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Matrix Transformation of Fibonacci Band Matrix on Generalized bv-space and its Dual Spaces

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abstract: In this paper we introduce a new sequence space $bv(\hat{F})$ by using the Fibonacci band matrix $\hat{F}$. We also establish a few inclusion relations concerning this space and determine its $\alpha-, \beta-, \gamma-$duals. Finally we characterize some matrix classes on the space $bv(\hat{F})$.

Key Words: Fibonacci numbers; $\alpha-$, $\beta-$, $\gamma-$duals; Matrix Transformations.

Contents

1. Introduction

Let ω be the space of all real-valued sequences. Any vector subspace of ω is called a *sequence space*. By l_∞, c, c_0 and l_p $(1 \leq p < \infty)$, we denote the sets of all bounded, convergent, null sequences and $p-$absolutely convergent series, respectively. Also we use the convensions that $e = (1, 1, ...)$ and $e^{(n)}$ is the sequence whose only non-zero term is 1 in the nth place for each $n \in \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, ...\}$.

Let $X$and Y be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real numbers a_{nk}, where $n, k \in \mathbb{N}$. We write $A = (a_{nk})$ instead of $A = (a_{nk})_{n,k=0}^{\infty}$. Then we say that A defines a matrix mapping from X into Y and we denote it by writing $A : X \to Y$ if for every sequence $x = (x_k)_{k=0}^{\infty} \in X$, the sequence $Ax = \{A_n(x)\}_{n=0}^{\infty}$, the A-transform of x, is in Y, where

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk}x_k \ \ (n \in \mathbb{N}). \tag{1.1}$$

For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. Also, if $x \in \omega$, then we write $x = (x_k)_{k=0}^{\infty}$.

By (X, Y), we denote the class of all matrices A such that $A : X \to Y$. Thus $A \in (X, Y)$ iff the series on the right-hand side of (1.1) converges for each $n \in \mathbb{N}$ and every $x \in X$ and we have $Ax \in Y$ for all $x \in X$.

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The approach constructing a new sequence space by means of matrix domain has recently employed by several authors.

The matrix domain X_A of an infinite matrix A in a sequence space X is defined by

$$X_A = \{x = (x_k) \in \omega : Ax \in X\}.$$

Let Δ denote the matrix $\Delta = (\Delta_{nk})$defined by

$$\Delta_{nk} = \begin{cases} (-1)^{n-k}, & n-1 \leq k \leq n; \\ 0, & 0 \leq k < n-1 \quad or \quad k > n. \end{cases}$$

We refer the reader to [2,3,4,5,11,16] for the concept of matrix domain.

Define the sequence $\{f_n\}_{n=0}^{\infty}$ of Fibonacci numbers given by the linear recurrence relations $f_0 = f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}, n \geq 2$. Fibonacci numbers have many interesting properties and applications. For example, the ratio sequences of Fibonacci numbers converges to the golden ratio which is important in sciences and arts. Also, some basic properties of Fibonacci numbers are given as follows:

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \frac{1+\sqrt{5}}{2} = \alpha \quad (golden\ ratio),$$
$$\sum_{k=0}^{n} f_k = f_{n+2} - 1 \quad (n \in \mathbb{N}),$$
$$\sum_k \frac{1}{f_k} \text{ converges},$$
$$f_{n-1}f_{n+1} - f_n^2 = (-1)^{n+1} \quad (n \geq 1) \text{ (Cassini formula)}.$$

Substituting for f_{n+1} in Cassini's formula yields $f_{n-1}^2 + f_n f_{n-1} - f_n^2 = (-1)^{n+1}$. We refer [1,6,8,10] for concepts of fibonacci numbers and related matrix domain.

A sequence space X is called a $FK-space$ if it is complete linear metric space with continuous coordinates $p_n : X \to \mathbb{R}(n \in \mathbb{N})$, where $\mathbb{R}$ denotes the real field and $p_n(x) = x_n$ for all $x = (x_k) \in X$ and every $n \in \mathbb{N}$. A $BKspace$ is a normed $FKspace$, that is a $BK - space$ is a Banach space with continuous coordinates. The sapce $l_p (1 \leq p < \infty)$ is a BK-sapce with

$$\| x \|_p = \left(\sum_{k=0}^{\infty} | x_k |^p \right)^{1/p}$$

and c_0, c and l_∞ are BK-spaces with

$$\| x \|_\infty = \sup_k | x_k | .$$

The sequence space λ is said to be solid iff
$\tilde{\lambda} = \{(u_k) \in \omega : \exists (x_k) \in \lambda \text{ such that } | u_k | \leq | x_k |, \forall\, k \in \mathbb{N}\} \subset \lambda.$

