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Inclusion And Equivalence Relations Between Absolute Nörlund And Absolute Weighted Mean Summability Methods

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ABSTRACT: In this paper, a set of conditions under which the absolute Nörlund summability method include in the absolute weighted mean method have been established. Three non-trivial examples to show that this inclusion holds have been given, and other three examples to show that even if both (N, r) and $(\overline{N}, q)$ are regular, the inclusion fails to holds have been constructed. The paper give two non-trivial examples to show that the equivalence of these two methods may holds. Finally, we give two examples to show that inclusion may holds in only one way without the other.

Key Words: Inclusion and Equivalence Relation, Absolute Nörlund Summability Method.

Contents

1. Introduction

Let A be a sequence-to sequence transformation

$$t_n = \sum_{k=0}^{\infty} A_{n,k} S_k \qquad ; \; n = 0, 1, 2, \cdots . \tag{1.1}$$

The sequence $\{S_n\}$ is said to be summable (A) to s if $t_n \longrightarrow s$ as $n \longrightarrow \infty$, and if in addition $\{t_n\}$, is of bounded variation, then $\{S_n\}$ is said to be absolutely summable (A) or summable $|A|$.

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We define the sequence of constants $\{C_n\}$ formally by means of the identity

$$\left(\sum_{n=0}^{\infty} r_n z^n\right)^{-1} = \sum_{n=0}^{\infty} c_n z^n \qquad ; \ c_{-n} = 0 \ (n \geq 0) \tag{1.2}$$

and will write $c(z)$ for $\sum_{n-0}^{\infty} c_n z^n$.

If for $n = 0, 1, 2, \cdots$

$$r_n > 0 \ , \ \frac{r_{n+1}}{r_n} \leq \frac{r_{n+2}}{r_{n+1}} \leq 1, \tag{1.3}$$

then we shall write $r_n \in \mu$.

Let (N, r) denote the Nörlund method in which the sequence $\{S_n\}$ is transformed into the sequence $\{t_n^r\}$, where

$$t_n^r = \frac{1}{R_n} \sum_{k=0}^{n} r_{n-k} S_k; \tag{1.4}$$

$$R_n = r_0 + r_1 + \cdots + r_n \neq 0 \ \ all \ n \geq 0$$

$$R_{-m} = r_{-m} = 0 \ ; \ (m > 0) \tag{1.5}$$

The special case in which $r_n = 1$ $(n \geq 0)$, then (N, r) reduces to a simple arithmetic mean of $(C, 1)$.

Each sequence $\{q_n\}$ for which $Q_n = q_0 + q_1 + \cdots, q_n \neq 0$ (all $n \geq 0$) for each n defines the weighted mean method $(\overline{N}, q)$ of the sequence $\{S_n\}$, where

$$t_n^{(q)} = \frac{1}{Q_n} \sum_{k=0}^{n} q_k S_k \qquad n = 0, 1, 2, \cdots \tag{1.6}$$

A method of summability is called regular, if it sums every convergent series to its ordinary sum. It follows from Toeplitz's Theorem ([6]; Theorem 2) that (N, r) is regular if, and only if,

$$\frac{r_n}{R_n} \to 0 \text{ as } n \to \infty, \tag{1.7}$$

and

$$\sum_{k=0}^{n} |r_k| = O\left(|R_n|\right) \tag{1.8}$$

And $(\overline{N}, q)$ is regular if, and only if,

$$|Q_n| \to 0 \text{ as } n \to \infty, \tag{1.9}$$

and

$$\sum_{k=0}^{n} |q_k| = O\left(|Q_n|\right) \tag{1.10}$$

Let A be a sequence-to sequence transformation given by (1.1). If whenever $\{S_n\}$ has a bounded variation it follows that $\{t_n\}$ has a bounded variation, and if the limits are preserved, we say that A is absolutely regular.

$(A) \subseteq (B)$ means that any series summable by (A) to sum S is necessary summable (B) to the same sum. (A) and (B) are equivalent if $(A) \subseteq (B)$ and $(B) \subseteq (A)$. For any sequence $\{u_n\}$ we shall write

$$\Delta u_n = u_n - u_{n+1} \tag{1.11}$$

2. Inclusion And Equivalence Relations

On inclusion and equivalence relations of different summability methods much work have been done already, see [1], [2], [3], [4], [5] and [7].

