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New Fractional Calculus and Application to the Fractional-order of Extended Biological Population Model

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Abstract: In this study, we propose a new algorithm to find exact solitary wave solutions of nonlinear time- fractional order of extended biological population model. The new algorithm basically illustrates how two powerful algorithms, conformable fractional derivative and the homogeneous balance method can be combined and used to get exact solutions of fractional partial differential equations. Next, the graphical behavior in two model will be discussed under the changing of the fractional value (α is fractional symbol). It show that with changing α (if α tends to one) the graphs of the solutions of fractional biological population model is near to graph of solution of biological population model in general form.

Key Words: fractional derivative, exact solution, biological population model.

Contents

1. Introduction

Searching of exacts solutions of NFPDEs in mathematical and other scientific applications is still quite challenging and needs new methods. Computing the exact solution of these equations is of considerable importance, because the exact solutions can help to understand the mechanism and complexity of phenomena that have been modeled by NPDEs with integer or fractional order.

Fractional differential equations are generalizations of classical differential equations of integer order. In recent decades, fractional differential equations have been the focus of many studies due to their frequent appearance in various applications

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in physics, biology, engineering, signal processing, systems identification, control theory, finance and fractional dynamics.
Many articles have investigated some aspects of fractional differential equations such as the existence and uniqueness of solutions to Cauchy-type problems, the methods for explicit and numerical solutions.
Among the investigations for fractional differential equations, research into seeking exact solutions and numerical solutions of fractional differential equations is an important topic. Many powerful and efficient methods have been proposed to obtain numerical solutions and exact solutions of fractional differential equations [1-3] and [13-23].
Recently, a new modification of Riemann-Liouville derivative is proposed by Jumarie[4,7]:

$$D_x^a f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-\varepsilon)^{-\alpha} \left(f(\varepsilon) - f(0)\right) d\varepsilon, \qquad 0 < \alpha < 1$$

and gave some basic fractional calculus formulae, for example, formulae (12) and (13) in [4]:

$$D_x^\alpha \left(u(x)\, v(x)\right) = v(x)\, D_x^\alpha \left(u(x)\right) + u(x)\, D_x^\alpha \left(v(x)\right), \tag{1.1}$$

$$D_x^\alpha \left(f\left(u(x)\right)\right) = f_u'(u)\, D_x^\alpha \left(u(x)\right) = D_x^\alpha f(u) \left(u_x^{'}\right)^\alpha, \tag{1.2}$$

The last formula (1.2) has been applied to solve the exact solutions to some nonlinear fractional order differential equations [9-12]. If this formula were true, then we could take the transformation $\xi = x - \frac{kt^\alpha}{\Gamma(1+\alpha)}$ and reduce the partial derivative $\frac{\partial^\alpha U(x,t)}{\partial t^\alpha}toU'(\xi)$. Therefore the corresponding fractional differential equations become the ordinary differential equations which are easy to study. But we must point out that Jumarie's basic formulae (1.1) and (1.2) are not correct, and therefore the corresponding results on differential equations are not true [8]. Fractional derivative is as old as calculus. The most popular definitions are:
(i) Riemann-Liouville definition: If$n$is a positive integer and $\alpha \in [n-1, n)$ the α^{th}derivative of f is given by

$$D_a^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}} dx$$

(ii) Caputo definition. For $\alpha \in [n-1, n)$ the α derivative of $f$is

$$D_a^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^n(x)}{(t-x)^{\alpha-n+1}} dx.$$

Now, all definitions are attempted to satisfy the usual properties of the standard derivative. The only property inherited by all definitions of fractional derivative is the linearity property. However, the following are the setbacks of one definition or another:

(i) The Riemann-Liouville derivative does not satisfy $D_a^\alpha(1) = 0\,(D_a^\alpha(1)) = 0$for the Caputo derivative), if$\alpha$is not a natural number.
(ii) All fractional derivatives do not satisfy the known product rule

$$D_a^\alpha(fg) = fD_a^\alpha(g) + gD_a^\alpha(f)$$

(iii) All fractional derivatives do not satisfy the known product rule

$$D_a^\alpha\left(\frac{f}{g}\right) = \frac{fD_a^\alpha(g) - gD_a^\alpha(f)}{g^2}$$