A sequence (b_n) in a normed space X is called a *Schauder basis* for X if every $x \in X$, there is a unique sequence (α_n) of scalars such that $x = \sum_n \alpha_n b_n$, i.e., $\lim_{m\to\infty} \| x - \sum_{n=0}^{m} \alpha_n b_n \| = 0$.

The $\alpha-$, $\beta-$, $\gamma-$duals of the sequence space X are respectively defined by

$$X^{\alpha} = \{a = (a_k) \in \omega : ax = (a_k x_k) \in l_1 , \forall\, x = (x_k) \in X\},$$
$$X^{\beta} = \{a = (a_k) \in \omega : ax = (a_k x_k) \in cs , \forall\, x = (x_k) \in X\},$$

and

$$X^{\gamma} = \{a = (a_k) \in \omega : ax = (a_k x_k) \in bs , \forall\, x = (x_k) \in X\},$$

where cs and bs are the sequence spaces of all convergent and bounded series, respectively (see [5,7,13,17]).

The space of all sequences of bounded variation defined by

$$bv = \left\{ x = (x_k) \in \omega : \sum_{k=1}^{\infty} |x_k - x_{k-1}| < \infty \right\},$$

which is a BK-space under the norm

$$\| x \|_{bv} = | x_0 | + \sum_{k=1}^{\infty} | x_k - x_{k-1} | \text{ for } x \in bv.$$

Now, let $A = (a_{nk})$ be an infinite matrix and list the following conditions:

$$\sup_{n,l} \left| \sum_{k=0}^{l} a_{nk} \right| < \infty \tag{1.2}$$

$$\lim_n a_{nk} = 0 , \forall\, k \tag{1.3}$$

$$\lim_n \sum_k a_{nk} = 0 \tag{1.4}$$

$$\lim_n a_{nk} \text{ exists } , \forall\, k \tag{1.5}$$

$$\lim_n \sum_k a_{nk} \text{ exists } , \forall\, k \tag{1.6}$$

$$\sup_n \left| \sum_k a_{nk} \right| < \infty \tag{1.7}$$

$$\sum_n \left| \sum_k a_{nk} \right| \text{ is convergent.} \tag{1.8}$$

$$\sup_l \sum_n \left| \sum_{k=0}^{l} a_{nk} \right| < \infty \tag{1.9}$$

$$\sum_n \left| \sum_k (a_{nk} - a_{n-1,k}) \right| \text{ is convergent.} \tag{1.10}$$

$$\sup_l \sum_n \left| \sum_{k=0}^{l} (a_{nk} - a_{n-1,k}) \right| < \infty \tag{1.11}$$

$$\sum_n \left| \sum_k a_{nk} \right|^r \text{ is convergent} \tag{1.12}$$

$$\sup_l \sum_n \left| \sum_{k=0}^{l} a_{nk} \right|^r < \infty. \tag{1.13}$$

We refer [9,12,14,15,17] for the concept of matrix transformations. Now, we may give the following lemma due to Stieglitz and Tietz [11] on the characterization of the matrix transformations between some sequence spaces.

Lemma 1.1. *The following statements hold:*

(a) $A \in (a_{nk}) \in (bv, c_0)$ *iff (1.2), (1.3), (1.4) holds.*

(b) $A \in (a_{nk}) \in (bv, c)$ *iff (1.2), (1.5), (1.6) holds.*

(c) $A \in (a_{nk}) \in (bv, l_\infty)$ *iff (1.2), (1.7) holds.*

(d) $A \in (a_{nk}) \in (bv, l_1)$ *iff (1.8), (1.9) holds.*

(e) $A \in (a_{nk}) \in (bv, bv)$ *iff (1.10), (1.11) holds.*

(f) $A \in (a_{nk}) \in (bv, l_p),\ p > 1$ *iff (1.12), (1.13) holds.*

2. The Fibonacci difference sequence space $bv(\hat{F})$

In this section, we have used the Fibonacci band matrix $\hat{\mathrm{F}} = (f_{nk})$ and introduce the sequence space $bv(\hat{\mathrm{F}})$. Also we present some inclusion theorems and construct the Schauder basis of the space $bv(\hat{\mathrm{F}})$.