3. Object Of The Paper

The author ([2], Theorem 6.1) obtained necessary and sufficient conditions for which $\left|(\overline{N}, q)\right| \subseteq |(N, r)|$. The object of this paper is to obtain a set of conditions for the other way round, and to give some non-trivial special cases to show that this inclusion may holds, and we will give some other special cases to show that this inclusion fails to hold even if both (N, r) and $(\overline{N}, q)$ are regular. Finally, we will give two examples involving the equivalence of these two methods, and another two examples to show that the inclusion may holds in only one way without the other. These results will be concluded in sections 5, 6 and 7.

4. Results Required

This section is devoted to results that are necessary for our purposes:

Theorem 4.1. *[8] The sequence-to-sequence transformation given by (1.1) is absolutely regular if, and only if,*

$$A_{n,k} \to 0 \text{ as } n \to \infty \quad \text{for each } k \tag{4.1}$$

$$\sum_{k=0}^{\infty} A_{n,k} \to 1 \text{ as } n \to \infty \tag{4.2}$$

and

$$\sum_{n=0}^{\infty} \left| \sum_{v=k}^{\infty} A_{n,v} - \sum_{v=k}^{\infty} A_{n+1,v} \right| = O(1)\ ; \quad k \to \infty \tag{4.3}$$

Theorem 4.2. *([2], Theorem 6.1) Suppose that $(\overline{N}, q)$ and (N, r) are regular, $q_n \neq 0$ (all $n \geq 0$), then $\left|(\overline{N}, q)\right| \subseteq |(N, r)|$ if, and only if,*

$$\sum_{n-k-1}^{\infty} \left| \Delta_n \frac{1}{R_n} \left(R_{n-k-1} + \frac{Q_r}{q_k} r_{n-k} \right) \right| = O(1), \qquad (\text{all } k \geq 0), \tag{4.4}$$

where

$$\Delta_n B_{n,k} = B_{n,k} - B_{n+1,k} \tag{4.5}$$

Lemma 4.3. *([2], Example 7.2) Let (N,r) be $(C,1)$, then $|(N,r)| \subseteq |(C,1)|$ if, and only if, $Q_n = O(nq_n)$.*

Theorem 4.4. *([6], Theorem 22) Let $r_n \in \mu$ and $\{c_n\}$ as defined in (1.2), then:*

1. $c_n > 0$, $c_n \leq 0$, $n = 1, 2, \cdots$.
2. $\sum\limits_{n=0}^{\infty} c_n z^n$ *is absolutely convergent for* $|z| \leq 1$.
3. $\sum\limits_{n=0}^{\infty} c_n > 0$, *except when* $\sum\limits_{n=0}^{\infty} r_n = \infty$ *in which case* $\sum\limits_{n=0}^{\infty} c_n = 0$.

5. Main Result

In this section we shall state and prove our main result:

Theorem 5.1. *Let $(\overline{N},q)$ and (N,r) are regular; then $|(N,r)| \subseteq \left|(\overline{N},q)\right|$ if, and only if,*

$$B_{n,v} \to 0 \text{ as } n \to \infty \tag{5.1}$$

and

$$\sum_{n=k-1}^{\infty} \left| \sum_{v=0}^{k-1} B_{n,v} - \sum_{v=0}^{k-1} B_{n+1,v} \right| = O(1) \tag{5.2}$$

where

$$B_{n,v} = \frac{R_v}{Q_n} \sum_{k=v}^{n} q_k c_{k-v} \qquad 0 \leq v \leq n \tag{5.3}$$

and

$$B_{n,v} = 0 \qquad \text{otherwise} \tag{5.4}$$

Further, if $r_n \in \mu$, then (5.2) is alone is necessary and sufficient condition for $|(N,r)| \subseteq \left|(\overline{N},q)\right|$.

Proof: Let $\{t_n^r\}$ and $\{t_n^q\}$ be respectively the (N,r) and $(\overline{N},q)$ transforms of $\{S_n\}$, then

$$t_n^r = \frac{1}{R_n} \sum_{k=0}^{n} r_{n-k} S_k \tag{5.5}$$

and

$$t_n^q = \frac{1}{Q_n} \sum_{k=0}^{n} q_k S_k \tag{5.6}$$

To prove the result, we need to find t_n^q in terms of t_n^r.