(iv) All fractional derivatives do not satisfy the known quotient rule:

$$D_a^\alpha(fog)(t) = f^\alpha(g(t))\,g^\alpha(t).$$

(v) All fractional derivatives do not satisfy the chain rule: $D^\alpha D^\beta f = D^{\alpha+\beta} f$ in general.
(vii) Caputo definition assumes that the function f is differentiable. Authors introduced a new definition of fractional derivative by using old definitions for fractional derivative [4-5] and [6] as follows:
for $\alpha \in [0,1)$, and $f : [0,\infty) \to$ Rlet

$$T_\alpha(f)(t) = \lim_{\xi\to 0}\frac{f\left(t + \xi t^{1-\alpha}\right) - f(t)}{\xi}$$

for $t > 0, \alpha \in (0,1)$. $T_\alpha$is called the conformable fractional derivative of $f$of order α.

Definition 1. Let $f^\alpha(t)$ stands for$T_\alpha(f)(t)$. Hence

$$f^\alpha(t) = \lim_{\xi\to 0}\frac{f\left(t + \xi t^{1-\alpha}\right) - f(t)}{\xi}$$

If $f$is α-differentiable in some$(0,a)$, $a > 0$, and $\lim_{t\to 0^+} f^\alpha(t)$exists, then by definition

$$f^\alpha(0) = \lim_{t\to 0^+} f^\alpha(t)$$

We should remark that $T_\alpha(t^\mu) = \mu t^{\mu-\alpha}$. Further, this definition coincides with the classical definitions of R-L and of Caputo on polynomials (up to a constant multiple).
One can easily show that $T_\alpha$satisfies all the properties in the theorem [7-8].
Theorem 2. Let $\alpha \in [0,1)$ and f, g beα-differentiable at a point t, Then:

$(i)\, T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)\,, for \quad all \quad a, b \in$ R.
$(ii)\, T_\alpha(t^\mu) = \mu t^{\mu-\alpha}, for \quad all \quad \mu \in$ R
$(iii)\, T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$
$(i\nu)\, T_\alpha\left(\frac{f}{g}\right) = \frac{fT_\alpha(g) - gT_\alpha(f)}{g^2}$

If, in addition, f is differentiable, then$T_\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}$.

Theorem3. Let $f : [0, \infty) \to$ Rbe a function such that f is differentiable and also differentiable. Let g be a function defined in the range of $f$and also differentiable; then, one has the following rule:

$$T_\alpha (fog)(t) = t^{1-\alpha} g'(t) f'(g(t)).$$

The above rule is referred to as Atangana beta-rule [4]. We will present new derivative for some special functions

$$(i)\, T_\alpha (e^{cx}) = cx^{1-\alpha} e^{cx}, c \in \mathrm{R}.$$
$$(ii)\, T_\alpha (\sin bx) = bx^{1-\alpha} \cos bx, b \in \mathrm{R}.$$
$$(iii)\, T_\alpha (\cos bx) = -bx^{1-\alpha} \sin bx, b \in \mathrm{R}.$$
$$(i\nu)\, T_\alpha \left(\frac{1}{\alpha} x^\alpha\right) = 1.$$

However, it is worth noting the following fractional derivatives of certain functions:

$$(i)\, T_\alpha \left(e^{\frac{1}{\alpha} t^\alpha}\right) = e^{\frac{1}{\alpha} t}.$$
$$(ii)\, T_\alpha \left(\sin \frac{1}{\alpha} t\right) = \cos \frac{1}{\alpha} t, .$$
$$(iii)\, T_\alpha \left(\cos \frac{1}{\alpha} t\right) = -\sin \frac{1}{\alpha} t, .$$

Definition 4. (Fractional Integral) Let $a \geq 0$and $t \geq a$. Also, let f be a function defined on $(a, t]$ and $\alpha \in f$. Then the $\alpha-$fractional integral of f is defined by,

$$I_a^\alpha (f)(t) = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx$$

if the Riemann improper integral exists. It is interesting to observe that the $-$fractional derivative and the $\alpha-$fractional integral are inverse of each other as given in [7-8].

