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Commutativity of Near-rings With Certain Constrains on Jordan Ideals

Abdelkarim Boua

ABSTRACT: The purpose of this paper is to study derivations satisfying certain differential identities on Jordan ideals of 3-prime near-rings. Moreover, we provide examples to show that hypothesis of our results are necessary.

Key Words: 3-prime near-rings, Jordan ideal, Derivations.

Contents

1. Introduction

Throughout this paper $\mathcal{N}$ will be a left near-ring with multiplicative center $Z(\mathcal{N})$; and usually $\mathcal{N}$ will be 3-prime, if for all $x, y \in \mathcal{N}$, $x\mathcal{N}y = 0$ implies $x = 0$ or $y = 0$. A near-ring $\mathcal{N}$ is called zero-symmetric if $x0 = 0$, for all $x \in \mathcal{N}$ (recall that right distributivity yields $0x = 0$). An additive mapping $d : \mathcal{N} \to \mathcal{N}$ is a derivation if $d(xy) = xd(y) + d(x)y$ for all $x, y \in \mathcal{N}$. For any $x, y \in \mathcal{N}$; as usual $[x, y] = xy - yx$ and $x \circ y = xy + yx$ will denote the well-known Lie product and Jordan product respectively. Recall that for $n \geq 2$, $\mathcal{N}$ is called n-torsion free if $nx = 0$ implies $x = 0$ for all $x \in \mathcal{N}$. For terminologies concerning near-rings we refer to G. Pilz [11]. An additive subgroup J of $\mathcal{N}$ is said to be a Jordan ideal of $\mathcal{N}$ if $j \circ n \in J$ and $n \circ j \in J$ for all $j \in J, n \in \mathcal{N}$ (For more details see reference [7]). The existing literature on 3-prime near-rings contains a number of theorems concerning multiplicative commutativity of near-rings. H. E. Bell, G. Mason, N. Argaç, A. A. M. Kamal, X. K. Wang and other have proved several results on commutativity of 3-prime near-rings with derivations (for reference see [1], [4], [10], [12]... etc.) Indeed, motivated by the notion of Jordan ideal introduced in near-rings (for reference [7], [8]) and the results of H. E. Bell, it is natural to continue this line of investigation for comparable results for 3-prime near-rings having derivations with Jordan ideals. In the present paper, we shall attempt to generalize the known result of H. E. Bell and study the commutativity of Jordan ideal in 3-prime near-rings satisfying certain identities involving the Jordan ideal.

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2. Some preliminaries

We begin with the following results which will be used extensively to prove our theorem. The first Lemma appears in [4] and [12].

Lemma 2.1. *Let $\mathcal{N}$ be a 3-prime near-ring.*

(i) *If $z \in Z(\mathcal{N})\backslash\{0\}$ and $xz \in Z(\mathcal{N})$, then $x \in Z(\mathcal{N})$.*

(ii) *If $\mathcal{N} \subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring.*

(iii) *If $\mathcal{N}$ is 2-torsion free and d is a derivation on $\mathcal{N}$ such that $d^2 = 0$, then $d = 0$.*

(iv) *If d is a derivation, then $x \in Z(\mathcal{N})$ implies $d(x) \in Z(\mathcal{N})$.*

Lemma 2.2. *Let $\mathcal{N}$ be a near-ring and d a derivation of $\mathcal{N}$. Then $\mathcal{N}$ satisfies the following partial distributive law*
$\Big(xd(y) + d(x)y\Big)z = xd(y)z + d(x)yz$ *for all $x, y, z \in \mathcal{N}$.*

Lemma 2.3. *Let d be an arbitrary additive endomorphism of $\mathcal{N}$. Then $d(xy) = xd(y) + d(x)y$ for all $x, y \in \mathcal{N}$ if and only if $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathcal{N}$. Therefore d is a derivation if and only if $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathcal{N}$.*

Recall that a map $d : \mathcal{N} \to \mathcal{N}$ is called a multiplicative derivation on N if $d(xy) = xd(y) + d(x)y$ for all $x, y \in \mathcal{N}$. Notice that any derivation on $\mathcal{N}$ is a multiplicative derivation.

