying that $\varepsilon_0 \leq \|w_n\|_{L^1(\Omega)}\to 0$, a contradiction.
Using $v=u(t)$ in (6.1) for all $u\in W_0^{1,x}L_\varphi(Q)$ with $\frac{|\nabla u|}{\lambda}\in K_\varphi(Q)$ and a.e. $t\in[0,T]$, we have

$$\|u(t)\|_{L^1(\Omega)} \leq \varepsilon\|\nabla u(t)\|_{L_\varphi(\Omega)} + C_\varepsilon\|u(t)\|_Y.$$

Since $\int_Q \varphi\Big(x,\frac{|\nabla u(x,t)|}{\lambda}\Big)dxdt < \infty$, we have thanks to Fubini's theorem $\int_\Omega \varphi\Big(x,\frac{|\nabla u(x,t)|}{\lambda}\Big)dx<\infty$ for a.e. $t\in[0,T]$ and then

$$\|\nabla u(t)\|_{L_\varphi(\Omega)} \leq \lambda\bigg(\int_\Omega \varphi\Big(x,\frac{|\nabla u(x,t)|}{\lambda}\Big)dx + 1\bigg),$$

which implies that

$$\|u(t)\|_{L_\varphi(\Omega)} \leq \varepsilon\lambda\bigg(\int_\Omega \varphi\Big(x,\frac{|\nabla u(x,t)|}{\lambda}\Big)dx + 1\bigg) + C_\varepsilon\|u(t)\|_Y.$$

Integrating this over $[0,T]$ yields

$$\|u\|_1 \leq \varepsilon\lambda\bigg(\int_Q \varphi\Big(x, \frac{|\nabla u|}{\lambda}\Big)dxdt + T\bigg) + C_\varepsilon \|u\|_{L^1(0,T,Y)}.$$

□

We also prove the following lemma which allows us to enlarge the space Y whenever necessary.

Lemma 6.2. *If F is bounded in $W_0^{1,x}L_\varphi(Q)$ and is relatively compact in $L^1(0,T,Y)$ then F is relatively compact in $L^1(Q)$ (and also in $E_\gamma(Q)$ for all Musielak function $\gamma \ll \varphi$).*

Proof Let $\varepsilon > 0$ be given. Let $C > 0$ be such that $\int_Q \varphi\Big(x, \frac{|\nabla f|}{C}\Big)dxdt \leq 1$ for all $f \in F$.
By the previous lemma, there exists $C_\varepsilon > 0$ such that for all $u \in W_0^1 L_\varphi(Q)$ with $\frac{|\nabla u|}{C} \in K_\varphi(Q)$,

$$\|u\|_{L^1(Q)} \leq \frac{2\varepsilon C}{4C(1+T)}\bigg(\int_\Omega \varphi\Big(x, \frac{|\nabla u|}{2C}\Big)dx + T\bigg) + C_\varepsilon\|u\|_{L^1(0,T,Y)}.$$

Moreover, there exists a finite sequence $(f_i)_i$ in F satisfying

$$\forall f \in F, \ \ \exists f_i \text{ such that } \|f - f_i\|_{L^1(0,T,Y)} \leq \frac{\varepsilon}{2C_\varepsilon}.$$

So that,

$$\|f - f_i\|_{L^1(Q)} \leq \frac{\varepsilon}{2(1+T)}\bigg(\int_Q \varphi(x, \frac{|\nabla f - \nabla f_i|}{2C} dxdt + T\bigg) + C_\varepsilon\|f - f_i\|_{L^1(0,T,Y)} \leq \varepsilon.$$

and hence F is relatively compact in $L^1(Q)$.
Since $\gamma \ll \varphi$ then by using Vitali's theorem, it is easy to see that F is relatively compact in $E_\gamma(Q)$. □

Remark 6.1. If $F \subset L^1(0,T,B)$ is such that $\Big\{\frac{\partial f}{\partial t} : f \in F\Big\}$ is bounded in $F \subset L^1(0,T,B)$ then $\|\tau_h f - f\|_{L^1(0,T,B)} \longrightarrow 0$ as $h \longrightarrow 0$ uniformly with respect to $f \in F$.

Lemma 6.3. *Let φ be a Musielak function. If F is bounded in $W^{1,x}L_\varphi(Q)$ and $\Big\{\frac{\partial f}{\partial t} : f \in F\Big\}$ is bounded in $W^{-1,x}L_\psi(Q)$, then F is relatively compact in $L^1(Q)$.*

Proof Let γ and θ be Musielak functions such that $\gamma \ll \varphi$ and $\theta \ll \varphi$ near infinity.

For all $0 < t_1 < t_2 < T$ and all $f \in F$, we have

$$\begin{aligned} \left\| \int_{t_1}^{t_2} f(t)dt \right\|_{W_0^1 E_\gamma(\Omega)} &\leq \int_0^T \|f(t)\|_{W_0^1 E_\gamma(\Omega)} dt \\ &\leq C_1 \|f\|_{W_0^{1,x} E_\gamma(Q)} \\ &\leq C_2 \|f\|_{W_0^{1,x} E_\varphi(Q)} \\ &\leq C. \end{aligned}$$

where we have used the following continuous imbedding

$$W_0^{1,x} L_\varphi(Q) \subset W_0^{1,x} E_\gamma(Q) \subset L^1(0, T, W_0^1 L_\varphi(\Omega)).$$

Since the imbedding $W_0^1 L_\gamma(\Omega) \subset L^1(\Omega)$ is compact we deduce that $(\int_{t_1}^{t_2} f(t)dt)_{f\in F}$ is relatively compact in $L^1(\Omega)$ and $W^{-1,1}(\Omega)$ as well.
On the other hand, $\left\{ \frac{\partial f}{\partial t} : f \in F \right\}$ is bounded in $W^{-1,x} L_\psi(Q)$ and $L^1(0, T, W^{-1,1}(\Omega))$ as well, since

$$W^{-1,x} L_\psi(Q) \subset W^{-1,x} E_\theta(Q) \subset L^1(0, T, W^{-1} E_\theta(\Omega)) \subset L^1(0, T, W^{-1,1}(\Omega)),$$

with continuous imbedding. By Remark 3 of [12], we deduce that $\|\tau_h f - f\|_{L^1(0,T,W^{-1,1}(\Omega))} \longrightarrow 0$ uniformly in $f \in F$ when $h \longrightarrow +\infty$ and by using Theorem 2 of [12], F is relatively compact in $L^1(0, T, W^{-1,1}(\Omega))$.
Since $L^1(\Omega) \subset W^{-1,1}(\Omega)$ with continuous imbedding we can apply Lemma 6.2 to conclude that F is relatively compact in $L^1(Q)$. □