Theorem 5: (Inverse property). Let $a \geq 0$, and$\alpha \in (0, 1)$. Also, let $f$be a continuous function such that $I_a^\alpha f$ exists. Then
$T_\alpha (I_a^\alpha f)(t) = f(t)$,for $t \geq a$.
In this paper, we obtain the exact solution of the time and space fractional derivatives cubic nonlinear Schrodinger equation by means of the homogeneous balance method. The homogeneous balance method is a powerful solution method for the computation of exact traveling wave solutions. This method is one of the most direct and effective algebraic methods for finding exact solutions of nonlinear fractional partial differential equations (FPDEs). The method is based on the homogeneous balance principle and the Jumarie's modified Riemann-Liouville derivative of fractional order [18-23].

2. Method Applied

We consider the following general nonlinear fractional differential equations:

$$G\left(u, D_t^\alpha u, D_x^\beta u, D_y^\psi u, D_t^\alpha D_t^\alpha u, D_t^\alpha D_x^\beta u, D_x^\beta D_x^\beta u, ...\right) = 0, \qquad 0 < \alpha, \beta, \psi < 1. \tag{2.1}$$

Where u is an unknown function, and $G$is a polynomial of u. In this equation, the partial fractional derivatives involving the highest order derivatives and the nonlinear terms are included. Next by using the new definition for traveling wave variable

$$u\left(x,t\right)=U\left(\xi\right)e^{i\left(k\frac{x^{\beta}}{\beta}+c\frac{t^{\alpha}}{\alpha}\right)},\qquad \xi=l\frac{x^{\beta}}{\beta}+\omega\frac{t^{\alpha}}{\alpha} \tag{2.2}$$

Where k,c,l and ω are non-zero arbitrary constants, we can rewrite Eq. (2.1) as the following nonlinear ODE:

$$Q\left(U,U',U'',U''',...\right)=0. \tag{2.3}$$

Where the prime denotes the derivation with respect toξ . If possible, we should integrate Eq. (2.5) term by term one or more times.
We assume that the solution of Eq. (2.1) is of the form

$$u(\xi)=\sum_{i=0}^{n}a_i\phi^i(\xi), \tag{2.4}$$

Where $a_i(i=1,2,..,n)$ are real constants to be determined later and ϕ satisfy the Riccati equation

$$\phi'=a\phi^2+b\phi+c \tag{2.5}$$

Eq. (2.5) admits the following solutions:
Case1: when $a=1,b=0$,the Riccati Eq. (2.5) has the following solutions

$$\begin{array}{l} \phi=-\sqrt{-c}\tanh\left(\sqrt{-c}\xi\right),\qquad c<0\\ \phi=-\frac{1}{\xi},\qquad c<0\\ \phi=\sqrt{c}\tan\left(\sqrt{c}\xi\right),\qquad c>0. \end{array} \tag{2.6}$$

Case2: Let $\phi=\sum_{i=0}^{n}b_i\tanh^i\xi$,Balancing $\phi'$with ϕ^2 in Eq.(2.5) gives $m=1$so

$$\phi=b_0+b_1\tanh\xi, \tag{2.7}$$

Substituting Eq. (2.7) into Eq. (2.5), we obtain the following solution of Eq. (2.5)

$$\phi=-\frac{1}{2a}\left(b+2\tanh\xi\right),\qquad ac=\frac{b^2}{4}-1. \tag{2.8}$$

Case3: We suppose that the Riccati Eq. (2.5) have the following solutions of the form:

$$\phi=A_0+\sum_{i=1}^{n}\sinh^{i-1}\left(A_i\sinh\omega+B_i\cosh\omega\right), \tag{2.9}$$