Lemma 2.4. [8, Corollary 3] *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and J be a nonzero Jordan ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a derivation d such that $d(J) = \{0\}$, then $d = 0$ or the element of J commute under the multiplication of $\mathcal{N}$.*

Lemma 2.5. [7, Lemma 2 & Lemma 3] *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and J a nonzero Jordan ideal of $\mathcal{N}$.*

(i) *If $j^2 = 0$ for all $j \in J$, then $J = \{0\}$.*

(ii) *If $J \subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring.*

Lemma 2.6. [10, Lemma 2.1] *A near-ring $\mathcal{N}$ admits a multiplicative derivation if and only if it is zero-symmetric.*

Using Lemma 2.6, we deduce that in all our results in the paper that $\mathcal{N}$ is a zero-symmetric near-ring.

Lemma 2.7. *Let $\mathcal{N}$ be a 3-prime near-ring and J a nonzero Jordan ideal of $\mathcal{N}$. If the element of J commute under the multiplication of $\mathcal{N}$, then J is commutative.*

Proof. Suppose that the element of J commute under the multiplication of $\mathcal{N}$, then $(i+j)(k+k) = (k+k)(i+j)$ for all $i, j, k \in J$ so that $\big((j+i)-(i+j)\big)k = 0$ for all $i, j, k \in J$. Replacing k by $k \circ n$ in the last expression, we get $\big((j+i)-(i+j)\big)nk = 0$ for all $i, j, k \in J$, $n \in \mathcal{N}$. Further, application of 3-primeness of $\mathcal{N}$ yields $j+i = i+j$ for all $i, j \in J$. Finally we conclude that J is commutative. □

3. Main Results

In this section, we give some new results and examples concerning the existence of Jordan ideal and derivations in near-rings. We begin this section by the following interesting results for near-rings.
In [4] H. E. Bell and G. Mason proved that a 2-torsion 3-prime near-ring $\mathcal{N}$ must be commutative if it admits a nonzero derivation d such that $d(\mathcal{N}) \subseteq Z(\mathcal{N})$. Our objective in the following theorem is to generalize and improve this result by treating the case of a Jordan ideal J of 3-prime near-ring $\mathcal{N}$ instead of $\mathcal{N}$. The following Theorem gives an analogous result for near-rings.

Theorem 3.1. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and J be a nonzero Jordan ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a derivation d such that $d(J) \subseteq Z(\mathcal{N})$, then $d = 0$ or J is commutative.*

Proof. Suppose that $d(j) \in Z(\mathcal{N})$ for all $j \in J$, then $d(j \circ j) = d(2j^2) \in Z(\mathcal{N})$ for all $j \in J$ which implies that $d(j)(j + j) + jd(j + j) \in Z(\mathcal{N})$ for all $j \in J$, by hypothesis this expression reduced to $4jd(j) \in Z(\mathcal{N})$ for all $j \in J$ and by application of Lemma 2.1 (i), we obtain

$$d(j) = 0 \text{ or } 4j \in Z(\mathcal{N}) \quad \text{for all } j \in J. \tag{3.1}$$

Suppose there exists $j_0 \in J$ such that $4j_0 \in Z(\mathcal{N})$, then $4j_0(j^2 + j^2) = j \circ 4j_0 j \in J$ for all $j \in J$. In view of the hypothesis, we find that

$$d\big(4j_0(j^2 + j^2)\big) = d(4j_0)(2j^2) + 4j_0 d(2j^2) \in Z(\mathcal{N}), \quad \text{for all } j \in J.$$

Using Lemma 2.2, the last upshot becomes $d(4j_0)(2j^2) \in Z(\mathcal{N})$ for all $j \in J$. Apply Lemma 2.1 (i), we obtain $d(4j_0) = 0$ or $2j^2 \in Z(\mathcal{N})$ for all $j \in J$ and 2-torsion freeness forces $d(j_0) = 0$ or $2j^2 \in Z(\mathcal{N})$ for all $j \in J$. It follows that the equation (3.1) can be written in the form

$$d(j) = 0 \text{ or } 2k^2 \in Z(\mathcal{N}) \quad \text{for all } j, k \in J. \tag{3.2}$$

