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Second Order Duality for Mathematical Programming Involving n-set Functions

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ABSTRACT: The notion of second order convexity and its generalization for n-set functions are introduced. Mond-weir type second order duality is formulated for the general mathematical programming problems involving n-set functions and proved the desired duality theorems. Further, counterexamples are provided in support of the present investigation.

Key Words: Mathematical programming, n-set functions, Second order generalized convexity, Second order Mond-Weir type duality.

Contents

1. Introduction

The objective function and the constraint functions of general optimization problems are considered as point functions. The point function $f : X \to Y$ maps a point of X to a point in Y. However, in real life situations there exist several functions which maps a set to a point. These functions are called set functions. The optimization problem involving set functions or n-set functions *i.e.* selection of measurable subsets from a given space is an interesting topic of research in the optimization field of recent time.

The idea of getting optimal selection of subsets for set functions appeared in many mathematical areas [7,8,25]. A few problems of this type have been confronted in designing of electrical insulators [3], fluid flows [2], optimal plasma confinements [33] and statistics [10]. It is observed that the theory and methods of the above works are suitable to solve many real life problems in several situations.

Morris [24] first revealed the complexity of the above problem, occurs inadequately structured feasible domain which were not open, nonconvex and, literally

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nowhere dense. And also developed a new theory to optimize the set functions. Further, the notions of differentiability of set functions and local and global convexity for them was established by him. Lin [20] first introduced the concepts of pseudoconvexity and quasiconvexity of set functions and established many properties of nonconvex differentiable n-set functions. Lai and Yang [14] derived optimality conditions for mathematical programming problems with set functions. The optimality conditions for n-set functions were established by Zalmai [34,35]. Second order optimality conditions for nonlinear programming problems involving set functions was studied by Chou *et al.* [4]. Corley [9] defined the concept of derivatives and convexity conditions for n-set functions. Many authors [6,14,15] were discussed the duality results of nonlinear programs with set functions. The above results were extended for n-set functions by Corley [9]. Zalmai [34] formulated Wolfe-type dual problems and proved the related duality results. Mond-weir type duality theorems of a nonlinear minimax programming problem involving n-set functions were established by Preda and Zalmai [26,35]. Multi-objective programs containing n-set functions have been discussed by Kim *et al.* [11,12,13], Chou *et al.* [5], Lai and Lin [16], Lin [17,18,19], Preda [26,27]. Preda and Stancu-Minasian [28,30,31] studied optimality and duality results for optimization problems containing vector-valued n-set functions. A new class of generalized convex n-set functions was introduced by Zalmai [36] and proved a number of parametric and semi-parametric sufficient efficiency conditions for multiobjective fractional subset selection programs. Many parametric and semi-parametric dual models and the duality results were established under these assumptions. The extended work of Zalmai [36] was reported by [21,22,32].

Chou *et al.* [4] discussed the second order differentiability of a set function, and obtained the necessary and sufficient conditions for optimality for a class of optimization problems. Bector and Chandra [1], and Mond and weir [23] separately introduced the second order pseudoconvex and quasiconvex functions for a twice differentiable point function. Preda [29] defined bonvexity and generalized bonvexity for twice differentiable n-set functions.

The study of second order duality is more effective due to the computational advantage over first order duality. However, to the best of our knowledge there is no literature available on second order duality models involving n-set functions. In the present investigation, second order convexity and second order generalized convexity are described for n-set functions with suitable illustrations. Further, second order Mond-weir type duality model for a general mathematical programming problem involving n-set functions is established and proved the desired duality results. Moreover, some counterexamples are given to justify the efficacy of our work.

2. Preliminaries

Suppose $(X, \mathcal{A}, \mu)$ is a finite and atomless measure space and $L_1(\mu)$ is separable. Let $\mathcal{A}^n = \{(S_1, ..., S_n) : S_i \in \mathcal{A}, i = 1, ..., n\}$ be the n-fold product of the σ-algebra $\mathcal{A}$ of subsets of a given set X. F, $G_1, ..., G_m$ are twice differentiable real valued functions on $\mathcal{A}^n$ and $\mathcal{B} \subset \mathcal{A}^n$. Now consider the following mathematical programming problem (P) involving n-set functions to find the minima of F on $\mathcal{B} = \{(S_1, ..., S_n) : G_j(S_1, ..., S_n) \leq 0), j \in \underline{m}\}$, *i.e.*,

$$\text{(P)} \ \min \ F(S) \tag{2.1}$$

$$\text{subject to } G_j(S) \leq 0, \ j \in \underline{m} \tag{2.2}$$

$$S = (S_1, ..., S_n) \in \mathcal{A}^n. \tag{2.3}$$

This type of problem appears in optimal selection of measurable subsets.
Throughout the paper, the set functions on $\mathcal{A}^n$ are called as n-set functions. But the fact is that $\mathcal{A}^n$ is only a semialgebra, not a σ-algebra. $(\mathcal{A}^n, d)$ is a pseudometric space under the pseudometric d defined by

$$d[(R_1, ..., R_n), (S_1, ..., S_n)] = \big[\sum_{i=1}^{n} \mu^2(R_i \Delta S_i)\big]^{\frac{1}{2}},$$

where Δ denotes symmetric difference. This pseudometric space $(\mathcal{A}^n, d)$ will serve as the domain for most of the functions which is used in this paper.
Let $f \in L_1(\mu)$ and $S \in \mathcal{A}$ with Characteristic (indicator) function $\mathcal{X}_s \in L_1(\mu)$. Then integral $\int_S f d\mu$ will be denoted by $\langle f, \mathcal{X}_s \rangle$. For $w \in L_1(\mu \times \mu)$ and $s_1, s_2 \in \mathcal{A}$, the integral $\int_{s_1 \times s_2} w$ is represented by $\langle w, \mathcal{X}_{s_1} \times \mathcal{X}_{s_2} \rangle$. diag w denotes the diagonal of w and defined a function on $\mathcal{A}$ as diag $w(s) = \langle w, \mathcal{X}_s \times \mathcal{X}_s \rangle$, $s \in \mathcal{A}$. Again, diag w is called $w^\star$-continuous if $\mathcal{X}_{s_n} \rightarrow^{w^\star} \mathcal{X}_s \Rightarrow$ diag $w(s_n) \rightarrow$ diag $w(s)$, where $\mathcal{X}_{s_n} \rightarrow^{w^\star} \mathcal{X}_s$ means $\langle f, \mathcal{X}_{s_n} \rangle \rightarrow \langle f, \mathcal{X}_s \rangle, \forall f \in L_1(\mu)$.
The notions of differentiability and convexity are defined for n-set functions as follows.