Where $\frac{d\omega}{d\xi}=\sinh\omega$ or$\frac{d\omega}{d\xi}=\cosh\omega$. It is easy to find that $n=1$by Balancing ϕ'withϕ^2. So we choose

$$\phi=A_0+A_1\sinh\omega+B_1\cosh\omega, \tag{2.10}$$

Where $\frac{d\omega}{d\xi} = \sinh\omega$, we substitute (2.10) and $\frac{d\omega}{d\xi} = \sinh\omega$, into (2.5) and set the coefficients of $\sinh^i\omega, \cosh^i\omega\,(i = 0, 1, 2; j0, 1)$ to zero. We obtain a set of algebraic equations and solving these equations we have the following solutions

$$A_0 = -\frac{b}{2a}, A_1 = 0, B_1 = \frac{1}{2a} \tag{2.11}$$

Where $c = \frac{b^2-4}{4a}$ and

$$A_0 = -\frac{b}{2a}, A_1 = \pm\sqrt{\frac{1}{2a}}, B_1 = \frac{1}{2a} \tag{2.12}$$

Where$c = \frac{b^2-1}{4a}$. To $\frac{d\omega}{d\xi} = \sinh\omega$ we have

$$\sinh\omega = -\csc h\xi, \cosh\omega = -\coth\xi \tag{2.13}$$

From (2.11)–(2.13), we obtain

$$\phi = -\frac{b + 2\coth\xi}{2a} \tag{2.14}$$

Where $c = \frac{b^2-4}{4a}$ and

$$\phi = -\frac{b \pm \csc h\xi + \coth\xi}{2a} \tag{2.15}$$

Where $c = \frac{b^2-1}{4a}$.
Next by substituting (6-15) into (2.1) along with (2.5), then the left hand side of Eq. (2.1) is converted into a polynomial in$F(\xi)$; equating each coefficient of the polynomial to zero yields a set of algebraic equations.
Now by solving the algebraic equations obtained in step 3, and substituting the results into (2.4), then we obtain the exact traveling wave solutions for Eq. (1.1).
The rest of this paper is organized as follows. In Sections 2, we use this method to obtain the exact solutions for the time and space fractional derivatives cubic nonlinear Schrodinger equation. Discussion and some conclusions are given in the last section.

3. Application of the homogeneous balance method to the fractional-order of extended biological population model

We consider a nonlinear fractional-order biological population model of the form

$$\begin{aligned} &\frac{\partial^\alpha\omega}{\partial t^\alpha} = \frac{\partial^2}{\partial x^2}\left(\omega^2\right) + \frac{\partial^2}{\partial y^2}\left(\omega^2\right) + h\left(\omega^2 - r\right) = 0, \\ &0 > t, \qquad 0 < \alpha \le 1, \end{aligned} \tag{3.1}$$

Where u denotes the population density, $h\left(\omega^2 - r\right)$represents the population supply due to births and deaths, h and r are constants, and is a parameter describing the order of the fractional time derivative. For our purpose, we introduce the following transformations:

$$\begin{aligned} &\omega\left(x,y,t\right)=U\left(\xi\right), \\ &\xi=x+iy+\tfrac{\nu t^{\alpha}}{\alpha}, \end{aligned} \tag{3.2}$$

Where c is a constant and$i^2=-1$. By substituting Eq. (3.2) into Eq. (3.1), Eq. (3.1) is reduced into an ODE

$$\nu U'+hU^2-hr=0, \tag{3.3}$$

By the same procedure as illustrated in Section 2, we can determine the value of m by balancing U^2 and U'in Eq. (3.3). We find$m=1$. We can suppose that the solution of Eq. (3.3) is in the form

$$U(\xi)=a_1\phi+a_0 \tag{3.4}$$

And from homogeneous balance method [9] we have

$$\phi'=a\varphi^2+b\phi+c \tag{3.5}$$

Substituting (3.4) along with (3.5) into (3.3) and collecting all the terms with the same power of together, equating each coefficient to zero, yield a set of algebraic equations. Solving these equations yields

$$a_1=\mp 2a\sqrt{\frac{r}{b^2-4ac}},\qquad a_0=\mp b\sqrt{\frac{r}{b^2-4ac}},\qquad v=\pm 2h\sqrt{\frac{r}{b^2-4ac}} \tag{3.6}$$

Substituting the result above into Eq. (3.6) and combining with homogeneous balance method [9], we can obtain the following exact solutions to Eq. (3.1).