Let f_n be the nth Fibonacci number for every $n \in \mathbb{N}$. Then we define the infinite matrix $\hat{\mathrm{F}} = (f_{nk})$ by

$$f_{nk} = \begin{cases} -\frac{f_{n+1}}{f_n}, & k = n-1; \\ \frac{f_n}{f_{n+1}}, & k = n; \\ 0, & 0 \le k < n-1 \quad or \quad k > n \end{cases}$$

where $n, k \in \mathbb{N}$.

Define the sequence $y = (y_n)$, which will be frequently used by the $\hat{F}-$transform of a sequence $x = (x_n)$, i.e., $y_n = \hat{F}_n(x)$, where

$$y_n = \begin{cases} \frac{f_0}{f_1}x_0 = x_0, & n = 0; \\ \dfrac{f_n}{f_{n+1}}x_n - \dfrac{f_{n+1}}{f_n}x_{n-1}, & n \geq 1 \end{cases} \tag{2.1}$$

where $n \in \mathbb{N}$.

Moreover, it is obvious that $\hat{F}$ is a triangle. Thus, it has a unique inverse $\hat{F}^{-1}$ and it is given by

$$\hat{f}^{-1}_{nk} = \begin{cases} \frac{f^2_{n+1}}{f_k f_{k+1}}, & 0 \leq k \leq n; \\ 0, & k > n \end{cases}$$

for all $n, k \in \mathbb{N}$. Therefore we have by (2.1) that

$$x_n = \sum_{k=0}^{n} \frac{f^2_{n+1}}{f_k f_{k+1}} y_k; (n \in \mathbb{N}). \tag{2.2}$$

Now, we introduce a new Fibonacci sequence space $bv(\hat{F})$ as follows

$$bv(\hat{F}) = \left\{ x = (x_k) \in \omega : \sum_{k=1}^{\infty} \left| \hat{F}_k(x) - \hat{F}_{k-1}(x) \right| < \infty \right\}.$$

In this section, we give some results related to the space $bv(\hat{F})$.

Theorem 2.1. *$bv(\hat{F})$ is a BK-space with norm*

$$\| x \|_{bv(\hat{F})} = | x_0 | + \sum_{k=1}^{n} | \hat{F}_k(x) - \hat{F}_{k-1}(x) | .$$

Proof. Since bv is a BK-space with respect to the norm

$$\| x \|_{bv} = | x_0 | + \sum_{k=1}^{\infty} | x_k - x_{k-1} | \text{ for } x \in bv$$

and the matrix $\hat{F}$ is triangular matrix. By Theorem 4.3.3 of Wilansky [17] gives the fact that the space $bv(\hat{F})$ is a BK space with $\| . \|_{bv(\hat{F})}$ norm. □

Remark 2.2. *The sequence space $bv(\hat{F})$ is of non-absolute type, because*

$$\| x \|_{bv(\hat{F})} \neq \| | x | \|_{bv(\hat{F})},$$

where $| x | = (| x_k |)$.

Theorem 2.3. *The inclusion $bv \subset bv(\hat{F})$ holds.*

Proof. Let $x = (x_k) \in bv$. Then we have

$$\sum_{k=1}^{\infty} |x_k - x_{k-1}| < \infty.$$

Since the inequalities $\frac{f_k}{f_{k+1}} \leq 1$ and $\frac{f_{k+1}}{f_k} \leq 2$ for every $k \in \mathbb{N}$, therefore we have

$$\begin{aligned}&\sum_{k=1}^{\infty} \mid \hat{F}_k(x) - \hat{F}_{k-1}(x) \mid \\ &= \left|\frac{x_1}{2} - 3x_0\right| + \sum_{k=2}^{\infty} \mid \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} - \frac{f_{k-1}}{f_k} x_{k-1} + \frac{f_k}{f_{k-1}} x_{k-2} \mid < \infty.\end{aligned}$$

So, $x \in bv(\hat{F})$. Hence $bv \subseteq bv(\hat{F})$.