Definition 2.1. [24] A set function $P : \mathcal{A} \rightarrow \mathbb{R}$ is said to be differentiable at $S^\star \in \mathcal{A}$ if there exists $DP_{S^\star} \in L_1(X, \mathcal{A}, \mu)$, called the derivative of P at $S^\star$, such that for each $S \in \mathcal{A}$,

$$P(S) = P(S^\star) + \langle DP_{S^\star}, \mathcal{X}_S - \mathcal{X}_{S^\star} \rangle + V_P(S, S^\star),$$

where $V_P(S, S^\star)$ is $o(d(S, S^\star))$, i.e., $\lim\limits_{d(S,S^\star) \rightarrow 0} \dfrac{V_P(S, S^\star)}{d(S, S^\star)} = 0.$

Definition 2.2. [4] A set function $P : \mathcal{A} \rightarrow R$ is said to be twice differentiable at $S^\star \in \mathcal{A}$ if it has a first derivative $DP_{S^\star}$ at $S^\star$, and there exists $D^2 P_{S^\star} \in L_1(\mu \times \mu)$, called the second derivative of P at $S^\star$ such that for each $S \in \mathcal{A}$,

$$P(S) = P(S^\star) + \langle DP_{S^\star}, \mathcal{X}_S - \mathcal{X}_{S^\star} \rangle + \langle D^2 P_{S^\star}, (\mathcal{X}_S - \mathcal{X}_{S^\star})^2 \rangle + E_P(S, S^\star),$$

where $E_P(S, S^\star) = o(d^2(S, S^\star))$, i.e., $\lim\limits_{d(S,S^\star) \rightarrow 0} \dfrac{E_P(S, S^\star)}{d^2(S, S^\star)} = 0.$

Definition 2.3. [9] An n-set function $Q : \mathcal{A}^n \to \mathbb{R}$ is said to have a partial derivative at $(S_1^\star, ..., S_n^\star) \in \mathcal{A}^n$ with respect to its ith argument if the function $P(S_i) = Q(S_1^\star, ..., S_{i-1}^\star, S_i^\star, S_{i+1}^\star, ..., S_n^\star)$ has derivative $DP_{S_i^\star}$, $i \in \underline{n}$. Now the ith partial derivatve of Q at $(S_1^\star, ..., S_n^\star)$ is defined to be $D_iQ_{S_1^\star,...,S_n^\star} = DP_{S_i^\star}$, $i \in \underline{n}$. If there exists $D_iQ_{S_1^\star,...,S_n^\star}, 1 \le i \le n$ we put $DQ_{S_1^\star,...,S_n^\star} = (D_1Q_{S_1^\star,...,S_n^\star}, ..., D_nQ_{S_1^\star,...,S_n^\star})$.

Definition 2.4. [9] An n-set function $Q : \mathcal{A}^n \to \mathbb{R}$ is said to be differentiable at $(S_1^\star, ..., S_n^\star)$ if all the partial derivatives $D_iQ_{S_1^\star,...,S_n^\star}$, $i \in \underline{n}$, exist and satisfy the requirement

$$Q(S_1, ..., S_n) = Q(S_1^\star, ..., S_n^\star) + \sum_{i=1}^{n} \langle D_iQ_{S_1^\star,...,S_n^\star}, \mathcal{X}_{S_i} - \mathcal{X}_{S_i^\star} \rangle \\ + W_Q((S_1, ..., S_n), (S_1^\star, ..., S_n^\star)),$$

where $W_Q((S_1, ..., S_n), (S_1^\star, ..., S_n^\star))$ is $o(d((S_1, ..., S_n), (S_1^\star, ..., S_n^\star)))$ for all $(S_1, ..., S_n) \in \mathcal{A}^n$.

Definition 2.5. [29] An n-set function $Q : \mathcal{A}^n \to \mathbb{R}$ is said to be twice differentiable at $S^\star = (S_1^\star, ..., S_n^\star) \in \mathcal{A}^n$ if it has the first derivative $DQ_{S^\star}$ at $S^\star$ and there exists $D_{ki}^2Q_{S^\star} \in L_1(\mu \times \mu), k, i \in \underline{n}$, such that for $S \in \mathcal{A}^n$ the function $q_{S^\star}(S) = \sum_{k,i=1}^{n} \langle D_{ki}^2Q_{S^\star}, (\mathcal{X}_{S_k} - \mathcal{X}_{S_k^\star}) \times (\mathcal{X}_{S_i} - \mathcal{X}_{S_i^\star}) \rangle$, called the second derivative of Q at $S^\star$, is $w^\star$ continuous and satisfies

$$Q(S) = Q(S^\star) + \sum_{i=1}^{n} \langle D_iQ_{S^\star}, \mathcal{X}_{S_i} - \mathcal{X}_{S_i^\star} \rangle \\ + \sum_{k,i=1}^{n} \langle D_{ki}^2Q_{S^\star}, (\mathcal{X}_{S_k} - \mathcal{X}_{S_k^\star}) \times (\mathcal{X}_{S_i} - \mathcal{X}_{S_i^\star}) \rangle + W_Q(S, (S^\star),$$

where $W_Q(S, S^\star) = o(d^2(S, S^\star))$, i.e., $\displaystyle\lim_{d(S,S^\star)\to 0} \frac{W_Q(S, S^\star)}{d^2(S, S^\star)} = 0$.

From Theorem 1 [4] it is obtained that the second derivatives are unique and Q is differentiable at $S^\star \in \mathcal{A}^n$. The second derivative, $q_{S^\star}(S)$, is the quadratic form defined by $(D_{ki}^2Q_{S^\star})_{k,i}$.

Example 2.6. An example of a twice differentiable set function is

$$F(S) = h(\int_S v_1 d\mu, ..., \int_S v_n d\mu)$$

where $h : \mathbb{R}^n \to \mathbb{R}$ is differentiable and $v_1, ..., v_n$ are in $L_1(\mu)$. Then its first derivative

$$DF_S = \sum_{i=1}^{n} D_ih\Big(\int_S v_1 d\mu, ..., \int_S v_n d\mu\Big) v_i$$

and its second derivative is

$$D^2 F_S = \sum_{j,i=1}^{n} D^2_{ji} h\Big(\int_S v_1 d\mu, ..., \int_S v_n d\mu\Big) v_i v_j,$$

where $D_i h$ and $D^2_{ji} h$ denote the ith first order partial derivative of h and ijth second order partial derivative of h, respectively.
The condition for local convexity of a twice differentiable set function given in Lemma 4 [4] i.e.
Let $F : \mathcal{A} \to \mathbb{R}$ be a set function which is twice differentiable at $S^\star \in \mathcal{A}$. If F is locally convex at $S^\star$ then there exists $\epsilon > 0$ such that $d(S^\star, S) < \epsilon$ implies $\langle D^2 F_{S^\star}, (\mathcal{X}_S - \mathcal{X}_{S^\star})^2\rangle \geq 0$, i.e., $D^2 F_{S^\star}$ is locally positive semidefinite.

3. Generalized Convexity

This section deals with the definition of the second order convexity and second order generalized convexity for twice differentiable n-set functions. Consider $F : \mathcal{A}^n \to \mathbb{R}$ as a twice differentiable n-set function throughout this section.