Observe that $r_0 \neq 0$, $\sum\limits_{n=0}^{\infty} r_n z^n = r(z)$, say, is non-zero in some neighborhood of the origin, we have $\dfrac{1}{r(z)} = c(z)$, say, is regular in some neighborhood of the origin, and so has a power series expansion $c(z) = \sum\limits_{n=0}^{\infty} c_n z^n$, which by

$$\sum_{n=0}^{\infty} R_n t^r z^n = \sum_{n=0}^{\infty} r_n z^n \sum_{n=0}^{\infty} S_n z^n = \left(\sum_{n=0}^{\infty} c_n z^n\right)^{-1} \sum_{n=0}^{\infty} S_n z^n, \tag{5.7}$$

so that

$$\sum_{n=0}^{\infty} S_n z^n = \sum_{n=0}^{\infty} c_n z^n \sum_{n=0}^{\infty} R_n t^r z^n. \tag{5.8}$$

Comparing the coefficient of in (5.8), we have

$$S_n = \sum_{k=0}^{\infty} c_{n-k} R^k t_k^r. \tag{5.9}$$

Using (5.9), it follows from (5.6) that

$$t_n^q = \sum_{v=0}^{n} B_{n.v} t_v^r, \tag{5.10}$$

where $B_{n.v}$ is given by (5.3) and (5.4)

The special case in which $S_n = 1$, $(n \geq 0)$, then (5.5), (5.6) and (5.10) imply that

$$\sum_{v=0}^{n} B_{n.v} t_v^r = 1. \tag{5.11}$$

This implies (4.2). Using (5.4), it follows from (5.11) that the left hand sides of (4.3) and (5.2) are equivalent, and Mears Theorem (5.1) implies the result. Next, if $r_n \in \mu$, then (N, r) is regular. Using Kaluza Theorem (5.3), it follows that$\{c_n\}$ is bounded and $c_c \to 0$ as $n \to \infty$. Using this and the regularity of $(\overline{N}, q)$, (5.1) holds and the proof is completed. □

Remark 5.2. *We remark that the condition $B_{n,n} = O(1)$ is necessary (but not sufficient) for (5.2) to be satisfied. It follows from Theorems 4.2 and 5.1 the following Lemma:*

Lemma 5.3. $|(N, r)| \sim \left|(\overline{N}, q)\right|$ *if, and only if (4.4), (5.1) and (5.2) are satisfied.*

6. Examples

In this section we will construct six examples to show in the first three that $|(N, r)| \subseteq \left|(\overline{N}, q)\right|$ in some non-trivial cases, and in examples 6.4 and 6.5 we will show that even if both (N, r) and $(\overline{N}, q)$ are regular, neither (5.1) nor (5.2) are satisfied, and in example **??** we will show that (5.1) is satisfied but (5.2) does not.

Example 6.1. *Let*

$$r_n = 1\ ,\ (n \geq 0), \tag{6.1}$$

and let

$$q_0 = 1 \quad \textit{and}\ q_n = n + \frac{1}{2} \ \ (n \geq 1), \tag{6.2}$$

then

$$|(C,1)| \subseteq \left|(\overline{N},q)\right|.$$

Proof: Using (6.1), we see that $r_n \in \mu$, which by Theorem 5.1 gives (5.1). Next, comparing the coefficient of z^{n-k} in the equation $r(z)c(z) = 1$, we see that

$$\sum_{k=v}^{n} r_{n-k}c_{k-v} = \begin{cases} 0 & , \quad n > v \\ 1 & , \quad n = v \end{cases} \tag{6.3}$$

Using (6.1), it follows from (6.3) that

$$c_0 = 1 \ , \ c_1 = -1 \text{ and } c_n = 0 \ , \ (n \geq 2). \tag{6.4}$$

Using (6.1), we have

$$Q_n = \frac{(n+1)^2+1}{2} \ , \ (n \geq 0). \tag{6.5}$$

The regularity of (N,r) and $(\overline{N},q)$ follows from (6.1), (6.2) and (6.5). Using (6.4) and (6.5), it follows from (5.3) that

$$B_{n,n} = \frac{(n+1)(2n+1)}{(n+1)^2+1} \ , \ n \geq 1, \tag{6.6}$$

$$B_{n,v} = -\frac{2(v+1)}{(n+1)^2+1} \ , \ 1 \leq v \leq n-1, \tag{6.7}$$

$$B_{n,0} = -\frac{1}{(n+1)^2+1} \ , \ n \geq 1, \tag{6.8}$$

and

$$B_{0,0} = 1. \tag{6.9}$$

Using (6.6)-(6.8), the left hand side of (5.2) reduces to:

$$\begin{aligned} \left|\sum_{v=0}^{k-1} B_{k-1,v} - \sum_{v=0}^{k-1} B_{k,v}\right| \quad &+ \quad \sum_{n=k}^{\infty}\left|B_{n,0} + \sum_{v=1}^{k-1} B_{k,v} - B_{n+1,0} - \sum_{v=1}^{k-1} B_{n+1,v}\right| \\ &= \quad |B_{k,k}| \\ &\quad\; +(k^2+k-1)\sum_{n=k}^{\infty}\left(\frac{1}{(n+1)^2+1} - \frac{1}{(n+2)^2+1}\right) \\ &= \quad \frac{(k+1)(2k+1)}{(k+1)^2+1} + (k^2+k-1)\cdot\frac{1}{(k+1)^2+1} \\ &= \quad \frac{3k^2+4k}{(k+1)^2+1} \\ &= \quad O(1), \end{aligned}$$

so (5.2) is satisfied, and Theorem 5.1 yields the result. □

Example 6.2. *Let*

$$r_n = \frac{1}{2^n}\ ,\ (n \geq 0) \tag{6.10}$$

and let

$$q_0 = 1 \tag{6.11}$$

$$q_n = n - \frac{1}{3}\ ,\ (n \geq 1), \tag{6.12}$$

then

$$|(N, r)| \subseteq \left|(\overline{N}, q)\right| .$$

.

Proof: Conditions (1.7) and (1.8) follow from (6.10), and conditions (1.9) and (1.10) follow from (6.11) and (6.12), and these imply that (N, r) and $(\overline{N}, q)$ are regular. Using (6.10), it follows from (1.4) and (6.3) that

$$R_n = 2 - \frac{1}{2^n}\ ,\ (n \geq 0) \tag{6.13}$$

and

$$c_0 = 1\ ,\ c_1 = -\frac{1}{2} \text{ and } c_n = 0\ ,\ (n \geq 2). \tag{6.14}$$

Using (6.11) and (6.12), we have

$$Q_n = \frac{1}{6}\left(3n^2 + n + 6\right)\ ,\ (n \geq 0) \tag{6.15}$$

Using (6.13)-(6.15), it follows from (5.3) that

$$B_{n,n} = \begin{cases} \dfrac{(2^{n+1} - 1)(3n - 1)}{3 \cdot 2^n Q_n} & ,\quad n \geq 1 \\ 1 & ,\quad n = 1 \end{cases} \tag{6.16}$$

$$B_{n,v} = \frac{(2^{v+1} - 1)(3v - 4)}{6 \cdot 2^v Q_n}\ ,\ 1 \leq v \leq n - 1, \tag{6.17}$$

and

$$B_{n,0} = -\frac{2}{3Q_n}\ \ n \geq 1. \tag{6.18}$$

Condition (5.1) follows from (6.15) and (6.17). The left hand side of (5.2) is equivalent to:

$$\left|\sum_{v=0}^{k-1} B_{k-1,v} - \sum_{v=0}^{k-1} B_{k,v}\right| + \sum_{n=k}^{\infty}\left|B_{n,0} + \sum_{v=1}^{k-1} B_{n,v} + B_{n+1,0} - \sum_{v=1}^{k-1} B_{n+1,v}\right| \tag{6.19}$$

Using (5.11), (6.13)-(6.18), we see that (6.19) reduces to:

$$\begin{aligned} |B_{k,k}| &+ \sum_{n=k}^{\infty}\left|\frac{2}{3}\left(\frac{1}{Q_n}-\frac{1}{Q_{n+1}}\right)+\frac{1}{6}\left(\frac{1}{Q_n}-\frac{1}{Q_{n+1}}\right)\sum_{v=1}^{k-1}\frac{(2^{v+1}-1)(3v-4)}{2^v}\right| \\ &\leq |B_{k,k}| \\ &\quad +\frac{2}{3}\sum_{n=k}^{\infty}\left|\frac{1}{Q_n}-\frac{1}{Q_{n+1}}\right|+\frac{1}{6}\left|\sum_{v=1}^{k-1}\frac{(2^{v+1}-1)(3v-4)}{2^v}\right|\sum_{n=k}^{\infty}\left|\frac{1}{Q_n}-\frac{1}{Q_{n+1}}\right| \\ &= A+B+C, \quad say, \end{aligned} \tag{6.20}$$

it is clear that

$$A+B=O(1),$$

and

$$\begin{aligned} C &= \frac{1}{Q_n}\left|-\frac{1}{4}+\frac{1}{6}\sum_{v=2}^{k-1}\left(6v-8-\frac{3v}{2^v}+\frac{4}{2^v}\right)\right| \\ &\leq \frac{1}{Q_n}\left\{\frac{1}{4}+\frac{1}{3}\sum_{v=2}^{k-1}(3v-4)\right\}=O(1) \end{aligned}$$