Case 1:

$$\begin{aligned} &\omega_{1-1}\left(x,y,t\right)=\pm 2a\sqrt{\tfrac{-cr}{b^2-4ac}}\times \\ &\tanh\left(\sqrt{-c}\left(x+iy\pm\frac{2h\sqrt{\frac{r}{b^2-4ac}}t^{\alpha}}{\alpha}\right)\right)\mp b\sqrt{\tfrac{r}{b^2-4ac}} \end{aligned}$$

And

$$\begin{aligned} &\omega_{2-1}\left(x,y,t\right)=\pm 2a\sqrt{\tfrac{r}{b^2-4ac}}\times \\ &\frac{1}{x+iy\pm\frac{2h\sqrt{\frac{r}{b^2-4ac}}t^{\alpha}}{\alpha}}\mp b\sqrt{\tfrac{r}{b^2-4ac}} \end{aligned}$$

And

$$\begin{aligned} &\omega_{3-1}\left(x,y,t\right)=\mp 2a\sqrt{\tfrac{cr}{b^2-4ac}}\times \\ &\tan\left(\sqrt{c}\left(x+iy\pm\frac{2h\sqrt{\frac{r}{b^2-4ac}}t^{\alpha}}{\alpha}\right)\right)\mp b\sqrt{\tfrac{r}{b^2-4ac}} \end{aligned}$$

Case 2:

$$\begin{aligned} &\omega_{1-2}\left(x,y,t\right)=\pm\sqrt{\tfrac{cr}{b^2-4ac}}\times \\ &\left(b+2\tanh\left(x+iy\pm\frac{2h\sqrt{\frac{r}{b^2-4ac}}t^{\alpha}}{\alpha}\right)\right)\mp b\sqrt{\tfrac{r}{b^2-4ac}} \end{aligned}$$

Case 3:

$$\omega_{1-3}(x,y,t)=\pm 2a\sqrt{\frac{r}{b^2-4ac}}\times \frac{b+2\coth\left(x+iy\pm\frac{2h\sqrt{\frac{r}{b^2-4ac}}t^{\alpha}}{\alpha}\right)}{2a}\mp b\sqrt{\frac{r}{b^2-4ac}},$$

And

$$\omega_{2-3}(x,y,t)=\pm 2a\sqrt{\frac{r}{b^2-4ac}}\times \frac{b\pm\operatorname{csc}h\left(x+iy\pm\frac{2h\sqrt{\frac{r}{b^2-4ac}}t^{\alpha}}{\alpha}\right)+\coth\left(x+iy\pm\frac{2h\sqrt{\frac{r}{b^2-4ac}}t^{\alpha}}{\alpha}\right)}{2a}\mp b\sqrt{\frac{r}{b^2-4ac}},$$

4. Figures captions

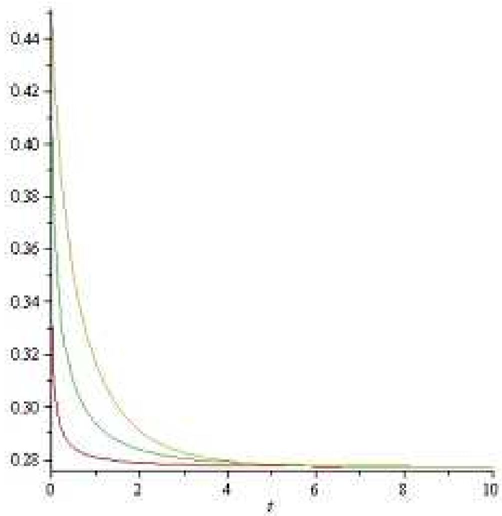

Figure 1: The variation of ω_{1-3} for $a=1, r=1, b=1, c=-1, k=0, y=0$ and different values of $\alpha=0.3$ (red Curve), $\alpha=0.5$ (green Curve), $\alpha=0.7$ (yellow Curve) in region,$x=-10...10, t=0..10$.