Definition 3.1. An n-set function $F : \mathcal{A}^n \to \mathbb{R}$ is called second order convex at $T = (T_1, ..., T_n) \in \mathcal{A}^n$ for each $S = (S_1, ..., S_n) \in \mathcal{A}^n, (S \neq T)$ if

$$F(S_1, ..., S_n) - F(T_1, ..., T_n) + \frac{1}{2} \sum_{k,i=1}^{n} \langle D^2_{ki} F_T, (\mathcal{X}_{S_k} - \mathcal{X}_{T_k}) \times (\mathcal{X}_{S_i} - \mathcal{X}_{T_i})\rangle \geq$$

$$\geq \sum_{i=1}^{n} \langle D_i F_T, \mathcal{X}_{S_i} - \mathcal{X}_{T_i}\rangle + \sum_{k,i=1}^{n} \langle D^2_{ki} F_T, (\mathcal{X}_{S_k} - \mathcal{X}_{T_k}) \times (\mathcal{X}_{S_i} - \mathcal{X}_{T_i})\rangle.$$

Definition 3.2. An n-set function $F : \mathcal{A}^n \to \mathbb{R}$ is termed as second order pseudo-convex at $T = (T_1, ..., T_n) \in \mathcal{A}^n$ for each $S = (S_1, ..., S_n) \in \mathcal{A}^n, (S \neq T)$ if

$$\sum_{i=1}^{n} \langle D_i F_T, \mathcal{X}_{S_i} - \mathcal{X}_{T_i}\rangle + \sum_{k,i=1}^{n} \langle D^2_{ki} F_T, (\mathcal{X}_{S_k} - \mathcal{X}_{T_k}) \times (\mathcal{X}_{S_i} - \mathcal{X}_{T_i})\rangle \geq 0$$

$$\Rightarrow F(S_1, ..., S_n) \geq F(T_1, ..., T_n) - \frac{1}{2} \sum_{k,i=1}^{n} \langle D^2_{ki} F_T, (\mathcal{X}_{S_k} - \mathcal{X}_{T_k}) \times (\mathcal{X}_{S_i} - \mathcal{X}_{T_i})\rangle,$$

or equivalently,

$$F(S_1, ..., S_n) < F(T_1, ..., T_n) - \frac{1}{2} \sum_{k,i=1}^{n} \langle D^2_{ki} F_T, (\mathcal{X}_{S_k} - \mathcal{X}_{T_k}) \times (\mathcal{X}_{S_i} - \mathcal{X}_{T_i})\rangle$$

$$\Rightarrow \sum_{i=1}^{n} \langle D_i F_T, \mathcal{X}_{S_i} - \mathcal{X}_{T_i}\rangle + \sum_{k,i=1}^{n} \langle D^2_{ki} F_T, (\mathcal{X}_{S_k} - \mathcal{X}_{T_k}) \times (\mathcal{X}_{S_i} - \mathcal{X}_{T_i})\rangle < 0.$$

Definition 3.3. An n-set function $F : \mathcal{A}^n \to \mathbb{R}$ is said to be strictly second order pseudoconvex at $T = (T_1, ..., T_n) \in \mathcal{A}^n$ for each $S = (S_1, ..., S_n) \in \mathcal{A}^n, (S \neq T)$ if

$$\sum_{i=1}^{n}\langle D_i F_T, \mathcal{X}_{S_i} - \mathcal{X}_{T_i}\rangle + \sum_{k,i=1}^{n}\langle D^2_{ki}F_T, (\mathcal{X}_{S_k} - \mathcal{X}_{T_k}) \times (\mathcal{X}_{S_i} - \mathcal{X}_{T_i})\rangle \geq 0$$

$$\Rightarrow F(S_1, ..., S_n) > F(T_1, ..., T_n) - \frac{1}{2}\sum_{k,i=1}^{n}\langle D^2_{ki}F_T, (\mathcal{X}_{S_k} - \mathcal{X}_{T_k}) \times (\mathcal{X}_{S_i} - \mathcal{X}_{T_i})\rangle,$$

or equivalently,

$$F(S_1, ..., S_n) \leq F(T_1, ..., T_n) - \frac{1}{2}\sum_{k,i=1}^{n}\langle D^2_{ki}F_T, (\mathcal{X}_{S_k} - \mathcal{X}_{T_k}) \times (\mathcal{X}_{S_i} - \mathcal{X}_{T_i})\rangle$$

$$\Rightarrow \sum_{i=1}^{n}\langle D_i F_T, \mathcal{X}_{S_i} - \mathcal{X}_{T_i}\rangle + \sum_{k,i=1}^{n}\langle D^2_{ki}F_T, (\mathcal{X}_{S_k} - \mathcal{X}_{T_k}) \times (\mathcal{X}_{S_i} - \mathcal{X}_{T_i})\rangle < 0.$$

Definition 3.4. An n-set function $F : \mathcal{A}^n \to \mathbb{R}$ is called second order quasiconvex at $T = (T_1, ..., T_n) \in \mathcal{A}^n$ for each $S = (S_1, ..., S_n) \in \mathcal{A}^n, (S \neq T)$ if

$$F(S_1, ..., S_n) \leq F(T_1, ..., T_n) - \frac{1}{2}\sum_{k,i=1}^{n}\langle D^2_{ki}F_T, (\mathcal{X}_{S_k} - \mathcal{X}_{T_k}) \times (\mathcal{X}_{S_i} - \mathcal{X}_{T_i})\rangle$$

$$\Rightarrow \sum_{i=1}^{n}\langle D_i F_T, \mathcal{X}_{S_i} - \mathcal{X}_{T_i}\rangle + \sum_{k,i=1}^{n}\langle D^2_{ki}F_T, (\mathcal{X}_{S_k} - \mathcal{X}_{T_k}) \times (\mathcal{X}_{S_i} - \mathcal{X}_{T_i})\rangle \leq 0,$$

or equivalently,

$$\sum_{i=1}^{n}\langle D_i F_T, \mathcal{X}_{S_i} - \mathcal{X}_{T_i}\rangle + \sum_{k,i=1}^{n}\langle D^2_{ki}F_T, (\mathcal{X}_{S_k} - \mathcal{X}_{T_k}) \times (\mathcal{X}_{S_i} - \mathcal{X}_{T_i})\rangle > 0$$

$$\Rightarrow F(S_1, ..., S_n) > F(T_1, ..., T_n) - \frac{1}{2}\sum_{k,i=1}^{n}\langle D^2_{ki}F_T, (\mathcal{X}_{S_k} - \mathcal{X}_{T_k}) \times (\mathcal{X}_{S_i} - \mathcal{X}_{T_i})\rangle.$$

With the help of following propositions the existance of second order pseudoconvexity and second order quasiconvexity of n-set functions is verified.

Proposition 3.5. [20] Let $\mathcal{S}$ be a convex subfamily of $\mathcal{A}^n$ and $f(S_1, ..., S_n) = v(\langle g_1, \mathcal{X}_{S_1}\rangle, ..., \langle g_n, \mathcal{X}_{S_n}\rangle)$, where $v : \mathbb{R}^n \to \mathbb{R}$ is a differentiable function, $g_1, ..., g_n \in L_1(X, \mathcal{A}, \mu)$ and $(S_1, ..., S_n) \in \mathcal{S}$.
(a) f is pseudoconvex set function, if v is pseudoconvex.
(b) f is quasiconvex set function, if v is quasiconvex.
(c) f is strictly quasiconvex set function, if v is strictly quasiconvex.

Proposition 3.6. Let $\mathcal{S}$ be a convex subfamily of $\mathcal{A}^n$ and

$$f(S_1, ..., S_n) = v(\langle g_1, \mathcal{X}_{S_1}\rangle, ..., \langle g_n, \mathcal{X}_{S_n}\rangle)$$

, where $v : \mathbb{R}^n \to \mathbb{R}$ is a differentiable function, $g_1, ..., g_n \in L_1(X, \mathcal{A}, \mu)$ and $(S_1, ..., S_n) \in \mathcal{S}$.
(a) f is second order pseudoconvex n-set function, if v is second order pseudoconvex.
(b) f is second order quasiconvex n-set function, if v is second order quasiconvex.
(c) f is second order strictly pseudoconvex n-set function, if v is second order strictly pseudoconvex.