Therefore $A+B+C$ is bounded, so (5.2) is satisfied and Theorem (5.1) yields the result. □

Example 6.3. *Let*

$$r_0=1\ ,\ r_1=-\frac{1}{2}\ \text{and}\ r_n=0\ ,\ (n\geq 0) \tag{6.21}$$

and let

$$q_n=e^n\ (n\geq 0), \tag{6.22}$$

then

$$|(N,r)|\subseteq\left|(\overline{N},q)\right|.$$

Proof: We will show that (5.1) and (5.2) are satisfied, and the result follows from Theorem (5.1).

Using (6.21) and (6.22), we have

$$R_0=1\ ,\ R_n=\frac{1}{2}\ ,\ (n\geq 1), \tag{6.23}$$

and

$$Q_n=\frac{e^{n+1}-1}{e-1}\ ,\ (n\geq 0), \tag{6.24}$$

Using (6.21)-(6.24), it follows that (1.7)-(1.10) are all satisfied which imply that (N,r) and $\left|(\overline{N},q)\right|$ are regular. Using (6.21), it follows from (6.3) that

$$c_n=\frac{1}{2^n}\ ,\ (n\geq 0). \tag{6.25}$$

Using (6.22), (6.24) and (6.25), it follows from (5.3) that

$$\begin{aligned} B_{n,v} &= \frac{2^v R_v(e-1)}{e^{n+1}-1} \sum_{u=v}^{n} \left(\frac{e}{2}\right)^u \quad ,0 \le v \le n \\ &= \frac{(e-1)e^{n+1}R_v 2^v}{(e-2)2^n(e^{n+1}-1)} - \frac{2(e-1)R_v 2^v}{(e-2)(e^{n+1}-1)} \ , \ 0 \le v \le n \end{aligned} \tag{6.26}$$

and (5.1) follows from (6.26). Next, using (6.23), we have

$$\sum_{v=0}^{k-1} R_v 2^v = 2^{k-1}, \tag{6.27}$$

and

$$\sum_{v=0}^{k-1} R_v e^v = \frac{e^k + e - 2}{2(e-1)}, \tag{6.28}$$

Using (6.26)-(6.28), we have

$$\sum_{v=0}^{k-1} B_{n,v} = \frac{e^{n+1}(e-1)2^{k-1}}{2^n(e-2)(e^{n+1}-1)} - \frac{e^k + e - 2}{(e-2)(e^{n+1}-1)} \tag{6.29}$$

$$= A_{n,k-1} \ , \ n \ge k-1, \ say. \tag{6.30}$$

Hence

$$\sum_{v=0}^{k-1} B_{n,v} = \sum_{v=0}^{k-1} B_{n+1,v} = A_{n,k-1} - A_{n+1,k-1}. \tag{6.31}$$

Since $A_{k-1,k-1} = 1$ and $\lim_{k\to\infty} A_{k^2,k-1} = 0$ condition (5.2) will be satisfied if we show that the right hand side of (6.31) is positive.

Write $A_{n,k} - A_{n+1,k-1}$ in the form

$$\begin{aligned} &\frac{e^{n+1}2^k(e-1)(e^{n+2}+e-2)}{(e-2)2^{n+2}(e^{n+1}-1)(e^{n+2}-1)} - \frac{e^{n+1}(e-1)(e^k+e-2)}{(e-2)(e^{n+1}-1)(e^{n+2}-1)} \\ = &\frac{e^{n+1}(e-1)\left\lfloor 2^k(e^{n+2}+e-2) - 2^{n+2}(e^k+e-2)\right\rfloor}{2^{n+2}(e-2)(e^{n+1}-1)(e^{n+2}-1)} \end{aligned} \tag{6.32}$$

Thus (5.2) will be satisfied if we show that

$$\frac{e^{n+2}+e-2}{2^{n+2}} - \frac{e^k+e-2}{2^k} > 0$$

Write

$$U_k = \frac{e^k+e-2}{2^k} \ , \ \text{it follows that}$$

$$U_{n+2} - U_k = D_{n+2} + D_{n+1} \cdots + D_{k+1}, \tag{6.33}$$

where

$$D_{n+2} = U_{n+2} - U_{n+1}. \tag{6.34}$$

Using (6.34), we have for $n \geq k-1$,

$$\begin{aligned} D_{n+1} &= \frac{e^{n+2}+e-2}{2^{n+2}} - \frac{e^{n+1}+e-2}{2^{n+1}} \\ &= \frac{(e-2)(e^{n+1}-1}{2^{n+2}} > 0 \ , \ 0 \leq k-1 \leq n. \end{aligned}$$

