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On Extended Generalized ϕ-Recurrent $(LCS)_{2n+1}$-Manifolds

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ABSTRACT: We introduce the notion of extended generalized ϕ-recurrent $(LCS)_{2n+1}$ -manifolds and study its various geometric properties with an example. Finally, we construct an example of 3-dimensional extended generalized ϕ -recurrent $(LCS)_{2n+1}$ -manifold which is neither ϕ-recurrent nor generalized ϕ-recurrent.

Key Words: Generalized recurrent $(LCS)_{2n+1}$-manifolds, Concircular curvature tensor, Extended generalized ϕ-recurrent $(LCS)_{2n+1}$- manifolds, Generalized ϕ -recurrent $(LCS)_{2n+1}$ -manifolds and Concircular curvature tensor .

Contents

1. Introduction

In 2003, Shaikh [14] introduced the notion of Lorentzian concircular structure manifolds (briefly $(LCS)_{2n+1}$-manifolds) with an example, which generalizes the notion of LP-Sasakian manifolds introduced by Matsumoto [7]. The notion of local symmetry of a Riemannian manifold has been weakened by many authors in several ways to a different extent. As a weaker version of local symmetry, Takahashi [19] introduced the notion of local ϕ-symmetry on a Sasakian manifold. Generalizing the notion of local ϕ-symmetry of Takahashi [19], De et al. [2] introduced the notion of ϕ-recurrent Sasakian manifold. Recently De et al. [3] introduced the notion of ϕ-recurrent Kenmotsu manifolds. The locally ϕ-symmetric LP-Sasakian manifolds is also studied by Shaikh and Baishya [15]. Again locally ϕ-symmetric and locally ϕ-recurrent $(LCS)_{2n+1}$-manifolds are respectively studied in [16] and [17]. The notion of generalized recurrent manifolds has been introduced by Dubey [6] and studied by De and Guha [4]. Again, the notion of generalized Ricci-recurrent manifolds has been introduced and studied by De et al. [5].

A Riemannian manifold $(M^n, g), n > 2$, is called generalized recurrent ([4], [6]), if its curvature tensor R satisfies the condition

$$\nabla R = A \otimes R + B \otimes G, \tag{1.1}$$

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where A and B are non vanishing 1-forms defined by $A(\cdot) = g(\cdot, \rho_1), B(\cdot) = g(\cdot, \rho_2)$ and the tensor G is defined by

$$G(X, Y)Z = g(Y, Z)X - g(X, Z)Y, \tag{1.2}$$

for all $X, Y, Z \in \chi(M); \chi(M)$ being the Lie algebra of smooth vector fields on M and ∇ denotes the operator of covariant differentiation with respect to metric g. The $1-$ forms A and B are called the associated 1-forms of the manifold. A Riemannian manifold $(M^n, g), n > 2$, is called generalized Ricci-recurrent [5] if its Ricci tensor S of type $(0, 2)$ satisfies the condition

$$\nabla S = A \otimes S + B \otimes g, \tag{1.3}$$

where A and B are non vanishing 1-forms defined in (1.1).

In 2007, *Özgür* [10] studied generalized recurrent Kenmotsu manifolds. Generalizing the notion of *Özgür* [10], recently Basari and Murathan [1] introduced the notion of generalized ϕ-recurrent Kenmotsu manifolds. Also, the notion of generalized ϕ-recurrency to Sasakian manifold, Lorentzian α-Sasakian manifolds and generalized Sasakian space-forms are respectively studied in ([11], [12], [22], [24], [26]). By extending the notion of generalized ϕ-recurrency, Prakasha [13] and Shaikh and Hui [18] introduced the notion of extended generalized ϕ-recurrency to β-Kenmotsu manifolds and Sasakian manifolds respectively. As a continuation of this here we have introduce the notion of extended generalized ϕ-recurrent $(LCS)_{2n+1}$-manifolds.

The paper is organized as follows. Section 2 deals with a brief account of $(LCS)_{2n+1}$-manifolds. In section 3, we study generalized ϕ-recurrent $(LCS)_{2n+1}$-manifolds and obtain a necessary and sufficient condition for a manifold to be a generalized Ricci-recurrent manifold. Also we study extended generalized concircularly ϕ-recurrent $(LCS)_{2n+1}$-manifolds and find the nature of its associated 1-forms. Finally; the last section is responsible for the existence of extended generalized ϕ-recurrent $(LCS)_{2n+1}$-manifolds.