Proof: (a) Suppose v is a second order pseudoconvex function. Let $(S_1, ..., S_n)$, $(T_1, ..., T_n) \in \mathcal{S}$, then it follows from the definition of partial derivatives,

$$D_i f_T = v_i(\langle g_1, \mathcal{X}_{T_1}\rangle, ..., \langle g_n, \mathcal{X}_{T_n}\rangle) g_i$$

$$D^2_{ki} f_T = v_{ki}(\langle g_1, \mathcal{X}_{T_1}\rangle, ..., \langle g_n, \mathcal{X}_{T_n}\rangle) g_k g_i,$$

where v_i and v_{ki} denote the ith and kith partial derivative of v, respectively.
If

$$\sum_{k,i=1}^{n} \langle D^2_{ki} f_T, (\mathcal{X}_{S_k} - \mathcal{X}_{T_k}) \times (\mathcal{X}_{S_i} - \mathcal{X}_{T_i})\rangle + \sum_{i=1}^{n} \langle D_i f_T, \mathcal{X}_{S_i} - \mathcal{X}_{T_i}\rangle \geq 0,$$

we have,
$\sum_{k,i=1}^{n} v_{ki}(\langle g_1, \mathcal{X}_{T_1}\rangle, ..., \langle g_n, \mathcal{X}_{T_n}\rangle)\langle g_k, \mathcal{X}_{S_k} - \mathcal{X}_{T_k}\rangle\langle g_i, \mathcal{X}_{S_i} - \mathcal{X}_{T_i}\rangle$
$+ \sum_{i=1}^{n} v_i(\langle g_1, \mathcal{X}_{T_1}\rangle, ..., \langle g_n, \mathcal{X}_{T_n}\rangle)\langle g_i, \mathcal{X}_{S_i} - \mathcal{X}_{T_i}\rangle \geq 0$

that is,

$$\begin{pmatrix} \langle g_1, \mathcal{X}_{S_1}\rangle - \langle g_1, \mathcal{X}_{T_1}\rangle \\ . \\ . \\ . \\ \langle g_n, \mathcal{X}_{S_n}\rangle - \langle g_n, \mathcal{X}_{T_n}\rangle \end{pmatrix}^T \nabla^2 v(g(x)) \begin{pmatrix} \langle g_1, \mathcal{X}_{S_1}\rangle - \langle g_1, \mathcal{X}_{T_1}\rangle \\ . \\ . \\ . \\ \langle g_n, \mathcal{X}_{S_n}\rangle - \langle g_n, \mathcal{X}_{T_n}\rangle \end{pmatrix}$$

$$+ \nabla v(\langle g_1, \mathcal{X}_{T_1}\rangle, ..., \langle g_n, \mathcal{X}_{T_n}\rangle) \begin{pmatrix} \langle g_1, \mathcal{X}_{S_1}\rangle - \langle g_1, \mathcal{X}_{T_1}\rangle \\ . \\ . \\ . \\ \langle g_n, \mathcal{X}_{S_n}\rangle - \langle g_n, \mathcal{X}_{T_n}\rangle \end{pmatrix} \geq 0,$$

where $g(x) = \langle g_1, \mathcal{X}_{T_1}\rangle, ..., \langle g_n, \mathcal{X}_{T_n}\rangle$. Since $v : \mathbb{R}^n \to \mathbb{R}$ is a second order pseudoconvex function it follows that,

$$f(S_1, ..., S_n) = v(\langle g_1, \mathcal{X}_{S_1}\rangle, ..., \langle g_n, \mathcal{X}_{S_n}\rangle) \geq v(\langle g_1, \mathcal{X}_{T_1}\rangle, ..., \langle g_n, \mathcal{X}_{T_n}\rangle)$$

$$- \frac{1}{2} \begin{pmatrix} \langle g_1, \mathcal{X}_{S_1} \rangle - \langle g_1, \mathcal{X}_{T_1} \rangle \\ . \\ . \\ . \\ \langle g_n, \mathcal{X}_{S_n} \rangle - \langle g_n, \mathcal{X}_{T_n} \rangle \end{pmatrix}^T \nabla^2 v(g(x)) \begin{pmatrix} \langle g_1, \mathcal{X}_{S_1} \rangle - \langle g_1, \mathcal{X}_{T_1} \rangle \\ . \\ . \\ . \\ \langle g_n, \mathcal{X}_{S_n} \rangle - \langle g_n, \mathcal{X}_{T_n} \rangle \end{pmatrix}$$

$$\geq f(T_1, ..., T_n) - \tfrac{1}{2} \sum_{k,i=1}^{n} \langle D^2_{ki} f_T, (\mathcal{X}_{S_k} - \mathcal{X}_{T_k}) \times (\mathcal{X}_{S_i} - \mathcal{X}_{T_i}) \rangle.$$

This shows that f is a second order pseudoconvex set function. □

The proof of (b) and (c) are similar to the prove of (a). Hence it is omitted.

Proposition 3.7. Suppose $f : \mathcal{A}^n \to \mathbb{R}$ be a convex n-set function. Then,
(a) f is second order pseudoconvex n-set function.
(b) f is second order strictly pseudoconvex n-set function.

Proof: (a) Since f is differentiable on $\mathcal{A}^n$ and convex then for all $T = (T_1, ..., T_n), S = (S_1, ..., S_n) \in \mathcal{A}^n$ we have,

$$f(S_1, ..., S_n) - f(T_1, ..., T_n) \geq \sum_{i=1}^{n} \langle D_i f_T, \mathcal{X}_{S_i} - \mathcal{X}_{T_i} \rangle,$$

again f is a twice differentiable n-set function and suppose it is locally convex at $T = (T_1, ..., T_n) \in \mathcal{A}^n$.
This implies $D^2 f_T$ is locally positive semidefinite. Therefore using these conditions in the definition of second order pseudoconvex n-set function, we immediately show that f is a second order pseudoconvex n-set function.
Similarly, if f is a twice differentiable convex n-set function then (b) holds.

Remark 3.8. Every twice differentiable convex n-set function is a second order pseudoconvex n-set function but the converse is not true, which can be easily verified by considering the set function $f : \mathcal{S} \to \mathbb{R}$ defined by $f(S) = \int_S g \; d\mu + (\int_S g \; d\mu)^3$ where $g \in L(X, \mathcal{A}, \mu)$ and $\mathcal{S}$ is a convex subfamily of $\mathcal{A}$.

Remark 3.9. If $f : \mathcal{S} \to \mathbb{R}$ is defined by $f(S) = (\int_S g \; d\mu)^3$, where $g \in L(X, \mathcal{A}, \mu)$ and $\mathcal{S}$ is a convex subfamily of $\mathcal{A}$, then f is not a convex set function but it is a second order quasiconvex function.