So that (5.2) is satisfied. □

Example 6.4. *Let*

$$r_0 = 1 \ , \ r_1 = -2 \ , \ r_n = 0 \ , \ (n \geq 2), \tag{6.35}$$

, and let

$$q_0 = 1 \ , \ (n \geq 0), \ i.e \ (\overline{N}, q) \ is \ (C, 1), \tag{6.36}$$

then (N, r) and $(\overline{N}, q)$ are regular, but neither (5.1) nor (5.2) satisfied.

Proof: $(C, 1)$ is known to be regular. Using (6.35), it follows from (1.7) and (1.8) that (N, r) is regular. Taking the special case in which $S_n = 1$ (all $n \geq 0$), it follows from (5.5) that $t_n^r = 1$ (all $n \geq 0$), and (5.9) reduces to

$$\sum_{k=0}^{n} c_{n-k} R_k = R_0 c_n + \sum_{k=1}^{n} c_{n-k} R_k$$

Using (6.35), we have $R_0 = 1 \ , \ R_n = -1 \ \ (n \geq 1)$, then

$$1 = c_n - \sum_{k=1}^{n} c_{n-k},$$

which implies that

$$c_n = 2^n \ , \ (n \geq 0) \tag{6.37}$$

Using (6.35)-(6.37), it follows from (5.3) that

$$\begin{aligned} B_{n,n} &= \frac{R_v}{n+1} \left(\sum_{k=v}^{n} 2^{k-v} \right) \\ &= \frac{R_v (2^{n-v+1} - 1)}{n+1} \\ &\not\to 0 \ \ as \ \ n \to \infty \end{aligned} \tag{6.38}$$

so that (5.1) is not satisfied. Using (6.38) we have that

$$\sum_{v=0}^{\infty} B_{n,v} = \frac{2^{n+1}}{n+1}\left(\sum_{v=0}^{k-1} \frac{R_v}{2^v}\right) - \frac{1}{n+1}\sum_{v=0}^{k-1} R_v,$$

which by (6.35) reduces to:

$$\sum_{v=0}^{\infty} B_{n,v} = \frac{1}{n+1}\left(2^{n-k+2} + k - 2\right) = U_{n,k-1}\ ,\ say. \tag{6.39}$$

Using (6.39) and note that $U_{n,k-1} < U_{n+1,k-1}$, it follows that the left hand side of (5.2) is equal to:

$$\lim_{k\longrightarrow\infty}\left(U_{k^2,k-1} - U_{k-1,k-1}\right) = \lim_{n\longrightarrow\infty} \frac{2^{k^2-k+2} + k - 2}{k^2+1} - 1 \neq O(1).$$

Therefore (5.2) is not satisfied. This completes the proof. □

Example 6.5. *Let*

$$r_n = 2n + 1\ ,\ (n \geq 0), \tag{6.40}$$

and let

$$q_n = 2^n\ ,\ (n \geq 0), \tag{6.41}$$

then (N, r) *and* $(\overline{N}, q)$ *are regular, but neither (5.1) nor (5.2) satisfied.*

Proof: Using (6.40) and (6.41), we have

$$R_n = (n+1)^2\ ,\ (n \geq 0), \tag{6.42}$$

and

$$Q_n = 2^{n+1} - 1\ ,\ (n \geq 0), \tag{6.43}$$

which imply that (1.7)-(1.10) are all satisfied, so (N, r) and $(\overline{N}, q)$ are regular. Using induction on n, it follows from (6.3) and (6.40) that

$$c_0 = 1\ ,\ c_1 = -3\ ,\ c_n = 4(-1)^n\ ,\ (n \geq 2), \tag{6.44}$$

Using (6.41)-(6.44); it follows from (5.3) that

$$B_{n,n} = \frac{R_n q_n}{Q_n} = \frac{(n+1)^2 \cdot 2^n}{2^{n+1} - 1}\ ,\ (n \geq 0), \tag{6.45}$$