2. Preliminaries

An $(2n+1)$-dimensional Lorentzian manifold M is smooth connected paracontact Hausdroff manifold with Lorentzian metric g, that is, M admits a smooth symmetric tensor field ϕ of type $(0, 2)$ such that for each point $p \in M$ the tensor $g_p : T_pM \times T_pM \to \Re$ is a non degenerate inner product of signature $(-, +, ..., +)$, where T_pM denotes the tangent space of M at p and $\Re$ is the real number space. A non-zero vector field $v \in T_pM$ is said to be time like (resp., non-spacelike, null, and spacelike) if it satisfies $g_p(v, v) < 0$ (resp.,$\leq 0, =, > 0$) [9].

Definition 2.1. *In a Lorentzian manifold (M, g) a vector field ρ defined by*

$$g(X, \rho) = A(X)$$

for any $X \in \chi(M)$ is said to be a concircular vector field if

$$(\nabla_X A)(Y) = \alpha \{ g(X, Y) + \omega(X)A(Y) \}$$

where α is a non-zero scalar and ω is a closed 1-form.

Let M be a Lorentzian manifold admitting a unit time like concircular vector field ξ, called the generator of the manifold. Then, we have

$$g(\xi, \xi) = -1, \tag{2.1}$$

Since, ξ is the unit concircular vector field, there exists a non-zero 1-form η such that

$$g(X, \xi) = \eta(X), \tag{2.2}$$

the equation of the following form holds

$$(\nabla_X \eta)\ (Y) = \alpha\ \{g(X, Y) + \eta\,(X)\,\eta(Y)\,\}\ (\alpha \neq 0) \tag{2.3}$$

since, ξ is a concircular field, therefore

$$\nabla_X \xi = \alpha\,\{X + \omega(X)\xi\}$$

where ω is a 1-form. Also since ξ is a unit vector field from (2.1), which implies $g(\nabla_X \xi, \xi) = 0$ and hence, we get from above equation

$$\omega(X) = \eta(X),$$

where $\eta(X) = g(X, \xi)$, Now

$$g(\alpha X, Y) + g(\alpha\eta(X)\xi, Y) = g(\nabla_X \xi, Y),$$

which implies

$$\alpha\,[g(x, Y) + \eta(X)\eta(Y)] = g(Y, \nabla_X \xi),$$

we have $(\nabla_X \eta)Y = X\eta(Y) - \eta(\nabla_X Y)$ hence

$$(\nabla_X \eta)Y = Xg(Y, \xi) - g(\nabla_Y, \xi)$$

since $(\nabla_X g)(Y, \xi) = 0$, we have

$$(\nabla_X \eta)Y = g(Y, \nabla_X \xi$$

now, we arrive at the result (2.3) for all vector fields X, Y, where ∇ denotes the operator of covariant differentiation with respect to the Lorentzan metric g and α is a non-zero scalar function satisfies

$$\nabla_X \alpha = (X\alpha) = d\alpha\,(X) = \rho\,\eta\,(X) \tag{2.4}$$

ρ being a certain scalar function given by $\rho = -(\xi\alpha)$. If we put

$$\phi X = \frac{1}{\alpha}\nabla_X \xi, \tag{2.5}$$

then from (2.3) and (2.5) we have

$$\phi X = X + \eta(X)\xi, \tag{2.6}$$

from which it follows that ϕ is a symmetric $(1,1)$-tensor and called the structure tensor of the manifold. Thus the Lorentzian manifold M together with the unit timelike concircular vector field ξ its associated 1-form η and $(1,1)$-tensor field ϕ is said to be a Lorentzian concircular structure manifold (briefly $(LCS)_{2n+1}$-manifolds) [4]. Especially, if we take $\alpha = 1$, then we can obtain the LP-Sasakian structure of Motsumoto [7].

In a $(LCS)_{2n+1}$-manifolds, the following relations hold [4]:

$$a)\ \eta(\xi) = -1,\ b)\ \phi\xi = 0,\ c)\ \phi^2 X = X + \eta(X)\xi, \tag{2.7}$$

$$d)\ \eta(\phi X) = 0,\ e)\ g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y),$$

$$\eta(R(X,Y)Z) = (\alpha^2 - \rho)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}, \tag{2.8}$$

$$R(X,Y)\xi = (\alpha^2 - \rho)\{\eta(Y)X - \eta(X)Y\}, \tag{2.9}$$

$$R(\xi,X)Y = (\alpha^2 - \rho)\{g(X,Y)\xi - \eta(Y)X\}, \tag{2.10}$$

$$R(\xi,X)\xi = (\alpha^2 - \rho)\{\eta(X)\xi + X\}, \tag{2.11}$$

$$(\nabla_X \phi)(Y) = \alpha\{g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\}, \tag{2.12}$$