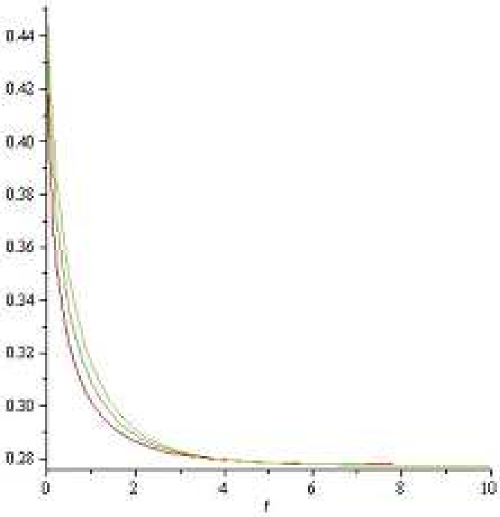

Figure 2: The variation of ω_{1-3} for $a = 1, r = 1, b = 1, c = -1, k = 0, y = 0$ and different values of $\alpha = 0.6$ (red Curve), $\alpha = 0.7$ (green Curve), $\alpha = 0.8$ (yellow Curve) in region,$x = -10...10, t = 0..10$.

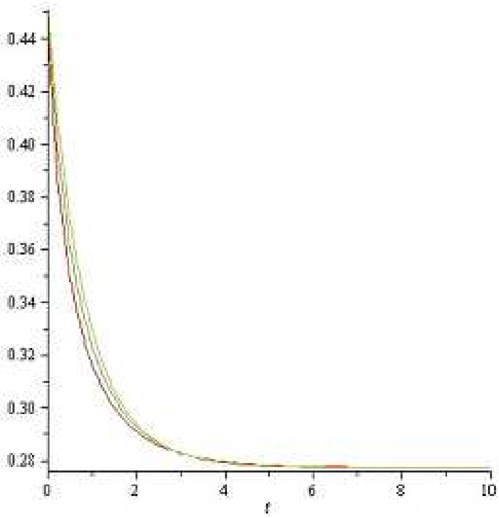

Figure 3: The variation of ω_{1-3} for $a = 1, r = 1, b = 1, c = -1, k = 0, y = 0$ and different values of $\alpha = 0.8$ (red Curve), $\alpha = 0.9$ (green Curve), $\alpha = 0.1$ (yellow Curve) in region,$x = -10...10, t = 0..10$.

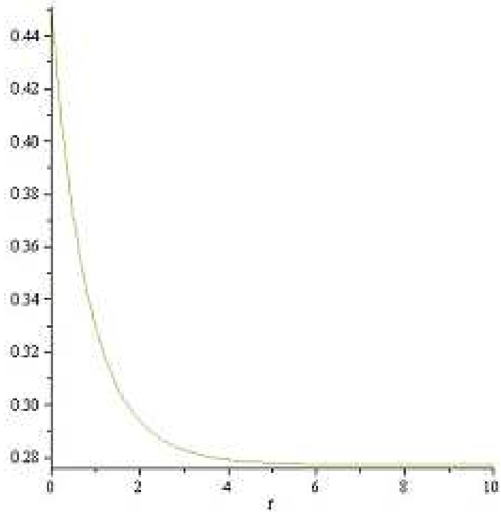

Figure 4: The variation of ω_{1-3} for $a = 1, r = 1, b = 1, c = -1, k = 0, y = 0$ and different values of $\alpha = 1$ in region,$x = -10...10, t = 0..10$.

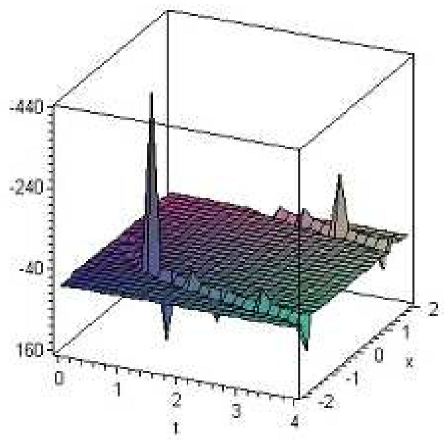

Figure 5: The variation of ω_{1-1} for $a = 1, r = 1, b = 1, c = -1, k = 0, y = 0$ and $\alpha = 0.1$ in region, $x = -2...2, t = 0..4$.