Theorem 3.10. (Kuhn-Tucker type necessary conditions [9]) Let $(X, \mathcal{A}, \mu)$ be a finite atomless measure space and let $F, G_1, ..., G_m : \mathcal{A}^n \to \mathbb{R}$, $j \in \underline{m}$, be differentiable at $T^\star = (T_1^\star, ..., T_n^\star) \in \mathcal{A}^n$. If $(T_1^\star, ..., T_n^\star)$ is a regular optimal solution of (P), i.e. if $(T_1^\star, ..., T_n^\star)$ is an optimal solution of (P) and for $(\widehat{T}_1, ..., \widehat{T}_n) \in \mathcal{A}^n$,

$$G_j(T_1^\star, ..., T_n^\star) + \sum_{i=1}^{n} \langle D_i G_{jT^\star}, \mathcal{X}_{\widehat{T}_i} - \mathcal{X}_{T_i^\star} \rangle < 0, \; j \in \underline{m} \; ,$$

then there exists $y^\star \in \mathbb{R}^m_+$ (nonnegative orthant of $\mathbb{R}^m$) such that,

$$\langle D_i F_{T^\star} + \sum_{j=1}^{m} y_j D_i G_{jT^\star}, \mathcal{X}_{S_i} - \mathcal{X}_{T_i^\star} \rangle \geq 0, \forall S_i \in \mathcal{A}, \; i \in \underline{n} \; ,$$

$$y_j G_j(T_1^\star, ..., T_n^\star) = 0, \; j \in \underline{m} \, ,$$

$$G_j(T_1^\star, ..., T_n^\star) \leq 0, \; j \in \underline{m} \, .$$

4. Second order Mond-Weir type duality

In this section, we consider the Mond-Weir type second order dual (SMWD) of a general mathematical programming problem (P) involving n-set functions. Weak, strong and strict converse duality theorems are established under generalized second order convexity assumptions.

$$\text{(SMWD)} \;\; \max \;\; F(U) - \frac{1}{2} \sum_{k,i=1}^{n} \langle D_{ki}^2 F_U, (\mathcal{X}_{S_k} - \mathcal{X}_{U_k}) \times (\mathcal{X}_{S_i} - \mathcal{X}_{U_i}) \rangle \tag{4.1}$$

$$\begin{aligned} \text{subject to } \langle D_i F_U, \mathcal{X}_{S_i} - \mathcal{X}_{U_i} \rangle &+ \sum_{j=1}^{m} \langle y_j D_i G_{jU}, \mathcal{X}_{S_i} - \mathcal{X}_{Ui} \rangle \\ &+ \sum_{k=1}^{n} \langle D_{ki}^2 F_U, (\mathcal{X}_{S_k} - \mathcal{X}_{U_k}) \times (\mathcal{X}_{S_i} - \mathcal{X}_{U_i}) \rangle \\ + \sum_{k=1}^{n} \sum_{j=1}^{m} &\langle y_j D_{ki}^2 G_{jU}, (\mathcal{X}_{S_k} - \mathcal{X}_{U_k}) \times (\mathcal{X}_{S_i} - \mathcal{X}_{U_i}) \rangle \geq 0, \\ &\forall S_i, U_i \in \mathcal{A}, i \in \underline{n}, \end{aligned} \tag{4.2}$$

$$\sum_{j=1}^{m} \{ y_j G_j(U) - \frac{1}{2} \sum_{k,i=1}^{n} \langle y_j D_{ki}^2 G_{jU}, (\mathcal{X}_{S_k} - \mathcal{X}_{U_k}) \times (\mathcal{X}_{S_i} - \mathcal{X}_{U_i}) \rangle \} \geq 0, \tag{4.3}$$

$$U = (U_1, ..., U_n) \in \mathcal{A}^n, \tag{4.4}$$

$$y = (y_1, ..., y_m) \in R_+^m. \tag{4.5}$$

Theorem 4.1. (Weak duality)
Suppose S and (U, y) are arbitrary feasible solutions of (P) and (SMWD), respectively, where $U = (U_1, ..., U_n) \in \mathcal{A}^n, S = (S_1, ..., S_n) \in \mathcal{A}^n$ and $y \in \mathbb{R}_+^m$. Again assume the following two conditions,

(A) $\sum_{j=1}^{m} y_j G_j(S)$ is second order quasiconvex at U,

(B) $F(S)$ is a second order pseudoconvex at U.

Then the following can not hold,

$$F(S) < F(U) - \frac{1}{2} \sum_{k,i=1}^{n} \langle D_{ki}^2 F_U, (\mathcal{X}_{S_k} - \mathcal{X}_{U_k}) \times (\mathcal{X}_{S_i} - \mathcal{X}_{U_i}) \rangle. \tag{4.6}$$

Proof: S and (U, y) are arbitrary feasible solution of (P) and (SMWD), respectively, and $y \in \mathbb{R}^m_+$. Now from inequality (2.2) and (4.3),

$$\sum_{j=1}^{m} y_j G_j(S) \leq 0 \leq \sum_{j=1}^{m} \{y_j G_j(U) \\ -\frac{1}{2} \sum_{k,i=1}^{n} \langle y_j D^2_{ki} G_{jU}, (\mathcal{X}_{S_k} - \mathcal{X}_{U_k}) \times (\mathcal{X}_{S_i} - \mathcal{X}_{U_i}) \rangle \}. \tag{4.7}$$

Second order quasiconvexity of $\sum_{j=1}^{m} y_j G_j(S)$ and condition (A) gives,

$$\sum_{i=1}^{n} \sum_{j=1}^{m} \langle y_j D_i G_{jU}, \mathcal{X}_{S_i} - \mathcal{X}_{Ui} \rangle \\ + \sum_{k,i=1}^{n} \sum_{j=1}^{m} \langle y_j D^2_{ki} G_{jU}, (\mathcal{X}_{S_k} - \mathcal{X}_{U_k}) \times (\mathcal{X}_{S_i} - \mathcal{X}_{U_i}) \rangle \leq 0. \tag{4.8}$$

From inequality (4.2) we obtain,

$$\langle D_i F_U, \mathcal{X}_{S_i} - \mathcal{X}_{U_i} \rangle + \sum_{k=1}^{n} \langle D^2_{ki} F_U, (\mathcal{X}_{S_k} - \mathcal{X}_{U_k}) \times (\mathcal{X}_{S_i} - \mathcal{X}_{U_i}) \rangle \geq \\ -[\sum_{j=1}^{m} \langle y_j D_i G_{jU}, \mathcal{X}_{S_i} - \mathcal{X}_{Ui} \rangle + \sum_{k=1}^{n} \sum_{j=1}^{m} \langle y_j D^2_{ki} G_{jU}, (\mathcal{X}_{S_k} - \mathcal{X}_{U_k}) \times (\mathcal{X}_{S_i} - \mathcal{X}_{U_i}) \rangle], \\ \forall S_i, U_i \in \mathcal{A}, i \in \underline{n}.$$

$$\Rightarrow \sum_{i=1}^{n} \langle D_i F_U, \mathcal{X}_{S_i} - \mathcal{X}_{U_i} \rangle + \sum_{k,\ i=1}^{n} \langle D^2_{ki} F_U, (\mathcal{X}_{S_k} - \mathcal{X}_{U_k}) \times (\mathcal{X}_{S_i} - \mathcal{X}_{U_i}) \rangle \\ \geq -[\sum_{i=1}^{n} \sum_{j=1}^{m} \langle y_j D_i G_{jU}, \mathcal{X}_{S_i} - \mathcal{X}_{Ui} \rangle \\ + \sum_{k,\ i=1}^{n} \sum_{j=1}^{m} \langle y_j D^2_{ki} G_{jU}, (\mathcal{X}_{S_k} - \mathcal{X}_{U_k}) \times (\mathcal{X}_{S_i} - \mathcal{X}_{U_i}) \rangle]. \tag{4.9}$$