Assume that $k^2 + k^2 = 2k^2 \in Z(\mathcal{N})$ for all $k \in J$. Then $d(k(2k^2)) = d(k \circ k^2) \in Z(\mathcal{N})$ for all $k \in J$. Therefore, $d(k)(2k^2) + kd(2k^2) \in Z(\mathcal{N})$ for all $k \in J$. Using the Lemma 2.2 and Lemma 2.1 (i), we find that $d(2k^2) = 0$ or $k \in Z(\mathcal{N})$ for all $k \in J$. By definition of d and 2-torsion freeness of $\mathcal{N}$, we arrive at $d(k)\mathcal{N}k = \{0\}$ or $k \in Z(\mathcal{N})$ for all $k \in J$. By 3-primeness of N, we conclude that $d(k) = 0$ or $k \in Z(N)$ for all $k \in J$, in this case (3.2) becomes

$$d(k) = 0 \text{ or } k \in Z(\mathcal{N}) \quad \text{for all } k \in J. \tag{3.3}$$

If there is $k_0 \in J$ such that $d(k_0) = 0$, then $k_0 d(j) + d(j)k_0 = d(k_0 \circ j) \in Z(\mathcal{N})$ for all $j \in J$ which means that $2k_0 d(j) \in Z(N)$ for all $j \in J$. By 2-torsion freeness of $\mathcal{N}$ and Lemma 2.1(i), we arrive at $d(j) = 0$ for all $j \in J$ or $k_0 \in Z(N)$. In this case (3.3) becomes

$$d(j) = 0 \text{ or } k \in Z(\mathcal{N}) \quad \text{for all } j, k \in J. \tag{3.4}$$

Using Lemma 2.4 and Lemma 2.7, we conclude that J is commutative. □

As a consequence of Theorem 3.1, we have the following result:

Corollary 3.2. [4, Theorem 2.1] *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring. If $\mathcal{N}$ admits a nonzero derivation d such that $d(\mathcal{N}) \subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring.*

Theorem 3.3. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and J be a nonzero Jordan ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a derivation d such that $d(J^2) = \{0\}$, then $d = 0$ or J is commutative.*

Proof. Suppose that $d(ij) = 0$ for all $i, j \in J$. Replacing i by $2i^2$, we get

$$\begin{aligned} 0 &= d\big((2i)^2 j\big) \\ &= d(i)(i+i)j + id((i+i)j) \quad \text{for all } \ i, j \in J \end{aligned}$$

this implies that $d(i)(2i)j = 0$ for all $i, j \in J$. Writing $j \circ n$ for j where $n \in \mathcal{N}$ in the above relation gives

$$d(i)(2i)jn + d(i)(2i)nj = 0 \quad \text{for all } \ i, j \in J, n \in \mathcal{N}$$

this equation reduced to

$$d(i)(2i)\mathcal{N}j = \{0\} \quad \text{for all } \ i, j \in J.$$

Since $J \neq \{0\}$, then by 3-primeness and 2-torsion freeness of $\mathcal{N}$, we arrive at $d(i)i = 0$ for all $i \in J$. Using this in the calculation of the expression $d(ji(i \circ n)) = 0$ for all $i, j \in J$, $n \in \mathcal{N}$, we find that $d(j)ini = 0$ for all $i, j \in J$ and using again the 3-primeness of $\mathcal{N}$, we obtain $d(j)i = 0$ for all $i, j \in J$. Taking $i \circ n$ instead of i in the last equation and invoking it again, we get $d(j)ni = 0$ for all $i, j \in J$ and by application the 3-primeness of $\mathcal{N}$ and $J \neq \{0\}$, we arrive at $d(j) = 0$ for all $j \in J$. By Lemma 2.4 and Lemma 2.7, we conclude that J is commutative. □

The following example demonstrates that the condition "3-primeness of $\mathcal{N}$" in Theorem 3.1 and Theorem 3.3 is crucial.