Further, since $x = (x_k) = (f_{k+1}^2)$ is in $bv(\hat{F}) - bv$, therefore $bv \subset bv(\hat{F})$. □

Theorem 2.4. *The inclusion $l_1 \subset bv(\hat{F})$ holds.*

Proof. We have

$$l_1 = \left\{ x \in \omega : \sum_{k=1}^{\infty} \mid x_k \mid < \infty \right\}.$$

Let $x \in l_1$. Then $\sum_{k=1}^{\infty} \mid x_k \mid < \infty$.

Now,

$$\sum_{k=2}^{\infty} \mid \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} - \frac{f_{k-1}}{f_k} x_{k-1} + \frac{f_k}{f_{k-1}} x_{k-2} \mid < \infty.$$

Therefore, $x \in bv(\hat{F})$. We have $x = (x_k) \in bv(\hat{F})$, where $x_k = 1$ for all $k \in \mathbb{N}$ but $x \notin l_1$. Hence $l_1 \subset bv(\hat{F})$. □

Lemma 2.5. *[2] Let λ be a BK-space including the space ϕ. Then, λ is solid if and only if $l_\infty \lambda \subset \lambda$.*

Theorem 2.6. *The space $bv(\hat{F})$ is not solid.*

Proof. Let the sequences $u = (u_k)$ and $v = (v_k)$ defined by $u_k = f_{k+1}^2$ and $v_k = (-1)^{k+1}$ for all $k \in \mathbb{N}$. Then it is clear that $u \in bv(F)$ and $v \in l_\infty$. Also, we have $uv = (-1)^{k+1} {f_{k+1}}^2$ is not in $bv(\hat{F})$, since $\hat{F}_k(uv) = 2(-1)^{k+1} f_k f_{k+1}$ for all $k \in \mathbb{N}$. This shows that $l_\infty bv(\hat{F})$ is not a subset of $bv(\hat{F})$. By applying Lemma 2.5, we get $bv(\hat{F})$ is not solid. □

Theorem 2.7. *The Fibonacci difference sequence space $bv(\hat{F})$ is linearly isomorphic to the bv space i.e. $bv(\hat{F}) \cong bv$.*

Proof. To prove this, we have to show that there exists a linear bijective mapping between $bv(\hat{F})$ and bv. Let us consider a mapping T defined from $bv(\hat{F})$ to bv by $Tx = \hat{F}(x) = y \in bv$ for every $x \in bv(\hat{F})$ where $x = (x_k)$ and $y = (y_k)$.

It is obvious that T is linear and for $x = 0$, we have $Tx = 0$. Hence T is injective.

Let $y = (y_k) \in bv$ and define the sequence $x = (x_k)$ by $x_k = \sum\limits_{j=0}^{k} \frac{f_{k+1}^2}{f_j f_{j+1}} y_j$, for $k \in \mathbb{N}$. Then by using (2.1) and (2.2), we have

$$\begin{aligned}
&\| x \|_{bv(\hat{F})}\\
&=| x_0 | + \sum_{k=1}^{n} | \hat{F}_k(x) - \hat{F}_{k-1}(x) |\\
&=| y_0 | + \sum_{k=1}^{\infty} | y_k - y_{k-1} |\\
&=\| y \|_{bv} < \infty.
\end{aligned}$$

Thus we have $x \in bv(\hat{F})$. Hence T is surjective and norm preserving. Consequently, T is a linear bijection which proves that $bv(\hat{F})$ and bv are linearly isomorphic. □

3. The $\alpha-$, $\beta-$ and $\gamma-$duals of the space $bv(\hat{F})$

In this section, we determine the $\alpha-$, $\beta-$ and $\gamma-$duals of the sequence space $bv(\hat{F})$. The following known results are fundamental for our investigation.

Lemma 3.1. *[11] Let $A = (a_{nk})$ be an infinite matrix. Then the following statement holds:*

(i) $A \in (bv : l_\infty)$ *iff* $\sup_n | \sum\limits_k a_{nk} | < \infty$, $\sup_{n,l} | \sum\limits_{k=0}^{l} a_{nk} | < \infty$.

(ii) $A \in (bv : c)$ *iff* $\sup_{n,l} | \sum\limits_{k=0}^{l} a_{nk} | < \infty$, $\lim\limits_n a_{nk}$ *exists for all* k *and* $\lim\limits_n \sum\limits_k a_{nk}$ *exists.*

(iii) $A \in (bv : l_1)$ *iff* $\sum\limits_n | \sum\limits_k a_{nk}$ *convergent,* $\sup_l \sum\limits_n | \sum\limits_{k=0}^{l} a_{nk} | < \infty$.