Now the inequality (4.8) and (4.9) gives,

$$\sum_{i=1}^{n} \langle D_i F_U, \mathcal{X}_{S_i} - \mathcal{X}_{U_i} \rangle + \sum_{k,i=1}^{n} \langle D^2_{ki} F_U, (\mathcal{X}_{S_k} - \mathcal{X}_{U_k}) \times (\mathcal{X}_{S_i} - \mathcal{X}_{U_i}) \rangle \geq 0. \tag{4.10}$$

Now assume that ineqality (4.6) holds, i.e.,

$$F(S) < F(U) - \frac{1}{2} \sum_{k,i=1}^{n} \langle D^2_{ki} F_U, (\mathcal{X}_{S_k} - \mathcal{X}_{U_k}) \times (\mathcal{X}_{S_i} - \mathcal{X}_{U_i}) \rangle,$$

which by virtue of (B), leads to

$$\sum_{i=1}^{n}\langle D_iF_U, \mathcal{X}_{S_i} - \mathcal{X}_{U_i}\rangle + \sum_{k,i=1}^{n}\langle D^2_{ki}F_U, (\mathcal{X}_{S_k} - \mathcal{X}_{U_k}) \times (\mathcal{X}_{S_i} - \mathcal{X}_{U_i})\rangle < 0. \quad (4.11)$$

Which contradicts the inequality (4.10). Hence, the inequality (4.6) can not hold.
□

Theorem 4.2. (Strong duality)
Suppose $\overline{S} = (\overline{S}_1, ..., \overline{S}_n) \in \mathcal{A}^n$ is a regular solution of (P) at which the Kuhn-Tucker type theorem is satisfied. Then there exists $\overline{y} \in \mathbb{R}^m_+$ such that the weak duality theorem holds between the primal (P) and the dual (SMWD), again assume that,

$$\sum_{k=1}^{n}\langle D^2_{ki}F_{\overline{S}}, (\mathcal{X}_{S_k} - \mathcal{X}_{\overline{S}_k}) \times (\mathcal{X}_{S_i} - \mathcal{X}_{\overline{S}_i})\rangle = 0, \forall S_i, \overline{S}_i \in \mathcal{A}, i \in \underline{n}, \quad (4.12)$$

and

$$\sum_{k=1}^{n}\sum_{j=1}^{m}\langle \overline{y}_j D^2_{ki}G_{j\overline{S}}, (\mathcal{X}_{S_k} - \mathcal{X}_{\overline{S}_k}) \times (\mathcal{X}_{S_i} - \mathcal{X}_{\overline{S}_i})\rangle = 0, \forall S_i, \overline{S}_i \in \mathcal{A}, i \in \underline{n}. \quad (4.13)$$

Then $(\overline{S}, \overline{y})$, is an optimal solution of (SMWD) and the corresponding optimal values of (P) and (SMWD) are equal.

Proof: Since $\overline{S}$ is a regular solution of (P) at which Kuhn-Tucker type theorem is satisfied, then there exists $\overline{y} \in \mathbb{R}^m_+$ such that

$$\langle D_iF_{\overline{S}} + \sum_{j=1}^{m} y_j D_iG_{j\overline{S}}, \mathcal{X}_{S_i} - \mathcal{X}_{\overline{S}_i}\rangle \geq 0, \forall S_i \in \mathcal{A},\ i \in \underline{n}, \quad (4.14)$$

$$\overline{y}_j G_j(\overline{S}) = 0, j \in \underline{m}, \quad (4.15)$$

$$G_j(\overline{S}) \leq 0, j \in \underline{m}, \quad (4.16)$$

so from (4.2) we get,

$$\sum_{j=1}^{m}\overline{y}_j G_j(\overline{S}) = 0.$$

from the assumptions (4.12) and (4.13),

$$\sum_{k=1}^{n}\langle D^2_{ki}F_{\overline{S}}, (\mathcal{X}_{S_k} - \mathcal{X}_{\overline{S}_k}) \times (\mathcal{X}_{S_i} - \mathcal{X}_{\overline{S}_i})\rangle = 0, \forall S_i, \overline{S}_i \in \mathcal{A}, i \in \underline{n},$$

and,

$$\sum_{k=1}^{n}\sum_{j=1}^{m}\langle \overline{y}_j D^2_{ki} G_{j\overline{S}}, (\mathcal{X}_{S_k} - \mathcal{X}_{\overline{S}_k}) \times (\mathcal{X}_{S_i} - \mathcal{X}_{\overline{S}_i})\rangle = 0, \forall S_i, \overline{S}_i \in \mathcal{A}, i \in \underline{n},$$

$$\Rightarrow \sum_{k=1}^{n}\langle D^2_{ki} F_{\overline{S}}, (\mathcal{X}_{S_k} - \mathcal{X}_{\overline{S}_k}) \times (\mathcal{X}_{S_i} - \mathcal{X}_{\overline{S}_i})\rangle$$
$$+\sum_{k=1}^{n}\sum_{j=1}^{m}\langle \overline{y}_j D^2_{ki} G_{j\overline{S}}, (\mathcal{X}_{S_k} - \mathcal{X}_{\overline{S}_k}) \times (\mathcal{X}_{S_i} - \mathcal{X}_{\overline{S}_i})\rangle = 0, \forall S_i, \overline{S}_i \in \mathcal{A}, i \in \underline{n}. \quad (4.17)$$

Using the two equations (4.12) and (4.13) we obtain,

$$\sum_{k,i=1}^{n}\langle D^2_{ki} F_{\overline{S}}, (\mathcal{X}_{S_k} - \mathcal{X}_{\overline{S}_k}) \times (\mathcal{X}_{S_i} - \mathcal{X}_{\overline{S}_i})\rangle = 0,$$

and,

$$\sum_{k,i=1}^{n}\sum_{j=1}^{m}\langle \overline{y}_j D^2_{ki} G_{j\overline{S}}, (\mathcal{X}_{S_k} - \mathcal{X}_{\overline{S}_k}) \times (\mathcal{X}_{S_i} - \mathcal{X}_{\overline{S}_i})\rangle = 0.$$

Therefore $(\overline{S}, \overline{y})$ is a optimal solution for (SMWD). If $\overline{S}$ is regular optimal solution of (P), and the corresponding optimal values of objective function of (P) and (SMWD) are equal, then the optimality of (SMWD) follows from weak duality theorem. □

Theorem 4.3. (Strict converse duality)
Suppose $\overline{S}$ and $(\overline{U}, \overline{y})$ are arbitrary feasible solutions of (P) and (SMWD), respectively, where $\overline{U} = (\overline{U}_1, ..., \overline{U}_n) \in \mathcal{A}^n$, $\overline{S} = (\overline{S}_1, ..., \overline{S}_n) \in \mathcal{A}^n$ and $\overline{y} \in \mathbb{R}^m_+$ such that,

$$F(\overline{S}) = F(\overline{U}) - \frac{1}{2}\sum_{k,i=1}^{n}\langle D^2_{ki} F_{\overline{U}}, (\mathcal{X}_{\overline{S}_k} - \mathcal{X}_{\overline{U}_k}) \times (\mathcal{X}_{\overline{S}_i} - \mathcal{X}_{\overline{U}_i})\rangle. \quad (4.18)$$

Again if one of the following conditions is satisfied.
(A) $\sum_{j=1}^{m} \overline{y}_j G_j(\overline{S})$ is second order quasiconvex at $\overline{U}$ and $F(\overline{S})$ is strictly second order pseudoconvex at $\overline{U}$.
(B) $\sum_{j=1}^{m} \overline{y}_j G_j(\overline{S})$ is strictly second order pseudoconvex at $\overline{U}$ and $F(\overline{S})$ is second order quasiconvex at $\overline{U}$.
Then $\overline{S} = \overline{U}$; i.e. $\overline{U}$ is an optimal solution of (P).