Lemma 6.4. *Let φ be a Musielak function.*
Let $(u_n)_n$ be a sequence of $W^{1,x} L_\varphi(Q)$ such that

$$u_n \rightharpoonup u \text{ weakly in } W^{1,x} L_\varphi(Q) \text{ for } \sigma(\Pi L_\varphi, \Pi L_\psi)$$

and

$$\frac{\partial u_n}{\partial t} = h_n + k_n \text{ in } \mathcal{D}'(Q)$$

with $(h_n)_n$ bounded in $W^{-1,x} L_\psi(Q)$ and $(k_n)_n$ bounded in the space $\mathcal{M}(Q)$ set of measures on Q.
then $u_n \longrightarrow u$ strongly in $L^1_{loc}(Q)$.
If further $u_n \in W_0^{1,x} L_\varphi(Q)$ then $u_n \longrightarrow u$ strongly in $L^1(Q)$.

Proof It is easily adapted from that given in [8] by using Theorem 4.4 and Remark 4.3 instead of Lemma 8 of [20]. □

7. Main results

For $k > 0$ we define the truncation at height k: $T_k : \mathbb{R} \longrightarrow \mathbb{R}$ by:

$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k. \\ k\frac{s}{|s|} & \text{if } |s| > k. \end{cases} \tag{7.1}$$

We note also

$$S_k(r) = \int_0^r T_k(\sigma)d\sigma = \begin{cases} \frac{r^2}{2} & \text{if } |r| \le k, \\ k|r| - \frac{r^2}{2} & \text{if } |r| > k. \end{cases} \tag{7.2}$$

We define

$$T_0^{1,\varphi}(Q) = \Big\{ u : \Omega \longrightarrow \mathbb{R} \ \text{ measurable such that } T_k(u) \in W_0^{1,x}L_\varphi(Q)\ \forall k > 0 \Big\}$$

We consider the following boundary value problem

$$(\mathcal{P}) \begin{cases} \frac{\partial u}{\partial t} + A(u) + g(x,t,u,\nabla u) = f - \operatorname{div}(F) & \text{in } Q, \\ u \equiv 0 & \text{on } \partial Q = \partial\Omega \times [0,T], \\ u(\cdot,0) = u_0 & \text{on } \Omega. \end{cases}$$

We will prove the following existence theorem

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$, φ and ψ be two complementary Musielak-Orlicz functions satisfying the assumptions of Lemma 4.1 and $\varphi(x,t)$ decreases with respect to one of coordinate of x, we assume also that (3.1)-(3.6) and (3.7) hold true. Then the problem $(\mathcal{P})$ has at least one entropy solution of the following sense

$$\begin{cases} u \in T_0^{1,\varphi}(Q) \cap W_0^{1,x}L_\varphi(Q),\ S_k(u) \in L^1(Q),\ g(.,u,\nabla u) \in L^1(Q) \\ \displaystyle\int_\Omega S_k(u(T) - v(T))dx + \Big\langle \frac{\partial v}{\partial t}, T_k(u-v) \Big\rangle + \int_Q a(x,t,u,\nabla u)\cdot \nabla T_k(u-v)dxdt \\ \displaystyle + \int_Q g(x,t,u,\nabla u) T_k(u-v)dxdt \\ \displaystyle \le \int_Q f T_k(u-v)dxdt + \int_Q F\cdot\nabla T_k(u-v)dxdt + \int_\Omega S_k(u_0 - v(0))dx \\ \forall v \in W_0^{1,x}L_\varphi(Q) \cap L^\infty(Q) \text{ such that } \dfrac{\partial v}{\partial t} \in W^{-1,x}L_\psi(Q) + L^1(Q). \end{cases}$$

Proof

Step 1 : Approximate problems

Consider the following approximate problem

$$(\mathcal{P}_n) \begin{cases} u_n \in W_0^{1,x}L_\varphi(Q), \quad u_n(.,0) = u_{0n} \text{ in } \partial Q = \partial\Omega \times [0,T], \\ \frac{\partial u_n}{\partial t} - \operatorname{div}(a(x,t,u_n,\nabla u_n)) + g_n(x,t,u_n,\nabla u_n) = f_n - \operatorname{div}(F) \quad \text{in } Q, \end{cases}$$

where we have set $g_n(x,t,s,\xi) = T_n(g(x,t,s,\xi))$. Moreover, the sequence$(f_n) \subset \mathcal{D}(Q)$ is such that $f_n \longrightarrow f$ strongly in $L^1(Q)$ and $\|f_n\|_{L^1(Q)} \le \|f\|_{L^1(Q)}$ and $(u_{0n}) \subset \mathcal{D}(\Omega)$ is such that $u_{0n} \longrightarrow u_0$ strongly in $L^1(\Omega)$ and $\|u_{0n}\|_{L^1(\Omega)} \le \|u_0\|_{L^1(\Omega)}$. Thanks to theorem 5.1 of [2], there exists at least one solution u_n of problem $(\mathcal{P}_n)$.

Step 2 : A priori estimates

In this section we denote by $c_i,\ i=1,2,...$ a constants not depends on k and n. For $k>0$, consider the test function $T_k(u_n)$ in $(\mathcal{P}_n)$, we have

$$\begin{aligned}\int_Q \frac{\partial u_n}{\partial t} T_k(u_n)dxdt &+ \int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u_n)dxdt \\ &+ \int_Q g_n(x,t,u_n,\nabla u_n)T_k(u_n)dxdt \\ &= \int_Q f_n T_k(u_n)dxdt + \int_Q F\cdot\nabla T_k(u_n)dxdt \\ &\leq \|f\|_{L^1(Q)}k + \int_Q F\cdot\nabla T_k(u_n)dxdt. \end{aligned} \tag{7.3}$$

On the one hand, let $0<p<\min(\alpha,1)$, (where α is the constant of (3.3)), then by using the Young's inequality, we have

$$\begin{aligned}\int_Q F\cdot\nabla T_k(u_n)dxdt &= \int_Q \frac{1}{p}F\cdot p\nabla T_k(u_n)dxdt \\ &\leq \int_Q \psi\Big(x,\frac{1}{p}|F|\Big)dxdt \\ &\quad + p\int_Q \varphi\Big(x,|\nabla T_k(u_n)|\Big)dxdt. \end{aligned} \tag{7.4}$$