$$B_{n,n-1} = -\frac{5R_{n-1}q_{n-1}}{Q_n} = -\frac{5n^2 \cdot 2^{n-1}}{2^{n+1} - 1}\ ,\ (n \geq 1), \tag{6.46}$$

$$\begin{aligned} B_{n,v} &= \frac{R_v}{Q_n}\left[-5.2^v + \sum_{\mu=v+2}^{n} q_\mu c_{\mu-(v+2)}\right], \ 0 \le v \le n-2 \\ &= \frac{(v+1)^2}{3(2^{n+1}-1)}\left[2^v - 2^{n+3}(-1)^{n-1}(-1)^{-v}\right] \ 0 \le v \le n-2, \end{aligned} \tag{6.47}$$

and

$$B_{n,v} = 0 \text{ otherwise.} \tag{6.48}$$

Using (6.47), we see that (5.1) is not satisfied. Next, observe that the first term of the left hand side of (5.2) is equivalent to:

$$\left|\sum_{v=0}^{k-1} B_{k-1,v} - \sum_{v=0}^{k-1} B_{k,v}\right|,$$

which by (5.11) reduces to $|B_{k,k}|$.

Using (6.45), we see that

$$|B_{k,k}| = \frac{(k+1)^2 \cdot 2^k}{2^{k+1}-1} \ne O(1).$$

Using remark (5.2), we see that (5.2) is not valid. This completes the proof. □

Example 6.6. *Let*

$$q_n = k^n \ , \ (n \ge 0) \ , \ |k| > 1 \tag{6.49}$$

and let (N, r) be $(C, 1)$, then both (N, r) and $(\overline{N}, q)$ are regular, and (5.1) is satisfied but (5.2) does not.

Proof: $(C, 1)$ is known to be regular, and (6.49) implies that

$$Q_n = \frac{k^{n+1}-1}{k-1} \ , \ n \ge 0, \tag{6.50}$$

so that the regularity of $(\overline{N}, q)$ follows from the satisfaction of (1.9) and (1.10). Next, using the assumptions, it follows from (6.1) and (6.3) that (6.4) is satisfied, and so (5.3) reduces to:

$$\begin{aligned} B_{n,v} &= \frac{(v+1)(k-1)}{k^{n+1}-1}\sum_{\mu=v}^{n} k^\mu c_{\mu-v} \\ &= \frac{(v+1)(k-1)}{k^{n+1}-1}\left[k^v - k^{v+1}\right] \ , \ 0 \le v \le n-1 \end{aligned} \tag{6.51}$$

and

$$B_{n,n} = \frac{(n+1)(k-1)k^n}{k^{n+1}} \ne O(1). \tag{6.52}$$

So that (5.1) follows from (6.51). Also, remark (5.2) together with (6.52) imply that (5.2) is not satisfied. □

7. Equivalence Relations

With the aid of Lemma (5.3), it is natural to give some examples to show that the equivalence may holds in some trivial and non-trivial cases. In this section we will construct two examples to show that $|(N,r)| \sim |\overline{N},q)|$, and two other examples to show that the inclusion may valid in only one way without the other.

Example 7.1. *Let $\{r_n\}$ be defined as in (6.1). Let $\{q_n\}$ be defined as in (6.2), then $|(N,r)| \sim |\overline{N},q)|$.*

Proof: Using the assumptions, it follows from example (6.1) that $|(C,1)| \subseteq |\overline{N},q)|$. Using (6.2), it follows from (6.5) that $Q-n = Q(nq_n)$, and lemma (**??**) implies that $|\overline{N},q)| \subseteq |(C,1)|$. This completes the proof. □

Example 7.2. *Let the sequences $\{r_n\}$ and $\{q_n\}$ be defined respectively as in (6.21) and (6.22), then $|(N,r)| \sim |\overline{N},q)|$.*

Proof: Using the assumptions, $|(N,r)| \subseteq |\overline{N},q)|$ follows from example (6.3). Using (6.21) and (6.22), we have

$$R_0 = 1\ ,\ R_n \frac{1}{2}\ ,\ (n \geq 1) \text{ and } Q_n = \frac{e^{n+1}-1}{e-1}. \tag{7.1}$$

Using (6.21), (6.22) and (7.1), we have

$$\Delta \frac{1}{R_n}\left(R_{n-k-1} + \frac{Q_k}{q_k} r_{n-k}\right) = 0\ ,\ \ n \geq k+2 \tag{7.2}$$