Proof: We assume that $\overline{S} \neq \overline{U}$. Since $\overline{S}$ and $(\overline{U}, \overline{y})$ are arbitrary feasible solutions of (P) and (SMWD), respectively, then we have

$$\sum_{j=1}^{m} \overline{y}_j G_j(\overline{S}) \leq 0 \leq \sum_{j=1}^{m} \{\overline{y}_j G_j(\overline{U})$$
$$-\frac{1}{2} \sum_{k,i=1}^{n} \langle \overline{y}_j D_{ki}^2 G_{j\overline{U}}, (\mathcal{X}_{\overline{S}_k} - \mathcal{X}_{\overline{U}_k}) \times (\mathcal{X}_{\overline{S}_i} - \mathcal{X}_{\overline{U}_i})\rangle\}. \tag{4.19}$$

Using second order quasiconvexity of $\sum_{j=1}^{m} \overline{y}_j \overline{G}_j(\overline{S})$ at $\overline{U}$ we get,

$$\sum_{i=1}^{n} \sum_{j=1}^{m} \langle \overline{y}_j D_i G_{j\overline{U}}, \mathcal{X}_{\overline{S}_i} - \mathcal{X}_{\overline{U}_i}\rangle$$
$$+ \sum_{k,i=1}^{n} \sum_{j=1}^{m} \langle y_j D_{ki}^2 G_{jU}, (\mathcal{X}_{\overline{S}_k} - \mathcal{X}_{\overline{U}_k}) \times (\mathcal{X}_{\overline{S}_i} - \mathcal{X}_{\overline{U}_i})\rangle \leq 0. \tag{4.20}$$

From inequality (4.2) we get

$$\langle D_i F_{\overline{U}}, \mathcal{X}_{\overline{S}_i} - \mathcal{X}_{\overline{U}_i}\rangle + \sum_{k=1}^{n} \langle D_{ki}^2 F_{\overline{U}}, (\mathcal{X}_{\overline{S}_k} - \mathcal{X}_{\overline{U}_k}) \times (\mathcal{X}_{\overline{S}_i} - \mathcal{X}_{\overline{U}_i})\rangle$$
$$\geq -\Big[\sum_{j=1}^{m} \langle \overline{y}_j D_i G_{j\overline{U}}, \mathcal{X}_{\overline{S}_i} - \mathcal{X}_{\overline{U}_i}\rangle$$
$$+ \sum_{k=1}^{n} \sum_{j=1}^{m} \langle \overline{y}_j D_{ki}^2 G_{j\overline{U}}, (\mathcal{X}_{\overline{S}_k} - \mathcal{X}_{\overline{U}_k}) \times (\mathcal{X}_{\overline{S}_i} - \mathcal{X}_{\overline{U}_i})\rangle\Big], \forall \overline{S}_i, \overline{U}_i \in \mathcal{A}, i \in \underline{n}$$

$$\Rightarrow \sum_{i=1}^{n} \Big[\langle D_i F_{\overline{U}}, \mathcal{X}_{\overline{S}_i} - \mathcal{X}_{\overline{U}_i}\rangle + \sum_{k=1}^{n} \langle D_{ki}^2 F_{\overline{U}}, (\mathcal{X}_{\overline{S}_k} - \mathcal{X}_{\overline{U}_k}) \times (\mathcal{X}_{\overline{S}_i} - \mathcal{X}_{\overline{U}_i})\rangle\Big] \geq$$
$$- \sum_{i=1}^{n} \Big[\sum_{j=1}^{m} \langle \overline{y}_j D_i G_{j\overline{U}}, \mathcal{X}_{\overline{S}_i} - \mathcal{X}_{\overline{U}_i}\rangle$$
$$+ \sum_{k=1}^{n} \sum_{j=1}^{m} \langle \overline{y}_j D_{ki}^2 G_{j\overline{U}}, (\mathcal{X}_{\overline{S}_k} - \mathcal{X}_{\overline{U}_k}) \times (\mathcal{X}_{\overline{S}_i} - \mathcal{X}_{\overline{U}_i})\rangle\Big]. \tag{4.21}$$

Now by the inequality (4.20) and (4.21) we have

$$\sum_{i=1}^{n} \Big[\langle D_i F_{\overline{U}}, \mathcal{X}_{\overline{S}_i} - \mathcal{X}_{\overline{U}_i}\rangle + \sum_{k=1}^{n} \langle D_{ki}^2 F_{\overline{U}}, (\mathcal{X}_{\overline{S}_k} - \mathcal{X}_{\overline{U}_k}) \times (\mathcal{X}_{\overline{S}_i} - \mathcal{X}_{\overline{U}_i})\rangle\Big] \geq 0$$

$$\Rightarrow \sum_{i=1}^{n} \langle D_i F_{\overline{U}}, \mathcal{X}_{\overline{S}_i} - \mathcal{X}_{\overline{U}_i}\rangle + \sum_{k,i=1}^{n} \langle D_{ki}^2 F_{\overline{U}}, (\mathcal{X}_{\overline{S}_k} - \mathcal{X}_{\overline{U}_k}) \times (\mathcal{X}_{\overline{S}_i} - \mathcal{X}_{\overline{U}_i})\rangle \geq 0. \tag{4.22}$$

If $F(\overline{S})$ is strictly second order pseudoconvex at $\overline{U}$ then,

$$F(\overline{S}) > F(\overline{U}) - \frac{1}{2}\sum_{k,i=1}^{n}\langle D^2_{ki}F_{\overline{U}}, (\mathcal{X}_{\overline{S}_k} - \mathcal{X}_{\overline{U}_k}) \times (\mathcal{X}_{\overline{S}_i} - \mathcal{X}_{\overline{U}_i})\rangle. \tag{4.23}$$

which contradicts equation (4.18).
When the hypothesis (B) holds, from inequality (4.19) we get,

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\langle \overline{y}_j D_i G_{j\overline{U}}, \mathcal{X}_{\overline{S}_i} - \mathcal{X}_{\overline{U}_i}\rangle + \sum_{k,i=1}^{n}\sum_{j=1}^{m}\langle y_j D^2_{ki}G_{jU}, (\mathcal{X}_{\overline{S}_k} - \mathcal{X}_{\overline{U}_k}) \times (\mathcal{X}_{\overline{S}_i} - \mathcal{X}_{\overline{U}_i})\rangle < 0. \tag{4.24}$$