Combining (7.3) and (7.4), we obtain

$$\begin{aligned}\int_Q \frac{\partial u_n}{\partial t} T_k(u_n)dxdt &+ \int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u_n)dxdt \\ + \int_Q g_n(x,t,u_n,\nabla u_n)T_k(u_n)dxdt &\leq c_1k+c_2+p\int_Q \varphi\Big(x,|\nabla T_k(u_n)|\Big)dxdt. \end{aligned} \tag{7.5}$$

Using now (3.5) and (3.3) which implies that

$$\int_Q \frac{\partial u_n}{\partial t} T_k(u_n)dxdt + \frac{\alpha-p}{\alpha}\int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u_n)dxdt \leq c_1k+c_2. \tag{7.6}$$

In other hand, the first term of the left hand side of the last inequality, reads as

$$\int_Q \frac{\partial u_n}{\partial t} T_k(u_n)dxdt = \int_\Omega S_k(u_n(T))dx - \int_\Omega S_k(u_{0n})dx,$$

Hence

$$\begin{aligned}&\int_\Omega S_k(u_n(T))dx + \frac{\alpha-p}{\alpha}\int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u_n)dxdt \\ &\leq c_1k+c_2+\int_\Omega S_k(u_{0n})dx.\end{aligned}$$

Using the fact that $S_k(\sigma) > 0$, $|S_k(u_{0n}| \leq k|u_{0n}|$, then (7.6) can be write as

$$\frac{\alpha - p}{\alpha} \int_Q a(x, t, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla T_k(u_n) dxdt \leq c_3 k + c_2. \tag{7.7}$$

Hence by using (3.3), we have

$$\int_Q \varphi\Big(x, |\nabla T_k(u_n)|\Big) dxdt \leq c_4 k + c_5.$$

By using the Lemma 4.5, we have

$$\int_Q \varphi\Big(x, \frac{|T_k(u_n)|}{c}\Big) dx \leq \int_Q \varphi\Big(x, |\nabla T_k(u_n)|\Big) dx \leq c_4 k + c_5, \tag{7.8}$$

where c is the constant of Lemma 4.5.
Then $(T_k(u_n))_n$ and $(\nabla T_k(u_n))_n$ are bounded in $L_\varphi(\Omega)$, hence $(T_k(u_n))_n$ is bounded in $W_0^1 L_\varphi(\Omega)$, there exist some $v_k \in W_0^1 L_\varphi(\Omega)$ such that

$$\begin{cases} T_k(u_n) \rightharpoonup v_k \quad \text{weakly in } W_0^1 L_\varphi(\Omega) \text{ for } \sigma(\Pi L_\varphi, \Pi E_\psi) \\ \qquad T_k(u_n) \longrightarrow v_k \quad \text{strongly in } E_\varphi(\Omega). \end{cases} \tag{7.9}$$

Step 3 : Convergence in measure of $(u_n)_n$

Let $k > 0$ large enough, by using (7.8), we have

$$\begin{aligned} meas\{|u_n| > k\} &\leq \frac{1}{\inf\limits_{x \in \Omega} \varphi(x, \frac{k}{\lambda})} \int_{\{|u_n| > k\}} \varphi(x, \frac{k}{\lambda}) dxdt \\ &\leq \frac{1}{\inf\limits_{x \in \Omega} \varphi(x, \frac{k}{\lambda})} \int_Q \varphi(x, \frac{1}{\lambda}|T_k(u_n)|) dxdt \\ &\leq \frac{c_4 k + c_5}{\inf\limits_{x \in \Omega} \varphi(x, \frac{k}{\lambda})} \qquad \forall n, \quad \forall k \geq 0. \end{aligned}$$

Where c_4 is a constant not dependent on k ,hence

$$meas\{|u_n| > k\} \leq \frac{c_4 k + c_5}{\inf\limits_{x \in \Omega} \varphi(x, \frac{k}{\lambda})} \longrightarrow 0 \text{ as } k \longrightarrow \infty.$$

For every $\lambda > 0$ we have

$$\begin{aligned} meas\{|u_n - u_m| > \lambda\} \quad &\leq \quad meas\{|u_n| > k\} \\ &+ \quad meas\{|u_m| > k\} \\ &+ \quad meas\{|T_k(u_n) - T_k(u_m)| > \lambda\}. \end{aligned} \tag{7.10}$$

Consequently, by (7.8) we can assume that $(T_k(u_n))_n$ is a Cauchy sequence in measure in Q.
Let $\varepsilon > 0$, then by (7.10) there exists some $k = k(\varepsilon) > 0$ such that

$$meas\{|u_n - u_m| > \lambda\} < \varepsilon, \qquad \text{for all } n, m \geq h_0(k(\varepsilon), \lambda).$$

Which means that $(u_n)_n$ is a Cauchy sequence in measure in Q, thus converge almost every where to some measurable functions u. Then

$$\begin{cases} T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in } W_0^{1,x}L_\varphi(Q) \text{ for } \sigma(\Pi L_\varphi, \Pi E_\psi) \\ T_k(u_n) \longrightarrow T_k(u) \quad \text{strongly in } E_\varphi(Q). \end{cases} \tag{7.11}$$

Step 4 : Boundedness of $(a(\cdot,\cdot,T_k(u_n),\nabla T_k(u_n)))_n$ ***in*** $(L_\psi(Q))^N$

Let $w \in (E_\varphi(Q)^N$ be arbitrary such that $\|w\|_{\varphi,Q} \leq 1$, by (3.2) we have

$$\Big(a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\frac{w}{\nu})\Big)(\nabla T_k(u_n) - \frac{w}{\nu}) > 0.$$

hence

$$\int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))\frac{w}{\nu}dxdt \leq \int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)dxdt - \int_Q a(x,t,T_k(u_n),\frac{w}{\nu})(\nabla T_k(u_n) - \frac{w}{\nu})dxdt. \tag{7.12}$$

Thanks to (7.7), we have

$$\int_Q a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)dxdt \leq c_3 k + c_2.$$

On the other hand, for λ large enough ($\lambda > \beta$), we have by using (3.1).