$$S(X,\xi) = 2n(\alpha^2 - \rho)\eta(X), \tag{2.13}$$

$$S(\phi X, \phi Y) = S(X,Y) + 2n(\alpha^2 - \rho)\eta(X)\eta(Y), \tag{2.14}$$

$$(X\rho) = d\rho(X) = \beta\eta(X). \tag{2.15}$$

We now state some curvature properties of $(LCS)_{2n+1}$-manifolds which will be frequently used later on.

Lemma 2.2. *From [17], Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a Lorentzian concircular structure manifold. Then for any vector fields X, Y, W the following relation holds:*

$$(\nabla_W R)(X,Y)\xi = (2\alpha\rho - \beta)\{\eta(Y)\eta(W)X - \eta(X)\eta(W)Y\}$$
$$+\alpha(\alpha^2 - \rho)\{g(Y,W)X - g(X,W)Y\} - \alpha R(X,Y)W.$$

3. Generalized ϕ-recurrent$(LCS)_{2n+1}$-manifolds

Definition 3.1. *A Lorentzian concircular structure manifold $M^{2n+1}(\phi,\xi,\eta,g)$, $n > 1$, is said to be an extended generalized ϕ-recurrent $(LCS)_{2n+1}$-manifolds if its curvature tensor R satisfies the condition*

$$\phi^2((\nabla_W R)(X,Y)Z) = A(W)\phi^2(R(X,Y)Z) + B(W)\phi^2(G(X,Y)Z), \tag{3.1}$$

for all $X, Y, Z, W \in \chi(M)$, where ∇ denotes the operator of covariant differentiation with respect to the metric g ,i.e. ∇ is the Riemannian connection; A and B are non-vanishing 1-form such that $A(X) = g(X,\rho_1), B(X) = g(X,\rho_2)$ and G is a tensor of type $(1,3)$ defined in (1.2). The 1-forms A and B are called the associated 1-forms of the manifold.

We consider a Lorentzian concircular structure manifold $M^{2n+1}(\phi,\xi,\eta,g), n>1$, which is extended generalized ϕ-recurrent. Then from (2.7) and (3.1) yields

$$(\nabla_W R)(X,Y)Z+\eta\left((\nabla_W R)(X,Y)Z\right)\xi = A(W)\,\{R(X,Y)Z+\eta(R(X,Y)Z)\xi\} \\ +B(W)\,\{G(X,Y)Z+\eta(G(X,Y)Z)\xi\}, \tag{3.2}$$

from which it follows that

$$\begin{aligned} &g((\nabla_W R)(X,Y)Z,U)+\eta\left((\nabla_W R)(X,Y)Z\right)\eta(U) \\ &= A(W)\,\{g(R(X,Y)Z,U)+\eta(R(X,Y)Z))\eta(U)\} \\ &+B(W)\,\{g\,(G(X,Y)Z,U)+\eta\,(G(X,Y)Z)\,)\,\eta(U)\}. \end{aligned} \tag{3.3}$$

Let $\{e_i : i=1,2,3,....,2n+1\}$ be an orthonormal basis of the tangent space at any point of the manifold. Replacing $X=U=e_i$ in (3.3) and taking summation over i, $1\le i\le 2n+1$, and then using (2.10), we have

$$\begin{aligned} &(\nabla_W S)(Y,Z)-g\left((\nabla_W R)\,(\xi,Y)\,Z,\xi\right) \\ &= A(W)\,\left\{\,S(Y,Z)-(\alpha^2-\rho)\{g(Y,Z)+\eta(Y)\,\eta(Z)\}\right\} \\ &+B(W)\,\{\,(2n-1)g\,(Y,Z)+\eta(Y)\eta(Z)\}. \end{aligned} \tag{3.4}$$

Also from (2.8), we obtain

$$\begin{aligned} g\left((\nabla_W R)(\xi,Y)\,Z,\xi\right) &= 2\,\alpha\,\rho\,\eta\,(W)\,\{g(Y,Z)+\eta(Y)\,\eta(Z)\,\} \\ &+(\alpha^2-\rho)\,\{g(W,Z)-\eta(W)\eta(Z)\,\}\,\eta(Y) \end{aligned} \tag{3.5}$$