Example 3.4. *Let $\mathcal{S}$ be a 2-torsion free noncommutative near-ring. We define $\mathcal{N}$, J and d by: $\mathcal{N} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathcal{S} \right\}$, $J = \left\{ \begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix} \mid m \in \mathcal{S} \right\}$ and $d\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$. It is obvious that $\mathcal{N}$ is 2-torsion free near-ring not 3-prime, J a nonzero Jordan ideal of $\mathcal{N}$ and d is a nonzero derivation such that $\{0\} = d(J) \subseteq Z(\mathcal{N})$ and $d(J^2) = \{0\}$. But J is not commutative.*

Theorem 3.5. *Let $\mathcal{N}$ be a 6-torsion free 3-prime near-ring and J be a nonzero Jordan ideal of $\mathcal{N}$. There is no derivation d of $\mathcal{N}$ such that $d^2(J) = J$.*

Proof. Suppose that $d^2(j) = j$ for all $j \in J$. Replacing j by $j \circ jn$ where $n \in \mathcal{N}$ and invoking the fact that $j \circ jn = j(j \circ n)$, we obtain $d^2(j(j \circ n)) = j(j \circ n)$ for all $j \in J, n \in \mathcal{N}$. Developing this expression, we arrive at

$$d^2(j)(j \circ n) + 2d(j)d(j \circ n) + jd^2(j \circ n) = j(j \circ n)$$

for all $j \in J, n \in \mathcal{N}$
which reduced to

$$2d(j)d(j \circ n) = -j(j \circ n) \quad \text{for all } \; j \in J, n \in \mathcal{N}. \tag{3.5}$$

Applying d to (3.5) and invoking our hypothesis, we obtain

$$2jd(j \circ n) + 2d(j)(j \circ n) = -d(j(j \circ n)) \quad \text{for all } \; j \in J, n \in \mathcal{N} \tag{3.6}$$

Applying d again to (3.6) and using our hypothesis, we conclude that $3j(j \circ n) = 0$ for all $j \in J$, $n \in \mathcal{N}$. By 6-torsion free of $\mathcal{N}$, we find that $j(j \circ n) = 0$ for all $j \in J$, $n \in \mathcal{N}$, which implies that $jnj = -j^2n$ for all $j \in J, n \in \mathcal{N}$. Replacing n by nm and using it again, we get

$$\begin{aligned} jnmj &= -j^2nm \\ &= j^2n(-m) \\ &= (-jnj)(-m) \\ &= jn(-j)(-m) \quad \text{for all } \; j \in J, m, n \in \mathcal{N} \end{aligned}$$

which means that

$$jn(mj - (-j)(-m)) = 0 \quad \text{for all } \; j \in J, m, n \in \mathcal{N}$$

Putting -j instead of j in the last expression, we arrive at

$$(-j)\mathcal{N}(-mj + jm) = \{0\} \quad \text{for all } \; j \in J, m \in \mathcal{N} \tag{3.7}$$

By 3-primeness of $\mathcal{N}$, we conclude that $j \in Z(\mathcal{N})$ for all $j \in J$. Replacing j by $2j^2$ in our hypothesis and using the 2-torsion freeness of $\mathcal{N}$, we get $d^2(j^2) = j^2$ for all $j \in J$ developing this expression by definition of d, we obtain

$$2d(j)d(j) + j^2 = 0 \quad \text{for all } \; j \in J \tag{3.8}$$

Applying d to (3.8), we get $6jd(j) = 0$ for all $j \in J$ and by 6-torsion freeness of $\mathcal{N}$, we obtain $jd(j) = 0$ for all $j \in J$ and with the help of the facts that $J \neq \{0\}$. Hence, the 3-primeness of $\mathcal{N}$ forces that $d(J) = \{0\}$, in this case, we obtain $J = \{0\}$; this leads to a contradiction. □

Theorem 3.6. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and J be a nonzero Jordan ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a derivation d satisfying one of the following conditions*

(i) *$d([j,n]) = 0$ for all $j \in J$, $n \in \mathcal{N}$, or*

(ii) *$d([j,n]) = [j,n]$ for all $j \in J$, $n \in \mathcal{N}$, or*

(iii) *$[d(j),n)] = [j,d(n)]$ for all $j \in J$, $n \in \mathcal{N}$,*

then $d = 0$ or J is commutative.