Theorem 3.2. *The α-dual of the sequence space $bv(\hat{F})$ is the set $d_1 \cap d_2$, where*

$$d_1 = \left\{ a = (a_k) \in \omega : \sum_n | \sum_k b_{nk} | \ \text{ is convergent} \right\},$$

$$d_2 = \left\{ a = (a_k) \in \omega : \sup_l \sum_n | \sum_{k=0}^{l} b_{nk} | < \infty \right\}$$

and the matrix $B = (b_{nk})$ is defined as follows

$$b_{nk} = \begin{cases} \frac{f_{n+1}^2}{f_k f_{k+1}} a_n, & 0 \le k \le n; \\ 0, & k > n \end{cases}$$

where $a = (a_n) \in \omega$.

Proof. Let $a = (a_n) \in \omega$. Also for every $x = (x_n) \in \omega$, we put $y = \hat{F}(x)$. Then from (2.1), it follows that $x_k = \sum_{j=0}^{k} \frac{f_{k+1}^2}{f_j f_{j+1}} y_j$ and

$$B_n(y) = \sum_{k=0}^{n} b_{nk} y_k = \sum_{k=0}^{n} \frac{f_{n+1}^2}{f_k f_{k+1}} a_n y_k = a_n x_n, \tag{3.1}$$

where $n \in \mathbb{N}$.

Thus, we observe by (3.1) that $ax = (a_n x_n) \in l_1$ whenever $x \in bv(\hat{F})$ if and only if $By \in l_1$ whenever $y \in bv$. Therefore, we derive by using the Lemma 3.1 that

$$\sum_n \mid \sum_k b_{nk} \mid \text{ is convergent and } \sup_l \sum_n \mid \sum_{k=0}^{l} b_{nk} \mid < \infty.$$

□

Theorem 3.3. *Define the sets d_3, d_4, d_5 and d_6 by*

$$d_3 = \left\{ a = (a_k) \in \omega : \sup_{n,l} \mid \sum_{k=0}^{l} d_{nk} \mid < \infty \right\}$$

$$d_4 = \left\{ a = (a_k) \in \omega : \lim_n d_{nk} \text{ exists }, \forall\ k \right\}$$

$$d_5 = \left\{ a = (a_k) \in \omega : \lim_n \sum_k d_{nk} \text{ exists} \right\}$$

and

$$d_6 = \left\{ a = (a_k) \in \omega : \sup_n \mid \sum_k d_{nk} \mid < \infty \right\}.$$

Then $\left(bv(\hat{F})\right)^{\beta} = d_3 \cap d_4 \cap d_5$ and $\left(bv(\hat{F})\right)^{\gamma} = d_3 \cap d_6$.

Proof. Let $a = (a_k) \in \omega$ and consider the equality

$$\sum_{k=0}^{n} a_k x_k = \sum_{k=0}^{n} a_k \left(\sum_{j=0}^{k} \frac{f_{k+1}^2}{f_j f_{j+1}} y_j \right) = \sum_{k=0}^{n} \left(\sum_{j=k}^{n} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j \right) y_k = D_n(y), \tag{3.2}$$

where $D = (d_{nk})$ is defined by

$$d_{nk} = \begin{cases} \sum\limits_{j=k}^{n} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j, & 0 \le k \le n \\ 0, & k > n \end{cases}$$

where $n, k \in \mathbb{N}$. Then we deduce from Lemma 3.1 that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in bv(\hat{F})$ iff $Dy \in c$ whenever $y \in bv$. Thus $a \in \left(bv(\hat{F})\right)^{\beta}$ iff $a \in d_3$, $a \in d_4$ and $a \in d_5$. Thus $\left(bv(\hat{F})\right)^{\beta} = d_3 \cap d_4 \cap d_5$. Similarly, we can show that $\left(bv(\hat{F})\right)^{\gamma} = d_3 \cap d_6$. □

4. Some matrix transformations related to the space $bv(\hat{F})$

For simplicity in notation, we write $\tilde{a}_{nk} = \sum\limits_{j=k}^{\infty} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj}$ for all $k, n \in \mathbb{N}$.

We used the following lemma to established our results.