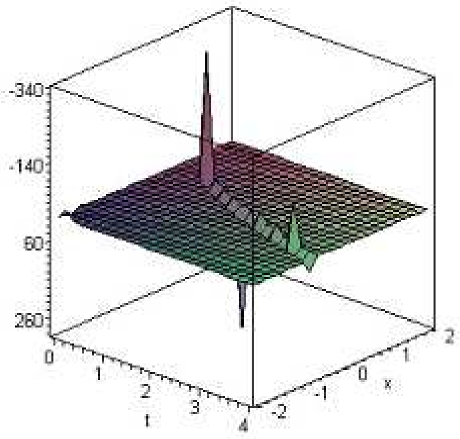

Figure 6: The variation of ω_{1-1} for $a = 1, r = 1, b = 1, c = -1, k = 0, y = 0$ and $\alpha = 0.7$ in region, $x = -2...2, t = 0..4$.

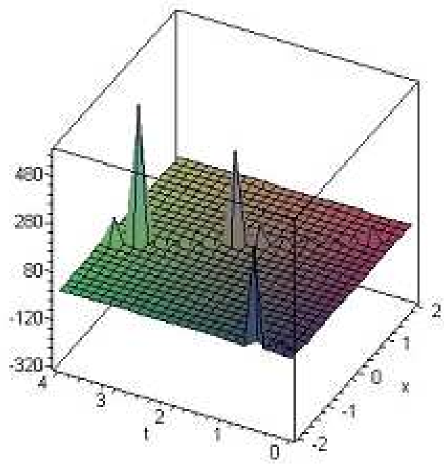

Figure 7: The variation of ω_{1-1} for $a = 1, r = 1, b = 1, c = -1, k = 0, y = 0$ and $\alpha = 0.9$ in region,$x = -2...2, t = 0..4$.

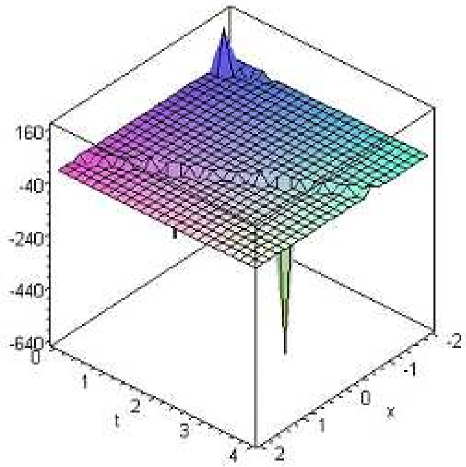

Figure 8: The variation of ω_{1-1} for $a = 1, r = 1, b = 1, c = -1, k = 0, y = 0$ and $\alpha = 1$ in region,$x = -2...2, t = 0..4$.

5. Conclusions and discussion

In these above graphs (Figs. 1-8) we have graphs of the solution related to ω_{1-1} and ω_{1-3}. These graphs show that with changing α (if αtends to one) the graphs of the solutions of fractional perturbed nonlinear Schrodinger equation with power law nonlinearity is near to graph of solution of perturbed nonlinear Schrodinger equation with power law nonlinearity in general form and finally for $\alpha = 1$ it coincide with the graph of the general form of perturbed nonlinear Schrodinger equation with power law nonlinearity.

In this paper, we introduced a new conformable fractional derivative to establish exact solutions for fractional partial differential and carried it out to obtain more new exact solutions of the extended biological population model. These exact solutions include hyperbolic function solutions and trigonometric function solutions.

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References

1. El-Sayed, AMA, Gaber, M: The Adomian decomposition method for solving partial differential equations of fractal order in finite domains. Phys. Lett. A 359, 175-182 (2006).

2. El-Sayed, AMA, Behiry, SH, Raslan, WE: Adomian's decomposition method for solving an intermediate fractional advection-dispersion equation. Comput. Math. Appl. 59, 1759-1765 (2010).