By the first dual constraints of (SMWD) we have,

$$\langle D_i F_{\overline{U}}, \mathcal{X}_{\overline{S}_i} - \mathcal{X}_{\overline{U}_i}\rangle + \sum_{k=1}^{n}\langle D^2_{ki}F_{\overline{U}}, (\mathcal{X}_{\overline{S}_k} - \mathcal{X}_{\overline{U}_k}) \times (\mathcal{X}_{\overline{S}_i} - \mathcal{X}_{\overline{U}_i})\rangle$$
$$\geq -[\sum_{j=1}^{m}\langle \overline{y}_j D_i G_{j\overline{U}}, \mathcal{X}_{\overline{S}_i} - \mathcal{X}_{\overline{U}_i}\rangle$$
$$+ \sum_{k=1}^{n}\sum_{j=1}^{m}\langle \overline{y}_j D^2_{ki}G_{j\overline{U}}, (\mathcal{X}_{\overline{S}_k} - \mathcal{X}_{\overline{U}_k}) \times (\mathcal{X}_{\overline{S}_i} - \mathcal{X}_{\overline{U}_i})\rangle], \forall \overline{S}_i, \overline{U}_i \in \mathcal{A}, i \in \underline{n}$$

$$\Rightarrow \sum_{i=1}^{n}[\langle D_i F_{\overline{U}}, \mathcal{X}_{\overline{S}_i} - \mathcal{X}_{\overline{U}_i}\rangle$$
$$+ \sum_{k=1}^{n}\langle D^2_{ki}F_{\overline{U}}, (\mathcal{X}_{\overline{S}_k} - \mathcal{X}_{\overline{U}_k}) \times (\mathcal{X}_{\overline{S}_i} - \mathcal{X}_{\overline{U}_i})\rangle] \tag{4.25}$$
$$\geq -\sum_{i=1}^{n}[\sum_{j=1}^{m}\langle \overline{y}_j D_i G_{j\overline{U}}, \mathcal{X}_{\overline{S}_i} - \mathcal{X}_{\overline{U}_i}\rangle \tag{4.26}$$
$$+ \sum_{k=1}^{n}\sum_{j=1}^{m}\langle \overline{y}_j D^2_{ki}G_{j\overline{U}}, (\mathcal{X}_{\overline{S}_k} - \mathcal{X}_{\overline{U}_k}) \times (\mathcal{X}_{\overline{S}_i} - \mathcal{X}_{\overline{U}_i})\rangle]. \tag{4.27}$$

So by the ineqalities (4.24) and (4.25) we have

$$\sum_{i=1}^{n}[\langle D_i F_{\overline{U}}, \mathcal{X}_{\overline{S}_i} - \mathcal{X}_{\overline{U}_i}\rangle + \sum_{k=1}^{n}\langle D^2_{ki}F_{\overline{U}}, (\mathcal{X}_{\overline{S}_k} - \mathcal{X}_{\overline{U}_k}) \times (\mathcal{X}_{\overline{S}_i} - \mathcal{X}_{\overline{U}_i})\rangle] > 0$$

$$\Rightarrow \sum_{i=1}^{n}\langle D_i F_{\overline{U}}, \mathcal{X}_{\overline{S}_i} - \mathcal{X}_{\overline{U}_i}\rangle + \sum_{k,i=1}^{n}\langle D^2_{ki}F_{\overline{U}}, (\mathcal{X}_{\overline{S}_k} - \mathcal{X}_{\overline{U}_k}) \times (\mathcal{X}_{\overline{S}_i} - \mathcal{X}_{\overline{U}_i})\rangle > 0. \tag{4.28}$$

Now if $F(\overline{S})$ is second order quasiconvex at $\overline{U}$, then

$$F(\overline{S}) > F(\overline{U}) - \frac{1}{2}\sum_{k,i=1}^{n}\langle D^2_{ki}F_{\overline{U}}, (\mathcal{X}_{\overline{S}_k} - \mathcal{X}_{\overline{U}_k}) \times (\mathcal{X}_{\overline{S}_i} - \mathcal{X}_{\overline{U}_i})\rangle, \tag{4.29}$$

which contradicts the equation (4.18). Therefore our assumption is wrong. So we get,

$$\overline{S} = \overline{U}.$$

□

A counterexample for an optimization problem is constructed, where the involved functions are not convex set functions. But the objective function is second order pseudoconvex set function and the constraint function is second order quasiconvex set function. It is also verified that the following example satisfy all duality results of the present investigation.

Example 4.4. Let $g \in L(X, \mathcal{A}, \mu)$. Suppose $\mathcal{S}$ is a convex subfamily of $\mathcal{A}$

$$\text{(P1)}\ \ \min\ \ F(S) = \int_S g\ d\mu + \Big(\int_S g\ d\mu\Big)^3$$

$$\text{subject to } G(S) = \Big(\int_S g\ d\mu\Big)^3$$

$$S \in \mathcal{S}.$$

Let $T \in \mathcal{S}$, then we have

$$\langle DF_T, \mathcal{X}_S - \mathcal{X}_T\rangle = 3\Big(\int_T g\ d\mu\Big)^2\langle g, \mathcal{X}_S - \mathcal{X}_T\rangle + \langle g, \mathcal{X}_S - \mathcal{X}_T\rangle,$$

$$\langle DG_T, \mathcal{X}_S - \mathcal{X}_T\rangle = 3\Big(\int_T g\ d\mu\Big)^2\langle g, \mathcal{X}_S - \mathcal{X}_T\rangle$$

and

$$\langle D^2F_T, (\mathcal{X}_S - \mathcal{X}_T) \times (\mathcal{X}_S - \mathcal{X}_T)\rangle = 6\Big(\int_T g\ d\mu\Big)\langle g, \mathcal{X}_S - \mathcal{X}_T\rangle\langle g, \mathcal{X}_S - \mathcal{X}_T\rangle$$

$$\langle D^2G_T, (\mathcal{X}_S - \mathcal{X}_T) \times (\mathcal{X}_S - \mathcal{X}_T)\rangle = 6\Big(\int_T g\ d\mu\Big)\langle g, \mathcal{X}_S - \mathcal{X}_T\rangle\langle g, \mathcal{X}_S - \mathcal{X}_T\rangle$$

Now we obtain the following second order Mond-Weir type dual problem (SMWD1) of (P1):

$$\text{(SMWD1)}\ \ \max\ \int_T g\ d\mu + \Big(\int_T g\ d\mu\Big)^3 - 3\Big(\int_T g\ d\mu\Big)\langle g, \mathcal{X}_S - \mathcal{X}_T\rangle\langle g, \mathcal{X}_S - \mathcal{X}_T\rangle$$

Subject to,

$$3(y+1)\Big(\int_T g\ d\mu\Big)^2\langle g, \mathcal{X}_S - \mathcal{X}_T\rangle$$

$$+\langle g, \mathcal{X}_S - \mathcal{X}_T\rangle + 6(y+1)\Big(\int_T g\ d\mu\Big)\langle g, \mathcal{X}_S - \mathcal{X}_T\rangle\langle g, \mathcal{X}_S - \mathcal{X}_T\rangle \geq 0,$$

$$y\Big(\int_S g\ d\mu\Big)^3 - 3y\Big(\int_T g\ d\mu\Big)\langle g, \mathcal{X}_S - \mathcal{X}_T\rangle\langle g, \mathcal{X}_S - \mathcal{X}_T\rangle \geq 0,$$

$$y \in \mathbb{R}_+ \ .$$

It can be easily seen that F is strictly second order pseudoconvex and yG is second order quasiconvex functions. Hence this example verifies all the above discussed theorems.

5. Conclusion

Second order convexity and its generalizations have been introduced for n-set functions as an extension of the concept of first order generalized convexity. Second order Mond-Weir type dual model have been constructed for a general mathematical programming problem involving n-set functions and desired duality theorems are established. Further, counter examples are given to justify our results. Since the concept of second order convexity extended to second order pseudoconvexity and second order quasiconvexity, the results presented in the paper are applicable to the larger class of nonconvex programming problems. The obtained results of the present investigation can also be generalized to a class of nondifferentiable programming problems involving n-set functions.

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