Using (7.2), we see that the left hand side of (4.4) is equivalent to:

$$\left|\frac{Q_k}{R_k q_k}\right| + \left|\frac{Q_k}{R_k q_k} - \frac{1}{R_{k+1}} - \frac{Q_k r_1}{R_{k+1} q_k}\right| + \left|\frac{1}{R_{k+1}} - \frac{Q_k r_1}{R_{k+1} q_k} - \frac{R_1}{R_{k+2}} - \frac{Q_k r_2}{R_{k+2} q_k}\right|$$
$$= \left|\frac{e^{k+1}-1}{\frac{1}{2}e^k}\right| + \left|\frac{c^{k+1}-1}{\frac{1}{2}e^k} - 2 + \frac{e^{k+1}-1}{e^k}\right| + \left|1 - \frac{e^{k+1}-1}{e^k}\right| = O(1)$$

and (4.4) is satisfied which implies that $|\overline{N},q)| \subseteq |(N,r)|$. This completes the proof. □

Example 7.3. *Let the assumptions on $\{r_n\}$ and $\{q_n\}$ be given as in (6.10)-(6.12), then $|(N,r)| \subseteq |\overline{N},q)|$ but the converse is not valid.*

Proof: The result that $|(N,r)| \subseteq |\overline{N},q)|$ follows from example (6.2). Next, observe that the first term of the left hand side of (4.4) is equivalent to $\dfrac{Q_k}{R_k q_k}$ which is necessary condition for (4.4) to be satisfied. Using (6.12), (6.13) and (6.15), we have

$$\frac{Q_k}{R_k q_k} = \frac{\frac{1}{2}k^2 + \frac{1}{6}k + 1}{\left(2 - \dfrac{1}{2^k}\right)\left(k - \dfrac{1}{3}\right)} \neq O(1)$$

Therefore, (4.4) is not satisfied, and so $\left|\overline{N}, q)\right| \not\subseteq |(N, r)|$. □

Example 7.4. *Let the assumptions on $\{r_n\}$ and $\{q_n\}$ be given as in (6.40) and (6.41), then $\left|\overline{N}, q)\right| \subseteq |(N, r)|$ but the converse is not valid.*

Proof: The proof of example (6.5) shows that $|(N, r)| \not\subseteq \left|\overline{N}, q)\right|$. We will show that (4.4) is satisfied, and Theorem (4.2) implies that $\left|\overline{N}, q)\right| \subseteq |(N, r)|$. Using (6.40)-(6.43), we see that the left hand side of (4.4) is equivalent to:

$$\sum_{n=k-1}^{\infty}\left|\frac{R_{n-k-1}}{R_n} + \frac{Q_k r_{n-k}}{R_n q_k} - \frac{R_{n-k}}{R_{n+1}} - \frac{Q_k r_{n-k}}{R_{n+1} q_k}\right|$$

$$\left|\frac{Q_k}{R_k q_k}\right| + \left|\frac{Q_k}{R_k q_k} - \frac{1}{R_{k+1}} - \frac{Q_k r_1}{R_{k+1} q_k}\right| + \left|\frac{1}{R_{k+1}} - \frac{Q_k r_k}{R_{k+1} q_k} - \frac{R_1}{R_{k+2}} - \frac{Q_k r_2}{R_{k+2} q_k}\right|$$

$$= \sum_{n=k+2}^{\infty} |F(n,k) - F(n+1,k)| \ , \ say, \tag{7.3}$$

where

$$F(n,k) = \frac{P_{n-k-1}}{R_n} + \frac{Q_k r_{n-k}}{q_k R_n} = \frac{(n-k)^2}{(n+1)^2} + \frac{(2^{k+1}-1)(2n-2k+1)}{2^k (n+1)^2}. \tag{7.4}$$

Use differentials, we see (after straightforward manipulations) that $F_x(x, k) > 0$, $(x \geq k+2)$. This implies that the quantity inside the absolute of sigma in (7.3) is negative. This implies that the left hand side of (4.4) is equivalent to:

$$A_k + \lim_{N \longrightarrow \infty} \left(F(N+1, k) - F(k+2, k)\right),$$

where A_k is the first three terms of (7.3) which is clearly bounded. Also,

$$\lim_{N \longrightarrow \infty} \left(\frac{(N-k+1)^2}{(N+2)^2} + \frac{(2^{k+1}-1)(2N-2k+3)}{2^k (N+2)^2}\right) - F(k+2,k) = O(1) \tag{7.5}$$

Therefore (4.4) is satisfied, and Theorem (4.2) yields the result. □

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