In view of (3.5), it follows from (3.4) that

$$\begin{aligned} &(\nabla_W S)(Y,Z) = A(W)S(Y,Z) \\ &+\left\{\,2\alpha\rho\eta(W)+(2n-1)B(W)-(\alpha^2-\rho)A(W)\right\}\,g(Y,Z) \\ &+\left\{\,2\alpha\rho\,\eta(W)+B\,(W)-(\alpha^2-\rho)A(W)\,\right\}\,\eta(Y)\eta(Z) \\ &+(\alpha^2-\rho)\,\{g(W,Z)-\eta(W)\,\eta(Z)\}\,\eta(Y). \end{aligned} \tag{3.6}$$

From (3.6), it follows that an extended generalized ϕ-recurrent $(LCS)_{2n+1}, n>1$, manifolds is a generalized Ricci-recurrent manifold if and only if

$$\begin{aligned} &\left\{\,2\,\alpha\,\rho\,\eta\,(W)+B(W)-(\alpha^2-\rho)\,A(W)\,\right\}\,\eta(Y)\,\eta(Z) \\ &+(\alpha^2-\rho)\,\left\{\,g(W,Z)-\eta\,(W)\,\eta(Z)\right\}\,\eta(Y)=0. \end{aligned} \tag{3.7}$$

This leads to the following:

Theorem 3.2. *An extended generalized ϕ-recurrent Lorentzian concircular structure manifold $M^{2n+1}(\phi,\xi,\eta,g), n>1$, is generalized Ricci-recurrent if and only if the relation (3.7) holds.*

Substituting $Z=\xi$ in (3.2), we obtain

$$(\nabla_W R)(X,Y)\xi = \left\{(\alpha^2-\rho)A(W)+B(W)\right\}\,(\,\eta(Y)X-\eta(X)Y\,) \tag{3.8}$$

By virtue of Lemma 2.2 and (3.8), we yields

$$\begin{aligned}&\alpha R(X,Y)W=\alpha\,(\alpha^2-\rho)\ \{g(Y,W)X-g(X,W)Y\}\\&+(2\alpha\rho-\beta)\ \{\eta(Y)\eta\,(W)X-\eta(X)\,\eta(W)Y\}\\&-\left\{\,(\alpha^2-\rho)\,A(W)+B(W)\right\}\ (\eta\,(Y)X-\eta\,(X)\,Y\,)\,.\end{aligned}\tag{3.9}$$

This leads to the following:

Theorem 3.3. *In an extended generalized ϕ-recurrent Lorentzian concircular structure manifold $M^{2n+1}(\phi,\xi,\eta,g), n>1$, the curvature tensor is of the form of (3.9).*

Also from (3.9), we have

$$\begin{aligned}&\alpha\,\widehat{R}\,(X,Y,W,U)=\alpha\,(\alpha^2-\rho)\ \{g(Y,W)g(X,U)-g(X,W)g(Y,U)\}\\&+(2\alpha\rho-\beta)\ \{\eta(Y)\eta\,(W)g(X,U)-\eta(X)\,\eta(W)g(Y,U)\}\\&-\left\{\,(\alpha^2-\rho)\,A(W)+B(W)\right\}\ (\eta\,(Y)g(X,U)-\eta\,(X)g(Y,U)\,)\,,\end{aligned}$$

where $\widehat{R}\,(X,Y,W,U)=g(R(X,Y)W,U)$. Setting $X=U=e_i$ in above and taking summation over $i,\ 1\leq i\leq 2n+1$, we yield that

$$\begin{aligned}&\alpha\,S(Y,W)=2n\alpha\ (\alpha^2-\rho)\ g(Y,W)+2\,n\,(2\alpha\rho-\beta)\ \eta(Y)\ \eta(W)\\&-2n\ \left\{\,(\alpha^2-\rho)\,A(W)+B(W)\right\}\ \eta(Y).\end{aligned}\tag{3.10}$$

This leads to the following:

Theorem 3.4. *In an extended generalized ϕ-recurrent Lorentzian concircular structure manifold $M^{2n+1}(\phi,\xi,\eta,g), n>1$, the Ricci tensor is of the form of (3.11).*