Proof. (i) Suppose that $d([j,n]) = 0$ for all $j \in J$, $n \in \mathcal{N}$. Replacing n by jn and using the definition of d, we get

$$\begin{aligned} 0 &= d([j,jn]) \\ &= d(j[j,n]) \\ &= d(j)[j,n] + jd([j,n]) \\ &= d(j)[j,n] \ \text{ for all } \ j \in J, n \in \mathcal{N} \end{aligned}$$

which means that

$$d(j)nj = d(j)jn \ \text{ for all } \ j \in J, n \in \mathcal{N}. \tag{3.9}$$

Taking nm instead of n in (3.9) and using it again, we get

$$d(j)nmj = d(j)njm \ \text{ for all } \ j \in J, m, n \in \mathcal{N}$$

this reduced to

$$d(j)\mathcal{N}[j,m] = \{0\} \ \text{ for all } \ j \in J, m \in \mathcal{N} \tag{3.10}$$

By 3-primeness of $\mathcal{N}$, we obtain $d(j) = 0$ or $j \in Z(\mathcal{N})$ for all $j \in J$. Using Lemma 2.2 (iv), the last two cases forces $d(J) \subseteq Z(\mathcal{N})$. Thus in view of Theorem 3.1, we conclude that $d = 0$ or J is commutative.
(ii) Assume that $d([j,n]) = [j,n]$ for all $j \in J$, $n \in \mathcal{N}$. Replacing n by jn in the latter equation and using the definition of d, we get

$$\begin{aligned} j[j,n] &= d(j[j,n]) \\ &= d(j)[j,n] + jd([j,n]) \\ &= d(j)[j,n] + j[j,n] \ \text{ for all } \ j \in J, n \in \mathcal{N} \end{aligned}$$

the above expression becomes

$$d(j)nj = d(j)jn \ \text{ for all } \ j \in J, n \in \mathcal{N}. \tag{3.11}$$

Putting nm instead of n in (3.11) and using it again, we get

$$d(j)nmj = d(j)njm \ \text{ for all } \ j \in J, m, n \in \mathcal{N}$$

this reduced to

$$d(j)\mathcal{N}[j,m] = \{0\} \ \text{ for all } \ j \in J, m \in \mathcal{N} \tag{3.12}$$

By 3-primeness of $\mathcal{N}$, we get $d(j) = 0$ or $j \in Z(\mathcal{N})$ for all $j \in J$. So from Lemma 2.1 (iv), the above two cases imply that $d(J) \subseteq Z(\mathcal{N})$ and using Theorem 3.1, we

deduce that $d = 0$ or J is commutative.
(iii) Suppose that

$$[d(j), n)] = [j, d(n)] \text{ for all } j \in J, n \in \mathcal{N} \tag{3.13}$$

Putting $d(j)n$ instead of n in (3.13), we get

$$\begin{aligned} [d(j), d(j)n] &= [j, d(d(j)n)] \\ &= jd(d(j)n) - d(d(j)n)j \\ &= jd(j)d(n) + jd^2(j)n - d^2(j)nj \\ &\quad -d(j)d(n)j \end{aligned}$$

for all $j \in J, n \in \mathcal{N}$
On the other hand, we have

$$\begin{aligned} [d(j), d(j)n] &= d(j)[d(j), n] \\ &= d(j)[j, d(n)] \\ &= d(j)jd(n) - d(j)d(n)j \end{aligned}$$

for all $j \in J, n \in \mathcal{N}$
Comparing the above expression, we find that

$$jd(j)d(n) + jd^2(j)n - d^2(j)nj = d(j)jd(n) \tag{3.14}$$

for all $j \in J, n \in \mathcal{N}$
Since $jd(j) = d(j)j$ for all $j \in J$ by (3.13), then (3.14) becomes

$$jd^2(j)n = d^2(j)nj \text{ for all } j \in J, n \in \mathcal{N} \tag{3.15}$$

Putting nm instead of n in (3.15) and using it again, we get

$$d^2(j)\mathcal{N}[j, m] = \{0\} \text{ for all } j \in J, m \in \mathcal{N}. \tag{3.16}$$