$$\begin{aligned}
&\int_Q \psi_x\Big(|\frac{a(x,t,T_k(u_n),\frac{w}{\nu})}{3\lambda}|\Big)dxdt \\
&\leq \int_Q \psi_x\Big(\frac{\beta\Big(d(x) + \psi_x^{-1}(\gamma(x,\nu|T_k(u_n)|)) + \psi_x^{-1}(\varphi(x,|w|))\Big)}{3\lambda}\Big)dxdt \\
&\leq \frac{\beta}{\lambda}\int_Q \psi_x\Big(\frac{h_1(x,t) + \psi_x^{-1}(\gamma(x,\nu|T_k(u_n)|)) + \psi_x^{-1}(\varphi(x,|w|))}{3}\Big)dxdt \\
&\leq \frac{\beta}{3\lambda}\Big(\int_Q \psi_x(h_1(x,t))dxdt + \int_Q \gamma(x,\nu|T_k(u_n)|)dxdt + \int_Q \varphi(x,|w|)dxdt\Big) \\
&\leq \frac{\beta}{3\lambda}\Big(\int_Q \psi_x(h_1(x,t))dxdt + \int_Q \gamma(x,\nu k)dxdt + \int_Q \varphi(x,|w|)dxdt\Big).
\end{aligned}$$

Now, since γ grows essentially less rapidly than φ near infinity ad by using the Remark 2.1, there exists $r(k) > 0$ such that $\gamma(x,\nu k) \leq r(k)\varphi(x,1)$ and so we have

$$\begin{aligned}
&\int_Q \psi_x\Big(\frac{a(x,t,T_k(u_n),\frac{w}{\nu})}{3\lambda}\Big)dxdt \\
&\leq \frac{\beta}{3\lambda}\Big(\int_Q \psi_x(h_1(x,t))dxdt + r(k)\int_Q \varphi(x,1)dxdt + \int_Q \varphi(x,|w|)dxdt\Big).
\end{aligned}$$

hence $a(x,t,T_k(u_n),\frac{w}{\nu})$ is bounded in $(L_\psi(Q))^N$.
Which implies that second term of the right hand side of (7.12) is bounded, consequently we obtain

$$\int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))w\,dxdt \le c_6(k), \quad \text{for all } w \in (L^\varphi(Q))^N \text{ with } \|w\|_{\varphi,Q} \le 1.$$

Hence by the theorem of Banach Steinhous the sequence $(a(x,t,T_k(u_n),\nabla T_k(u_n)))_n$ remains bounded in $(L_\psi(Q))^N$.
Which implies that, for all $k>0$ there exists a function $h_k \in (L_\psi(Q))^N$ such that

$$a(x,t,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup h_k \text{ weakly-star in } (L_\psi(Q))^N \text{ for } \sigma(\Pi L_\psi, \Pi E\varphi). \tag{7.13}$$

Step 5 : Modular convergence of truncations

For the sake of simplicity, we will write only $\varepsilon(n,j,\mu,s)$ to mean all quantities (possibly different) such that

$$\lim_{n\to+\infty}\lim_{j\to+\infty}\lim_{\mu\to+\infty}\lim_{s\to+\infty}\varepsilon(n,j,\mu,s)=0.$$

Since $T_k(u) \in W_0^{1,x}L_\varphi(Q)$ then there exists a sequence $(\alpha_k^j) \subset D(Q)$ such that $(\alpha_k^j) \longrightarrow T_k(u)$ for the modular convergence in $W_0^{1,x}L_\varphi(Q)$. For the remaining of this article, χ_s and $\chi_{j,s}$ will denoted respectively the characteristic functions of the sets $Q_s = \{(x,t)\in Q : |\nabla T_k(u(x,t))| \le s\}$ and $Q_{j,s} = \{(x,t)\in Q : |\nabla T_k(\alpha_k^j(x,t))| \le s\}$.
Taking now $T_\eta(u_n - T_k(\alpha_k^j)_\mu)$ as test function in $(\mathcal{P}_n)$, we get

$$\begin{aligned}
&\int_Q \frac{\partial u_n}{\partial t} T_\eta(u_n - T_k(\alpha_k^j)_\mu)dxdt \\
&+\int_Q a(x,t,u_n,\nabla u_n)\cdot\nabla T_\eta(u_n - T_k(\alpha_k^j)_\mu)dxdt \\
&+\int_Q g_n(x,t,u_n,\nabla u_n)T_\eta(u_n - T_k(\alpha_k^j)_\mu)dxdt \\
&\le \|f\|_1\eta + \int_{\{|T_\eta(u_n - T_k(\alpha_k^j)_\mu)|<\eta\}} F\cdot\nabla T_\eta(u_n - T_k(\alpha_k^j)_\mu)dxdt.
\end{aligned}$$

Let $0<p<\min(1,\alpha)$, by Young's inequality, we have

$$\begin{aligned}
&\int_Q \frac{\partial u_n}{\partial t} T_\eta(u_n - T_k(\alpha_k^j)_\mu)dxdt + \int_Q a(x,t,u_n,\nabla u_n)\cdot\nabla T_\eta(u_n - T_k(\alpha_k^j)_\mu)dxdt \\
&+\int_Q g_n(x,t,u_n,\nabla u_n)T_\eta(u_n - T_k(\alpha_k^j)_\mu)dxdt \\
&\le \|f\|_1\eta + \int_{\{|T_\eta(u_n - T_k(\alpha_k^j)_\mu)|<\eta\}} \psi(x,\frac{|F|}{p})dxdt \\
&+p\int_Q \varphi(x,|\nabla T_\eta(u_n - T_k(\alpha_k^j)_\mu)|)dxdt.
\end{aligned}$$

Using now (3.3) on the last term of the last inequality, we get

$$\int_Q \frac{\partial u_n}{\partial t} T_\eta(u_n - T_k(\alpha_k^j)_\mu)dxdt + \int_Q a(x,t,u_n,\nabla u_n)\cdot\nabla T_\eta(u_n - T_k(\alpha_k^j)_\mu)dxdt$$
$$+\int_Q g_n(x,t,u_n,\nabla u_n)T_\eta(u_n - T_k(\alpha_k^j)_\mu)dxdt$$
$$\leq \|f\|_1\eta + \int_{\{|T_\eta(u_n - T_k(\alpha_k^j)_\mu)|<\eta\}} \psi(x,\frac{|F|}{p})dxdt$$
$$+\frac{p}{\alpha}\int_Q a(x,t,T_{k+\eta}(u_n),\nabla T_{k+\eta}(u_n))\nabla u_n dxdt.$$

Which implies that,

$$\int_Q \frac{\partial u_n}{\partial t} T_\eta(u_n - T_k(\alpha_k^j)_\mu)dxdt \tag{7.14}$$
$$+\frac{\alpha-p}{\alpha}\int_Q a(x,t,T_{k+\eta}(u_n),\nabla T_{k+\eta}(u_n))\nabla u_n dxdt$$
$$+\int_Q g_n(x,t,u_n,\nabla u_n)T_\eta(u_n - T_k(\alpha_k^j)_\mu)dxdt$$
$$\leq c_1\eta + \int_{\{|T_\eta(u_n - T_k(\alpha_k^j)_\mu)|<\eta\}} \psi(x,\frac{|F|}{p})dxdt. \tag{7.15}$$