3. Daftardar-Gejji, V, Jafari, H: Adomian decomposition: a tool for solving a system of fractional differential equations. J. Math. Anal. Appl. 301(2), 508-518 (2005)

4. Jumarie G. Modified Riemann-Liouville derivative and fractional Taylor se- ries of nondifferentiable functions further results. Computers and Mathe- matics with Applications, 2006, 51(9): 1367-1376.

5. R. Khalil , M. Al Horani , A. Yousef, M. Sababheh, A new definition of fractional derivative, Journal of Computational and Applied Mathematics 264 (2014) 65–70.

6. A. Neirameh, Topological Solutions for Nonlinear Schrodinger Equation which in the Dimensionless Form International Journal of Modern Mathematical Sciences, 2014, 12(2): 55-63.

7. G. Jumarie, Modified Riemann-Liouville derivative and fractional Taylor series of nondifferentiable functions further results. Computers and Mathematics with Applications. 51 :1367-1376 (2006)

8. Liu, C-s., Counterexamples on Jumarie's two basic fractional calculus formulae, Communications in Nonlinear Science and Numerical Simulation (2014), doi: http://dx.doi.org/10.1016/j.cnsns.2014.07.022.

9. M. Abu Hammad, R. Khalil, CONFORMABLE FRACTIONALHEAT DIFFERENTIAL EQUATION, International Journal of Pure and Applied Mathematics, Volume 94 No. 2 2014, 215-221.

10. R. Khalil , M. Al Horani , A. Yousef, M. Sababheh, A new definition of fractional derivative, Journal of Computational and Applied Mathematics 264 (2014) 65–70

11. Liu, S.K., Fu, Z.T., Liu, S.D. and Zhao, Q. (2001) Jacobi Elliptic Function Expansion Method and Periodic Wave Solutions of Nonlinear Wave Equations. Physics Letters A, 289, 69-74.

12. Parkes, E.J. and Duffy, B.R. (1996) An Automated Tanh-Function Method for Finding Solitary Wave Solutions to Non-Linear Evolution Equations. Computer Physics Communications, 98, 288-300.

13. A. Neirameh, Topological soliton solutions to the coupled Schrodinger-Boussinesq equation by the SEM, Optik - International Journal for Light and Electron Optics, Volume 126, Issue 23, December 2015, Pages 4179-4183.

14. A. Neirameh, Binary simplest equation method to the generalized Sinh-Gordon equation, Optik - International Journal for Light and Electron Optics, Volume 126, Issue 23, December 2015, Pages 4763-4770.

15. N. Taghizadeh, A. Neirameh, New complex solutions for some special nonlinear partial differential systems, Computers and Mathematics with Applications, Volume 62, Issue 4, August 2011, Pages 2037-2044

16. A. Neirameh, New analytical solutions for the coupled nonlinear Maccari's system, Alexandria Engineering Journal, In Press, Corrected Proof, Available online 30 July 2016.

17. A. Neirameh, Exact solutions of the generalized Sinh-Gordon equation, Computational Mathematics and Mathematical Physics, July 2016, Volume 56, Issue 7, pp 1336-1342.

18. Thabet Abdeljawad, On conformable fractional calculus, Journal ofComputational and Applied Mathematics, Vol. 279, 1 May 2015, 57-66.

19. Thabet Abdeljawad, Mohammad Al Horani, Roshdi Khalil, Conformablefractional semi group operators, Journal of Semigroup Theory andApplications vol. 2015 (2015) Article ID. 7.

20. I. Podlubny, Fractional Di_erential Equations, Academic Press, 1999.

21. S. Samko, A. A. Kilbas, Marichev, Fractional Integrals andDerivatives: Theory and Applications, Gordon and Breach, Yverdon, 1993.

22. A. A. Kilbas, M. H. Srivastava, J. J. Trujillo, Theory and Applicationof Fractional Differential Equations, North Holland Mathematics Studies 204,2006

23. A. A. Kilbas, M. Saigo, On solution of integral equation ofAbel-Volterra type, Di_. Integr Equat., 8 (5), (1995) 993-1011.

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