By 3-primeness of $\mathcal{N}$, (3.16) becomes

$$d^2(j) = 0 \text{ or } j \in Z(N) \text{ for all } j \in J. \tag{3.17}$$

If there is an element $j_0 \in J$ such that $d^2(j_0) = 0$. Replacing j by j_0 in (3.13) after applying d to it, we obtain $j_0 d^2(n) = d^2(n)j_0$ for all $n \in \mathcal{N}$, in this case, (3.17) becomes

$$jd^2(n) = d^2(n)j \text{ for all } j \in J, n \in \mathcal{N}.$$

Taking $d(n)$ instead of n in (3.13), we arrive at

$$d(j)d(n) = d(n)d(j) \text{ for all } j \in J, n \in \mathcal{N}. \tag{3.18}$$

Replacing n by $d(n)m$ in (3.18) and using the same again, we obtain

$$d^2(n)md(j) = d(j)d^2(n)m \text{ for all } j \in J, n, m \in \mathcal{N}. \tag{3.19}$$

Putting mt instead of m in (3.19) and using it again, we also have

$$d^2(n)\mathcal{N}[d(j),t] = \{0\} \text{ for all } j \in J, n, t \in \mathcal{N}. \tag{3.20}$$

By 3-primeness of $\mathcal{N}$, we arrive at $d^2(\mathcal{N}) = \{0\}$ or $d(J) \subseteq Z(\mathcal{N})$. Now Lemma 2.1 (iii) and Theorem 3.1, forces us to conclude that, $d = 0$ or J is commutative. □

As a direct consequence of Theorem 3.6, we obtain the following results.

Corollary 3.7. [3, Theorem 4.1] *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring. If $\mathcal{N}$ admits a nonzero derivation d such that $d([x,y]) = 0$ for all $x, y \in \mathcal{N}$, then $\mathcal{N}$ is a commutative ring.*

Corollary 3.8. [5, Theorem 2.2] *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring. If $\mathcal{N}$ admits a nonzero derivation d such that $d([x,y]) = [x,y]$ for all $x, y \in \mathcal{N}$, then $\mathcal{N}$ is a commutative ring.*

Corollary 3.9. [6, Theorem 2.1] *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring. If $\mathcal{N}$ admits a nonzero derivation d such that $[d(x), y] = [x, d(y)]$ for all $x, y \in \mathcal{N}$, then $\mathcal{N}$ is a commutative ring.*

The following examples show that the "3-primeness of $\mathcal{N}$" in the Theorem 3.6 (i) and Theorem 3.6 (ii) can not be omitted.

Example 3.10. *Let S be a 2-torsion free left near-ring. Define $\mathcal{N}, J, d$ by:*

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} \mid x, y, z \in S \right\}, \quad J = \left\{ \begin{pmatrix} 0 & 0 & m \\ 0 & 0 & 0 \\ 0 & n & 0 \end{pmatrix} \mid m, n \in S \right\} \text{ and}$$

$d\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} = \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. *Then it can be seen easily that $\mathcal{N}$ is a left near-ring which is not 3-prime, J is a nonzero Jordan ideal of $\mathcal{N}$ and d is a derivation on $\mathcal{N}$ such that $d([j,n]) = 0$ for all $j \in J$, $n \in \mathcal{N}$. However, J is not commutative.*

Example 3.11. *Let S be a 2-torsion free left near-ring. Define $\mathcal{N}, J, d$ by:*

$$\mathcal{N} = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \mid x, y \in S \right\}, \quad J = \left\{ \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \mid m \in S \right\} \text{ and}$$

$d\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}$ *Then it can be seen easily that $\mathcal{N}$ is a left near-ring which is not 3-prime, J is a nonzero Jordan ideal of $\mathcal{N}$ and d is a derivation on $\mathcal{N}$ such that $d([j,n]) = [j,n]$ for all $j \in J$, $n \in \mathcal{N}$. However, J is not commutative.*

Theorem 3.12. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and J be a nonzero Jordan ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a derivation d satisfying one of the following conditions*

(i) *$d([j,n]) = j \circ n$ for all $j \in J$, $n \in N$, or*

(ii) *$d(j \circ n) = [j,n]$ for all $j \in J$, $n \in N$,*

then $d = 0$ or J is commutative.