Lemma 4.1. *[2] Let $C = (c_{nk})$ be defined via a sequence $a = (a_k) \in \omega$ and the inverse matrix $V = (v_{nk})$ of the triangle matrix $U = (u_{nk})$ by*

$$c_{nk} = \begin{cases} \sum\limits_{j=k}^{n} a_j v_{jk}, & 0 \le k \le n; \\ 0, & k > n \end{cases}$$

for all $k, n \in \mathbb{N}$. Then for any sequence space λ,

$$\lambda_U^{\gamma} = \{a = (a_k) \in \omega : C \in (\lambda, l_\infty)\}$$

and

$$\lambda_U^{\beta} = \{a = (a_k) \in \omega : C \in (\lambda, c)\}.$$

Theorem 4.2. *Let μ be an arbitrary subset of ω. Then $A = (a_{nk}) \in (bv(\hat{F}), \mu)$ iff*

$$D^{(m)} = \left(d_{nk}^{(m)}\right) \in (bv, c) \text{ for all } n \in \mathbb{N}, \tag{4.1}$$

$$D = (d_{nk}) \in (bv, \mu), \tag{4.2}$$

where

$$d_{nk}^{(m)} = \begin{cases} \sum\limits_{j=1}^{m} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj}, & 0 \le k \le m; \\ 0, & k > m \end{cases}$$

and $d_{nk} = \sum\limits_{j=k}^{\infty} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj}$ for all $k, m, n \in \mathbb{N}$.

Proof. To prove this theorem, we follow the similar way due to Kirişçi and Başar [9]. Let $A = (a_{nk}) \in (bv(\hat{F}), \mu)$ and $x = (x_k) \in bv(\hat{F})$. From (2.2), we have $x_k = \sum\limits_{j=0}^{k} \frac{f_{j+1}}{f_j f_{j+1}} y_j$ for all $k \in \mathbb{N}$. From (3.2) we get

$$\begin{aligned}\sum_{k=0}^{m} a_{nk}x_k =& \sum_{k=0}^{m} a_{nk}\left(\sum_{j=0}^{k} \frac{f_{k+1}^2}{f_j f_{j+1}} y_j\right)\\ =& \sum_{k=0}^{m}\left(\sum_{j=k}^{m} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj}\right) y_k\\ =& \sum_{k=0}^{m} d_{nk}^{(m)} y_k\\ =& D_n^{(m)}(y), \end{aligned} \tag{4.3}$$

for all $m, n \in \mathbb{N}$. Since Ax exists, $D^{(m)} \in (bv, c)$. As $m \to \infty$ in the equality (4.3), we obtain $Ax = Dy$ which implies $D \in (bv, \mu)$.

Conversely, suppose (4.1) and (4.2) holds and take any $x \in bv(\hat{F})$. Then, we have $(d_{nk}) \in bv^\beta$ which gives together with (4.1) that $A_n = (a_{nk})_{k\in\mathbb{N}} \in (bv(\hat{F}))^\beta$ for all $n \in \mathbb{N}$. Thus, Ax exists. Therefore, we derive by equality (4.3) as $m \to \infty$ that $Ax = Dy$ and this shows that $A \in (bv(\hat{F}), \mu)$. □

Now, we list the following conditions,

$$\sup_{n,l}\left|\sum_{k=0}^{l} d_{nk}\right| < \infty \tag{4.4}$$

$$\lim_n d_{nk} = 0 \text{ , } \forall\ k \tag{4.5}$$

$$\lim_n \sum_k d_{nk} = 0 \tag{4.6}$$

$$\lim_n d_{nk} \text{ exists , } \forall\ k \tag{4.7}$$

$$\lim_n \sum_k d_{nk} \text{ exists , } \forall\ k \tag{4.8}$$

$$\sup_n\left|\sum_k d_{nk}\right| < \infty \tag{4.9}$$

$$\sum_n\left|\sum_k d_{nk}\right| \text{ is convergent} \tag{4.10}$$

$$\sup_l \sum_n\left|\sum_{k=0}^{l} d_{nk}\right| < \infty \tag{4.11}$$

$$\sum_n \left| \sum_k (d_{nk} - d_{n-1,k}) \right| \text{ is convergent} \tag{4.12}$$

$$\sup_l \sum_n \left| \sum_{k=0}^{l} (d_{nk} - d_{n-1,k}) \right| < \infty \tag{4.13}$$