The first term of the left hand side of the last equality reads as

$$\int_Q \frac{\partial u_n}{\partial t} T_\eta(u_n - T_k(\alpha_k^j)_\mu)dxdt = \int_Q \Big(\frac{\partial u_n}{\partial t} - \frac{\partial T_k(\alpha_k^j)_\mu}{\partial t}\Big) T_\eta(u_n - T_k(\alpha_k^j)_\mu)dxdt$$
$$+\int_Q \frac{\partial T_k(\alpha_k^j)_\mu}{\partial t} T_\eta(u_n - T_k(\alpha_k^j)_\mu)dxdt.$$

The second term of the last equality can be easily to see that is positive and the third term can be written as

$$\int_Q \frac{\partial T_k(\alpha_k^j)_\mu}{\partial t} T_\eta(u_n - T_k(\alpha_k^j)_\mu)dxdt = \mu\int_Q (T_k(\alpha_k^j) - T_k(\alpha_k^j)_\mu)T_\eta(u_n - T_k(\alpha_k^j)_\mu)dxdt,$$

thus by letting $n,j \longrightarrow +\infty$, and since $(\alpha_k^j) \longrightarrow T_k(u)$ a.e. in Q and by using Lebesgue Theorem,

$$\int_Q (T_k(\alpha_k^j) - T_k(\alpha_k^j)_\mu)T_\eta(u_n - T_k(\alpha_k^j)_\mu)dxdt = \int_Q (T_k(u) - T_k(u)_\mu)\cdots$$
$$\cdots T_\eta(u - T_k(u)_\mu)dxdt + \varepsilon(n,j).$$

Consequently

$$\int_Q \frac{\partial T_k(\alpha_k^j)_\mu}{\partial t} T_\eta(u_n - T_k(\alpha_k^j)_\mu)dxdt \geq \varepsilon(n,j).$$

Then, (7.14) can be write as

$$\frac{\alpha-p}{\alpha}\int_Q a(x,t,u_n,\nabla u_n)T_\eta(u_n-T_k(\alpha_k^j)_\mu)dxdt$$
$$+\int_Q g_n(x,t,u_n,\nabla u_n)T_\eta(u_n-T_k(\alpha_k^j)_\mu)dxdt \le c_1\eta$$
$$+\int_{\{|T_\eta(u_n-T_k(\alpha_k^j)_\mu)|<\eta\}}\psi(x,\frac{|F|}{p})dxdt+\varepsilon(n,j). \tag{7.16}$$

On the other hand,

$$\int_Q a(x,t,u_n,\nabla u_n)T_\eta(u_n-T_k(\alpha_k^j)_\mu)dxdt$$
$$=\int_{\{|u_n-T_k(\alpha_k^j)_\mu|<\eta\}} a(x,t,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla T_k(\alpha_k^j)_\mu\chi_{j,s})dxdt$$
$$+\int_{\{|u_n|>k\}\cap\{|u_n-T_k(\alpha_k^j)_\mu|<\eta\}} a(x,t,u_n,\nabla u_n)\cdot\nabla u_n dxdt$$
$$-\int_{\{|u_n|>k\}\cap\{|u_n-T_k(\alpha_k^j)_\mu|<\eta\}} a(x,t,u_n,\nabla u_n)\cdot\nabla T_k(\alpha_j^k)_\mu\chi_{\{|\nabla T_k(\alpha_j^k)|>s\}}dxdt$$

Thus, by using the fact that

$$\int_{\{|u_n|>k\}\cap\{|u_n-T_k(\alpha_k^j)_\mu|<\eta\}} a(x,t,u_n,\nabla u_n)\cdot\nabla u_n dxdt\ge 0$$

We have

$$\frac{\alpha-p}{\alpha}\int_{\{|u_n-T_k(\alpha_k^j)_\mu|<\eta\}} a(x,t,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla T_k(\alpha_k^j)_\mu\chi_{j,s})dxdt$$
$$+\int_Q g_n(x,t,u_n,\nabla u_n)T_\eta(u_n-T_k(\alpha_k^j)_\mu)dxdt$$
$$\le c_1\eta+\int_{\{|T_\eta(u_n-T_k(\alpha_k^j)_\mu)|<\eta\}}\psi(x,\frac{|F|}{p})dxdt$$
$$+\frac{\alpha-p}{\alpha}\int_{\{|u_n|>k\}\cap\{|u_n-T_k(\alpha_k^j)_\mu|<\eta\}} a(x,t,u_n,\nabla u_n)\cdot\nabla T_k(\alpha_j^k)_\mu\chi_{\{|\nabla T_k(\alpha_j^k)|>s\}}dxdt$$
$$+\varepsilon(n,j) \tag{7.17}$$

Now, using (3.5) and the fact that $T_\eta(u_n-T_k(\alpha_k^j)_\mu)$ has the same sign of u_n on the set $\{|u_n| > k\}$, we get

$$\begin{aligned}
&\frac{\alpha - p}{\alpha}\int_{\{|u_n - T_k(\alpha_k^j)_\mu| < \eta\}} a(x,t,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n) - \nabla T_k(\alpha_k^j)_\mu \chi_{j,s})dxdt \\
&+\int_{\{|u_n|\le k\}} g_n(x,t,u_n,\nabla u_n)T_\eta(u_n - T_k(\alpha_k^j)_\mu)dxdt \\
&\le c_1\eta + \int_{\{|T_\eta(u_n - T_k(\alpha_k^j)_\mu)| < \eta\}} \psi(x,\frac{|F|}{p})dxdt \\
&+\frac{\alpha - p}{\alpha}\int_{\{|u_n|>k\}\cap\{|u_n - T_k(\alpha_k^j)_\mu| < \eta\}} a(x,t,u_n,\nabla u_n)\cdot\nabla T_k(\alpha_j^k)_\mu \chi_{\{|\nabla T_k(\alpha_j^k)|>s\}}dxdt \\
&+\varepsilon(n,j)
\end{aligned} \tag{7.18}$$