Proof. (i) Suppose that $d([j,n]) = j \circ n$ for all $j \in J$, $n \in N$. Replacing n by jn, we get

$$\begin{aligned} j \circ jn &= d([j,jn]) \\ &= d(j[j,n]) \\ &= jd([j,n]) + d(j)[j,n] \\ &= j(j \circ n) + d(j)[j,n] \\ &= j \circ jn + d(j)[j,n] \text{ for all } j \in J, n \in \mathcal{N} \end{aligned}$$

which implies that

$$d(j)nj = d(j)jn \text{ for all } j \in J, n \in \mathcal{N} \tag{3.21}$$

Since (3.21) is the same as (3.9), then using the same techniques as used after (3.9), we conclude that $d = 0$ or J is commutative. In this case, for $n = i$, we obtain $i \circ j = 0$ for all $i, j \in J$ which implies that $2ij = 0$ for all $i, j \in J$, using the 2-torsion freeness, we get $ij = 0$ for all $i, j \in J$. Using $j \circ n$ in the place of j, where $n \in \mathcal{N}$ in the previous relation and with the help of the fact that N is 3-prime, we conclude that $J = \{0\}$; leading to a contradiction.
(ii) Assume that $d(j \circ n) = [j,n]$ for all $j \in J$, $n \in \mathcal{N}$. Replacing n by jn in the last equation and using it again, we get

$$\begin{aligned} j[j,n] &= d(j(j \circ n)) \\ &= jd(j \circ n) + d(j)(j \circ n) \\ &= j[j,n] + d(j)(j \circ n) \text{ for all } j \in J, n \in \mathcal{N} \end{aligned}$$

the above expression becomes

$$d(j)nj = -d(j)jn \text{ for all } j \in J, n \in \mathcal{N} \tag{3.22}$$

Taking nm instead of n in (3.22) and using it again, we have for all $j \in J, m, n \in \mathcal{N}$

$$\begin{aligned} d(j)nmj &= -d(j)jnm \\ &= d(j)jn(-m) \\ &= d(j)n(-j)(-m) \end{aligned}$$

Putting $-j$ instead of j in the last expression, we obtain

$$d(-j)\mathcal{N}[j,m] = \{0\} \text{ for all } j \in J, m \in \mathcal{N} \tag{3.23}$$

By 3-primeness of $\mathcal{N}$, we get $d(-j) = 0$ or $j \in Z(\mathcal{N})$ for all $j \in J$ this gives $d(J) \subseteq Z(\mathcal{N})$ by Lemma 2.1(iv). Now using the Theorem 3.1, we conclude that then

$d=0$ or J is commutative. In this case, for $n=j\circ jm$, we obtain $2d(j(j\circ jm))=0$ for all $j\in J$, $m\in\mathcal{N}$, which implies that by 2-torsion freeness $d(j(j\circ jm))=0$ for all $j\in J, m\in\mathcal{N}$, by the simple calculation, we find that $j^2[j,m]=0$ for all $j\in J, m\in\mathcal{N}$. Putting nm instead of m in the previous relation and using the same again with the 3-primeness of $\mathcal{N}$, we conclude that $j^2=0$ or $j\in Z(\mathcal{N})$ for all $j\in J$ and by Lemma 2.5 (i), we get $J\subseteq Z(\mathcal{N})$. The application of Lemma 2.5 (ii) assures that $\mathcal{N}$ is a commutative ring. In this case, returning to our hypothesis, we get $2d(jn)=0$ for all $j\in J, n\in\mathcal{N}$ and by the 2-torsion freeness and definition of d, we find that $d(j)n+jd(n)=0$ for all $j\in J, n\in\mathcal{N}$. Replacing n by nj in the last equation, we get $d(j)\mathcal{N}j=\{0\}$ for all $j\in J$, by 3-primeness of $\mathcal{N}$, we obtain $d(J)=\{0\}$. By Lemma 2.4 and Lemma 2.7, we conclude that $d=0$ or J is commutative. □