$$\sum_n \left| \sum_k d_{nk} \right|^r \text{ is convergent} \tag{4.14}$$

$$\sup_l \sum_n \left| \sum_{k=0}^{l} d_{nk} \right|^r < \infty \tag{4.15}$$

$$\sup_{n,l} \left| \sum_{k=0}^{l} d_{nk}^{(m)} \right| < \infty \tag{4.16}$$

$$\lim_n d_{nk}^{(m)} \text{ exists } , \forall\ k \tag{4.17}$$

$$\lim_n \sum_k d_{nk}^{(m)} \text{ exists } , \forall\ k. \tag{4.18}$$

Combining Theorem 4.2 and Lemma 1.1, we derive the following results.

Corollary 4.3. *Let $A = (a_{nk})$ be an infinite matrix. Then the following statements hold:*

(a) $A \in (bv(\hat{F}), c_0)$ *if and only if (4.4), (4.5), (4.6), (4.16), (4.17), (4.18).*

(b) $A \in (bv(\hat{F}), c)$ *if and only if (4.4), (4.7), (4.8), (4.16), (4.17), (4.18).*

(c) $A \in (bv(\hat{F}), l_\infty)$ *if and only if (4.4), (4.9), (4.16), (4.17), (4.18).*

(d) $A \in (bv(\hat{F}), l_1)$ *if and only if (4.10), (4.11), (4.16), (4.17), (4.18).*

(e) $A \in (bv(\hat{F}), bv)$ *if and only if (4.12), (4.13), (4.14), (4.17), (4.18).*

(f) $A \in (bv(\hat{F}), l_p),\ p > 1$ *if and only if (4.14), (4.15), (4.16), (4.17), (4.18).*

References

1. A. Alotaibi, M. Mursaleen, B.AS. Alamri, S.A. Mohiuddine, Compact operators on some Fibonacci difference sequence spaces, J. Inequal. Appl. 2015, 2015:203.
2. B. Altay, F. Başar, Certain topological properties and duals of the domain of a triangle matrix in a sequence space, J. Math. Anal. Appl. 336(2007) 632-645.
3. B. Altay, F. Başar, M. Mursaleen, Some generalizations of the space bv_p of p-bounded variation sequences, Nonlinear Anal. TMA 68(2008) 273-287.
4. C. Aydin, F. Başar, Some new sequence spaces which include the spaces l_p and l_∞, Demonstr. Math. 38(3)(2005) 641-656.
5. F. Başar, Summability Theory and Its Applications, Bentham Science Publishers, Istanbul, 2012.

6. A. Das, B. Hazarika, Some properties of generalized Fibonacci difference bounded and p-absolutely convergent sequences, Bol. Soc. Parana. Mat., 36(1)(2018), 37-50.

7. P.K. Kamthan, M.Gupta, Sequence Spaces and Series, Marcel Dekker Inc., New York and Basel, 1981.

8. E.E. Kara, Some topological and geometrical properties of new Banach sequence spaces, J. Inequa. Appl. 2013, 2013:38

9. M. Kirişçi, F. Başar, Some new sequence spaces derived by the domain of generalized difference matrix, Comput. Math. Appl. 60(2010) 1229-1309.

10. T. Koshy, Fibonacci and Lucas Numbers with applications, Wiley, 2001.

11. Michael Stieglitz, Hubert Tietz, Matrixtransformationen von Folgenräumen Eine Ergebnisübersicht, Math. Z. 154(1977) 1-16.

12. SK Mishra, Matrix maps involving certain sequence spaces, Indian J. Pure Appl. Math. 24(2)(1993) 125-132.

13. M. Mursaleen, AK Gaur, AH Saifi, Some new sequence spaces and their duals and matrix transformations, Bull. Calcutta Math. Soc. 88(3)(1996) 207-212.

14. M. Mursaleen, Generalized spaces of difference sequences, J. Math. Anal. Appl. 203(3)(1996) 738-745.

15. M. Mursaleen, AK Noman, On some new sequence spaces of non-absolute type related to the spaces l_p and l_∞ I. Filomat 25(2)(2011) 33-51.

16. B. C. Tripathy, Matrix Transformation Between Series and Sequences, Bull. Malaysian Math. Soc. 21(1998) 17-20.

17. A. Wilansky, Sumability Through Functional Analysis.North-Holland Mathematics Studies, vol. 85. Elsevier Amsterdam, 1984.

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