Hence, by using (3.4), we get

$$\begin{aligned}
&\frac{\alpha - p}{\alpha}\int_{\{|u_n - T_k(\alpha_k^j)_\mu| < \eta\}} a(x,t,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n) - \nabla T_k(\alpha_k^j)_\mu \chi_{j,s})dxdt \\
&\le c_1\eta + \int_{\{|T_\eta(u_n - T_k(\alpha_k^j)_\mu)| < \eta\}} \psi(x,\frac{|F|}{p})dxdt \\
&+\frac{\alpha - p}{\alpha}\int_{\{|u_n|>k\}\cap\{|u_n - T_k(\alpha_k^j)_\mu| < \eta\}} a(x,t,u_n,\nabla u_n)\cdot\nabla T_k(\alpha_j^k)_\mu \chi_{\{|\nabla T_k(\alpha_j^k)|>s\}}dxdt \\
&+\varepsilon(n,j) \\
&+\int_{\{|u_n|\le k\}} b_k\Big(h_2(x,t) + \varphi(x,|\nabla T_k(u_n)|)\Big)|T_\eta(u_n - T_k(\alpha_k^j)_\mu)|dxdt,
\end{aligned} \tag{7.19}$$

where $b_k = \sup\{b(s) : |s| \le k\}$.
Using now (7.8), there exists a constant $c_3 > 0$ depends on k such that

$$\begin{aligned}
&\int_{\{|u_n - T_k(\alpha_k^j)_\mu| < \eta\}} a(x,t,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n) - \nabla T_k(\alpha_k^j)_\mu \chi_{j,s})dxdt \\
&\le c_3\eta + \int_{\{|T_\eta(u_n - T_k(\alpha_k^j)_\mu)| < \eta\}} \psi(x,\frac{|F|}{p})dxdt \\
&+\int_{\{|u_n|>k\}\cap\{|u_n - T_k(\alpha_k^j)_\mu| < \eta\}} a(x,t,u_n,\nabla u_n)\cdot\nabla T_k(\alpha_j^k)_\mu \chi_{\{|\nabla T_k(\alpha_j^k)|>s\}}dxdt \\
&+\varepsilon(n,j).
\end{aligned} \tag{7.20}$$

Since $a(x,t,T_{k+\eta}(u_n),\nabla T_{k+\eta}(u_n)) \rightharpoonup h_{k+\eta}$ weakly-star in $(L_\psi(Q))^N$ for $\sigma(\Pi L_\psi, \Pi E_\varphi)$, then

$$\int_{\{|u_n|>k\}\cap\{|u_n-T_k(\alpha_k^j)_\mu|<\eta\}} a(x,t,u_n,\nabla u_n)\cdot\nabla T_k(\alpha_j^k)_\mu\chi_{\{|\nabla T_k(\alpha_j^k)|>s\}}dxdt$$
$$= \int_{\{|u|>k\}\cap\{|u-T_k(\alpha_k^j)_\mu|<\eta\}} h_{k+\eta}\cdot\nabla T_k(\alpha_j^k)_\mu\chi_{\{|\nabla T_k(\alpha_j^k)|>s\}}dxdt+\varepsilon(n).$$

Now, letting j to infinity, we obtain

$$\int_{\{|u_n|>k\}\cap\{|u_n-T_k(\alpha_k^j)_\mu|<\eta\}} a(x,t,u_n,\nabla u_n)\cdot\nabla T_k(\alpha_j^k)_\mu\chi_{\{|\nabla T_k(\alpha_j^k)|>s\}}dxdt$$
$$= \int_{\{|u|>k\}\cap\{|u-T_k(u)_\mu|<\eta\}} h_{k+\eta}\cdot\nabla T_k(u)_\mu\chi_{\{|\nabla T_k(u)|>s\}}dxdt+\varepsilon(n,j).$$

Hence, we get

$$\int_{\{|u_n|>k\}\cap\{|u_n-T_k(\alpha_k^j)_\mu|<\eta\}} a(x,t,u_n,\nabla u_n)\cdot\nabla T_k(\alpha_j^k)_\mu\chi_{\{|\nabla T_k(\alpha_j^k)|>s\}}dxdt$$
$$= \int_{\{|u|>k\}\cap\{|u-T_k(u)|<\eta\}} h_{k+\eta}\cdot\nabla T_k(u)\chi_{\{|\nabla T_k(u)|>s\}}dxdt+\varepsilon(n,j,\mu)$$
$$=\varepsilon(n,j,\mu,s).$$

Then (7.20) becomes

$$\int_{\{|u_n-T_k(\alpha_k^j)_\mu|<\eta\}} a(x,t,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla T_k(\alpha_k^j)_\mu\chi_{j,s})dxdt$$
$$\leq c_3\eta+\int_{\{|T_\eta(u_n-T_k(\alpha_k^j)_\mu)|<\eta\}}\psi(x,\frac{|F|}{p})dxdt+\varepsilon(n,j,\mu,s). \tag{7.21}$$

On the other hand, remark that

$$\int_{\{|u_n-T_k(\alpha_k^j)_\mu|<\eta\}} a(x,t,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla T_k(\alpha_k^j)_\mu\chi_{j,s})dxdt$$
$$=\int_{\{|u_n-T_k(\alpha_k^j)_\mu|<\eta\}} a(x,t,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla T_k(\alpha_k^j)\chi_{j,s})dxdt$$
$$+\int_{\{|u_n-T_k(\alpha_k^j)_\mu|<\eta\}} a(x,t,T_k(u_n),\nabla T_k(u_n))\cdots$$
$$\cdots(\nabla T_k(\alpha_j^k)\chi_{j,s}-\nabla T_k(\alpha_k^j)_\mu\chi_{j,s})dxdt \tag{7.22}$$

for the second term of the last inequality, we have obviously that

$$\int_{\{|u_n-T_k(\alpha_k^j)_\mu|<\eta\}} a(x,t,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(\alpha_j^k)-\nabla T_k(\alpha_k^j)_\mu\chi_{j,s})dxdt$$
$$=\varepsilon(n,j,\mu,s).$$

Then (7.21) becomes

$$\int_{\{|u_n-T_k(\alpha_k^j)_\mu|<\eta\}} a(x,t,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla T_k(\alpha_k^j)\chi_{j,s})dxdt$$
$$\leq c_3\eta+\int_{\{|T_\eta(u_n-T_k(\alpha_k^j)_\mu)|<\eta\}}\psi(x,\frac{|F|}{p})dxdt+\varepsilon(n,j,\mu,s). \tag{7.23}$$

Hence by letting η to zero, we get

$$\int_{\{|u_n-T_k(\alpha_k^j)_\mu|<\eta\}} a(x,t,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla T_k(\alpha_k^j)\chi_{j,s})dxdt$$
$$\leq \varepsilon(n,j,\mu,s,\eta). \tag{7.24}$$