As a consequences, we get the following results:

Corollary 3.13. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring. Then $\mathcal{N}$ admits no nonzero derivation such that $d([x,y])=x\circ y$ for all $x,y\in\mathcal{N}$.*

Corollary 3.14. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring. Then $\mathcal{N}$ admits no nonzero derivation such that $d(x\circ y)=[x,y]$ for all $x,y\in\mathcal{N}$.*

Theorem 3.15. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring with $Z(\mathcal{N})\neq\{0\}$ or and J be a nonzero Jordan ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a nonzero derivation d such that $d(j)\circ n=j\circ d(n)$ for all $j\in J$, $n\in\mathcal{N}$, then J is commutative.*

Proof. Suppose that $d(j)\circ n=j\circ d(n)$ for all $j\in J$, $n\in N$. In particular, for $n\in Z(\mathcal{N})$ and by 2-torsion freeness, we obtain $d(j)n=jd(n)$ for all $j\in J$. Replacing n by nm where $m\in Z(\mathcal{N})$ in the last expression and using it again, we get

$$\begin{aligned} d(j)nm &= jd(nm)\\ &= jd(n)m+jnd(m)\\ &= d(j)nm+jnd(m)\end{aligned}$$

for all $j\in J, m,n\in Z(\mathcal{N})$ which implies that $n\mathcal{N}j\mathcal{N}d(m)=\{0\}$ for all $j\in J, m,n\in Z(\mathcal{N})$. Since $J\neq\{0\}$ and $\mathcal{N}$ is 3-prime, we obtain $d(Z(\mathcal{N}))=\{0\}$. Returning to our hypothesis, we obtain $d(j)\circ n=0$ for all $j\in J, n\in\mathcal{N}$ this means that $2d(j)n=0$ for all $j\in J, n\in Z(\mathcal{N})$ and by 2-torsion freeness, we can conclude that $d(J)\mathcal{N}Z(\mathcal{N})=\{0\}$. Using the fact that $Z(\mathcal{N})\neq\{0\}$ and the 3-primeness of $\mathcal{N}$, we deduce that J is commutative. □

As a consequence, we get the following result:

Corollary 3.16. [6, Theorem 2.7] *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring with $Z(\mathcal{N})\neq\{0\}$. Then $\mathcal{N}$ admits no nonzero derivation such that $d(x)\circ y=x\circ d(y)$ for all $x,y\in\mathcal{N}$.*

The following example proves that the "3-primeness of $\mathcal{N}$" in Theorem 3.5, Theorem 3.6(iii), Theorem 3.12 and Theorem 3.15 can not be omitted.

Example 3.17. *Let S be a 2-torsion free left near-ring and let*

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} \mid x, y, z \in S \right\} \text{ and } J = \left\{ \begin{pmatrix} 0 & m & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid m \in S \right\}.$$

Define $d : \mathcal{N} \to \mathcal{N}$ by $d \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then it can be seen easily that $\mathcal{N}$ is a left near-ring which is not 3-prime with $Z(\mathcal{N}) \neq \{0\}$, J is a nonzero Jordan ideal of $\mathcal{N}$ and the maps d is a derivation on $\mathcal{N}$ such that:

(i) $d^2(J) = J$

(ii) $[d(j), n] = [j, d(n)]$

(iii) $d([j, n]) = j \circ n$

(iv) $d(j \circ n) = [j, n]$

(v) $d(j) \circ n = j \circ d(n)$ *for all* $j \in J, n \in \mathcal{N}$.

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Abdelkarim Boua,
Abdelmalek Essaadi University,
Polydisciplinary Faculty,
Departement Of Mathematics,
Rabat road, Km 2-
Larache P. O. Box. 745-92004,
Larache;Morocco.
E-mail address: abdelkarimboua@yahoo.fr, karimoun2006@yahoo.fr