Now, let $0<\theta<1$, by applying the Young's inequality with $p=\frac{1}{\theta}$ and $\frac{1}{1-\theta}$, $y_n=(x,t,T_k(u_n),\nabla T_k(u_n))$, $y=(x,t,T_k(u_n),\nabla T_k(u))$, we get

$$\int_{Q_\tau\cap\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}}\Big(\Big[a(y_n)-a(y)\Big]\times\Big[\nabla T_k(u_n)-\nabla T_k(u)\Big]\Big)^\theta dxdt$$
$$=\int_{Q_\tau}\Big(\Big[a(y_n)-a(y)\Big]\times\Big[\nabla T_k(u_n)-\nabla T_k(u)\Big]\Big)^\theta\chi_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}}dxdt$$
$$\leq c\text{meas}\Big\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\Big\}^{\frac{1}{1-\theta}}$$
$$+c\Big(\int_{Q_\tau\cap\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}}\Big[a(y_n)-a(y)\Big]\times\Big[\nabla T_k(u_n)-\nabla T_k(u)\Big]dxdt\Big)^\theta. \tag{7.25}$$

But we have for $s>\tau$, $y_\chi=(x,t,T_k(u_n),\nabla T_k(u)\chi_s)$ and $y_\alpha=(x,t,T_k(u_n),\nabla T_k(\alpha_j^k)\chi_{j,s})$, we have

$$\int_{Q_\tau\cap\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}}\Big[a(y_n)-a(y)\Big]\times\Big[\nabla T_k(u_n)-\nabla T_k(u)\Big]dxdt$$
$$\leq\int_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}}\Big[a(y_n)-a(y_\chi)\Big]\times\Big[\nabla T_k(u_n)-\nabla T_k(u)\chi_s\Big]dxdt$$
$$\leq\int_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}}\Big[a(y_n)-a(y_\alpha)\Big]\times\Big[\nabla T_k(u_n)-\nabla T_k(\alpha_j^k)\chi_{j,s}\Big]dxdt$$
$$+\int_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}}a(y_n)\Big[\nabla T_k(\alpha_j^k)\chi_{j,s}-\nabla T_k(u)\chi_s\Big]dxdt$$

$$
\begin{aligned}
&+\int_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}} \Big[a(y_\alpha)-a(y_\chi)\Big]\nabla T_k(u_n)dxdt \\
&-\int_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}} a(y_\alpha)\nabla T_k(\alpha_j^k)\chi_{j,s}dxdt \\
&+\int_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}} a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)\nabla T_k(u)\chi_s dxdt \\
&= J_1+J_2+J_3+J_4+J_5. \qquad (7.26)
\end{aligned}
$$

We shall go to limit as n, j, μ and s to infinity in the last fifth integrals of the last side.
Starting by J_1, one has

$$
J_1 \le \varepsilon(n,j,\mu,\eta) - \int_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}} a(y_\alpha)\Big[\nabla T_k(u_n)-\nabla T_k(\alpha_j^k)\chi_{j,s}\Big]dxdt.
$$

Since $a(y_\alpha)$ converge strongly to $a(x,t,T_k(u),\nabla T_k(\alpha_j^k)\chi_{j,s})$ in $(E_\psi(Q))^N$ and $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ weakly in $(L_\varphi(Q))^N$, then

$$
\begin{aligned}
&\int_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}} a(y_\alpha)\Big[\nabla T_k(u_n)-\nabla T_k(\alpha_j^k)\chi_{j,s}\Big]dxdt \\
&= \int_{\{|T_k(u)-T_k(\alpha_j^k)_\mu|<\eta\}} a(x,t,T_k(u),\nabla T_k(\alpha_j^k)\chi_{j,s})\Big[\nabla T_k(u)-\nabla T_k(\alpha_j^k)\chi_{j,s}\Big]dxdt \\
&+\varepsilon(n).
\end{aligned}
$$

which gives by letting $j \longrightarrow \infty$, $\mu \longrightarrow \infty$ and $s \longrightarrow \infty$ respectively

$$
\begin{aligned}
&\int_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}} a(y_\alpha)\Big[\nabla T_k(u_n)-\nabla T_k(\alpha_j^k)\chi_{j,s}\Big]dxdt \\
&= \int_{\{|T_k(u)-T_k(u)_\mu|<\eta\}} a(x,t,T_k(u),\nabla T_k(u)\chi_s)\Big[\nabla T_k(u)-\nabla T_k(u)\chi_s\Big]dxdt \\
&+\varepsilon(n,j) \\
&= \int_Q a(x,t,T_k(u),\nabla T_k(u)\chi_s)\Big[\nabla T_k(u)-\nabla T_k(u)\chi_s\Big]dxdt+\varepsilon(n,j,\mu) \\
&= \varepsilon(n,j,\mu,s).
\end{aligned}
$$

Finally, we get

$$
J_1 = \varepsilon(n,j,\mu,s,\eta). \qquad (7.27)
$$

Similarly, we get

$$
J_2 = J_3 = J_4 = J_5 = \varepsilon(n,j,\mu,s,\eta). \qquad (7.28)
$$

Combining (7.25)-(7.28), we get

$$
\lim_{n\to+\infty}\int_{Q_\tau}\Big(\Big[a(y_n)-a(y)\Big]\times\Big[\nabla T_k(u_n)-\nabla T_k(u)\Big]\Big)^\theta dxdt = 0.
$$

and, like a same argument in [3], we have

$$\nabla T_k(u_n) \longrightarrow \nabla T_k(u) \text{ as } n \longrightarrow +\infty \text{ for the modular convergence,} \tag{7.29}$$

Step 6 : Compactness of the nonlinearities

In this step, we need to prove that

$$g_n(x,t,u_n,\nabla u_n) \longrightarrow g(x,t,u,\nabla u) \text{ strongly in } L^1(Q). \tag{7.30}$$

By virtue of (7.29), one has

$$g_n(x,t,u_n,\nabla u_n) \longrightarrow g(x,t,u,\nabla u) \text{ a.e. in } Q. \tag{7.31}$$

Let E be measurable subset of Q and let $m > 0$. Using (3.3) and (3.4), we can write

$$\begin{aligned}
&\int_E |g_n(x,t,u_n,\nabla u_n)|dxdt \\
&= \int_{E\cap\{|u_n|\le m\}} |g_n(x,t,u_n,\nabla u_n)|dxdt + \int_{E\cap\{|u_n|>m\}} |g_n(x,t,u_n,\nabla u_n)|dxdt \\
&\le b(m)\int_E h_2(x,t)dxdt + b(m)\int_E a(x,t,T_m(u_n),\nabla T_m(u_n))\cdot\nabla T_m(u_n)dxdt \\
&+\frac{1}{m}\int_E g_n(x,t,u_n,\nabla u_n)u_n dxdt.
\end{aligned}$$

Taking u_n as a test function in $(\mathcal{P}_n)$ and using the same argument as in step 2, there exists a constant $c > 0$ such that

$$\int_E g_n(x,t,u_n,\nabla u_n)u_n dxdt \le c.$$

Then, we have

$$\lim_{m\to+\infty}\frac{1}{m}\int_E g_n(x,t,u_n,\nabla u_n)u_n dxdt = 0.$$

Thanks to (7.29) the sequence $(a(x,t,T_m(u_n),\nabla T_m(u_n))\cdot T_m(u_n))_n$ is equi-integrable, the fact which allows us to get

$$\lim_{|E|\to 0}\sup_n\int_E a(x,t,T_m(u_n),\nabla T_m(u_n))\cdot\nabla T_m(u_n)dxdt = 0.$$

This shows that $g_n(x,t,u_n,\nabla u_n)$ is equi-integrable. Thus, Vitali's theorem implies that $g(x,t,u,\nabla u) \in L^1(Q)$ and

$$g_n(x,t,u_n,\nabla u_n) \longrightarrow g(x,t,u,\nabla u) \text{ strongly in } L^1(Q).$$

Step 7 : Passage to the limit

Let $v \in W_0^{1,x}L_\varphi(Q)$ such that $\frac{\partial v}{\partial t} \in W^{-1,x}L_\psi(Q) + L^1(Q)$.
There exists a prolongation $\overline{v}$ of v such that (see the proof of lemma)

$$\begin{cases} \overline{v} = v \quad \text{on } Q, \\ \overline{v} \in W_0^{1,x}L_\varphi(\Omega\times\mathbb{R}) \cap L^1(\Omega\times\mathbb{R}) \cap L^\infty(\Omega\times\mathbb{R}), \\ \text{and} \quad \frac{\partial \overline{v}}{\partial t} \in W^{-1,x}L_\psi(\Omega\times\mathbb{R}) + L^1(\Omega\times\mathbb{R}). \end{cases}$$

By theorem , there exists a sequence $(w_j)_j$ in $D(\Omega\times\mathbb{R})$ such that $w_j \longrightarrow \overline{v}$ in $W_0^{1,x}L_\varphi(\Omega\times\mathbb{R})$ and $\frac{\partial w_j}{\partial t} \longrightarrow \frac{\partial \overline{v}}{\partial t}$ in $W^{-1,x}L_\psi(\Omega\times\mathbb{R}) + L^1(\Omega\times\mathbb{R})$ for the modular convergence and $\|w_j\|_{\infty,Q} \leq (N+2)\|v\|_{\infty,Q}$.
Using $T_k(u_n - w_j)\chi_{[0,\tau]}$ as a test function in $(\mathcal{P}_n)$, then for every $\tau \in [0,T]$, one has

$$\begin{aligned} &\int_{Q_\tau} \frac{\partial u_n}{\partial t} T_k(u_n - w_j)dxdt \\ &\quad + \int_{Q_\tau} a(x,t,u_n,\nabla u_n)\cdot \nabla T_k(u_n - w_j)dxdt \\ &\qquad + \int_{Q_\tau} g_n(x,t,u_n,\nabla u_n) T_k(u_n - w_j)dxdt \\ &\qquad\quad \leq \int_{Q_\tau} f_n T_k(u_n - w_j)dxdt \\ &\qquad\qquad + \int_{Q_\tau} F\cdot \nabla T_k(u_n - w_j)ddxdt. \end{aligned} \tag{7.32}$$

For the first term of (7.32), we get

$$\begin{aligned} \int_{Q_\tau} \frac{\partial u_n}{\partial t} T_k(u_n - w_j)dxdt &= \left[\int_\Omega T_k(u_n - w_j)dx\right]_0^\tau \\ &\quad + \int_{Q_\tau} \frac{\partial w_j}{\partial t} T_k(u_n - w_j)dxdt \\ &= \left[\int_\Omega T_k(u - w_j)dx\right]_0^\tau \\ &\quad + \int_{Q_\tau} \frac{\partial w_j}{\partial t} T_k(u - w_j)dxdt + \varepsilon(n) \\ &= \int_{Q_\tau} \frac{\partial u}{\partial t} T_k(u - w_j)dxdt. \end{aligned}$$

for the second term of (7.32), we have if $|u_n| > \lambda$ then $|u_n - w_j| \geq |u_n| - \|w_j\|_\infty > k$, therefore $\{|u_n - w_j| \le k\} \subseteq \{|u_n| \le k + (N+2)\|v\|_\infty\}$, which implies that, we get

$$\begin{aligned}
&\liminf_{n\to+\infty} \int_Q a(x,t,u_n,\nabla u_n)\nabla T_k(u_n - w_j)\,dxdt \\
&\ge \int_Q a(y_\|v\|)(\nabla T_{k+(N+2)\|v\|_\infty}(u) - \nabla w_j)\chi_{\{|u-v|\le k\}}\,dxdt, \\
&= \int_Q a(x,t,u,\nabla u)(\nabla u - \nabla w_j)\chi_{\{|u-w_j|\le k\}}\,dxdt \\
&= \int_Q a(x,t,u,\nabla u)\nabla T_k(u - w_j)\,dxdt,
\end{aligned} \tag{7.33}$$

where $y_\|v\| = (x,t,T_{k+(N+2)\|v\|_\infty}(u), \nabla T_{k+(N+2)\|v\|_\infty}(u))$. Consequently, y using the strong convergence of $(g_n(x,t,u_n,\nabla u_n))_n$ and $((f_n))_n$, one has

$$\begin{aligned}
&\int_{Q_\tau} \frac{\partial u}{\partial t} T_k(u - w_j)dxdt \\
&\quad + \int_{Q_\tau} a(x,t,u,\nabla u)\cdot\nabla T_k(u - w_j)dxdt \\
&\quad\quad + \int_{Q_\tau} g(x,t,u,\nabla u)T_k(u - w_j)dxdt \\
&\quad\quad\quad \le \int_{Q_\tau} fT_k(u - w_j)dxdt \\
&\quad\quad\quad\quad + \int_{Q_\tau} F\cdot\nabla T_k(u - w_j)ddxdt.
\end{aligned} \tag{7.34}$$

Thus , by using the modular convergence of j, we achieve this step.
As a conclusion of Step 1 to Step 7, the proof of Theorem 7 is complete.

□

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