Again, in view of lemma 2.2, equation (3.2) can be written as

$$\begin{aligned}&(\nabla_W R)(X,Y)Z=\begin{bmatrix}(2\alpha\rho-\beta)\{g(X,Z)\eta(Y)\eta(W)-g(Y,Z)\eta(X)\eta(W)\}\\+\alpha(\alpha^2-\rho)\{g(X,Z)g(Y,W)-g(Y,Z)g(X,W)\}\\-\alpha g(R(X,Y)W,Z)\end{bmatrix}\xi\\&+A\,(W)\ \{R(X,Y)Z+\eta(R(X,Y)Z)\xi\}\\&+B\,(W)\ \{G(X,Y)Z+\eta(G(X,Y)Z)\xi\}\,,\end{aligned}\tag{3.11}$$

Conversely, applying ϕ^2 on both sides of (3.12), we get the relation (3.1). This leads to the following:

Theorem 3.5. *A Lorentzian concircular structure manifold $M^{2n+1}(\phi,\xi,\eta,g)$, $n>1$, is an extended generalized ϕ -recurrent if and only if the relation (3.12) holds.*

Definition 3.6. *A Lorentzian concircular structure manifold $M^{2n+1}(\phi,\xi,\eta,g)$, $n>1$, is said to be an extended generalized concircularly ϕ-recurrent $(LCS)_{2n+1}$-manifolds if its concircular curvature tensor C satisfies the condition*

$$\phi^2((\nabla_W C)(X,Y)Z)=A(W)\phi^2(C(X,Y)Z)+B(W)\phi^2(G(X,Y)Z),\tag{3.12}$$

for all $X, Y, Z, W \in \chi(M)$, where ∇ denotes the operator of covariant differentiation with respect to the metric g, i.e. ∇ is the Riemannian connection; A and B are non-vanishing 1-forms defined in (1.1) and G is a tensor of type $(1,3)$ defined in (1.2).

The concircular curvature tensor C of type $(1,3)$ is given by [20]

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{2n(2n+1)}G(X,Y)Z, \tag{3.13}$$

where r is the scalar curvature of the manifold.
Let us consider an extended generalized concircularly ϕ-recurrent Lorentzian concircular structure manifold $M^{2n+1}(\phi,\xi,\eta,g), n>1$. Then from (2.7) and (3.13) yields

$$\begin{aligned}&(\nabla_W C)(X,Y)Z + \eta\left((\nabla_W C)(X,Y)\,Z\right)\xi\\ &= A\,(W)\,\{C(X,Y)Z + \eta(C(X,Y)Z)\xi\}\\ &+B\,(W)\,\{G(X,Y)Z + \eta(G(X,Y)Z)\xi\}\,,\end{aligned} \tag{3.14}$$

from which it follows that

$$\begin{aligned}&g((\nabla_W C)(X,Y)Z,U) + \eta\left((\nabla_W C)(X,Y)\,Z\right)\eta(U)\\ &= A\,(W)\,\{g(C(X,Y)Z,U) + \eta(C(X,Y)Z))\eta(U)\}\\ &+B\,(W)\,\{\,g\,(G(X,Y)Z,U) + \eta\,(G(X,Y)Z)\,)\,\eta(U)\}\,.\end{aligned} \tag{3.15}$$

Taking contraction of (3.16) over X and U, we get

$$\begin{aligned}&(\nabla_W S)(Y,Z) - \tfrac{dr(W)}{2n+1}g(Y,Z) + g\left((\nabla_W C)(\xi,Y)Z,\xi\right)\\ &= A(W)\left\{S(Y,Z) - \tfrac{r}{2n+1}g(Y,Z) + \eta\left((C(\xi,Y)Z\right)\right\}\\ &+B(W)\,\{(2n-1)\,g(Y,Z) - \eta(Y)\,\eta(Z)\,\}\end{aligned} \tag{3.16}$$

In view of (3.5) and (3.14), we have

$$\begin{aligned}&g((\nabla_W C)(\xi,Y)Z,\xi) = \left\{2\alpha\rho(W) + \tfrac{dr(W)}{2n(2n+1)}\right\}\{g(Y,Z) + \eta(Y)\eta(Z)\}\\ &+(\alpha^2-\rho)\eta(Y)\,\{g(Z,W) + \eta(Z)\eta(W)\}\,.\end{aligned} \tag{3.17}$$

Also from (2.10) and (3.14), we get

$$\eta(C(\xi,Y)Z) = \left\{\frac{r}{2n(2n+1)} - (\alpha^2-\rho)\right\}\,\{\,g\,(Y,Z) + \eta\,(Y)\,\eta(Z)\} \tag{3.18}$$

By virtue of (3) and (3.18), equation (3.17) reduces to

$$\begin{aligned}&(\nabla_W S)(Y,Z) = A(W)S(Y,Z) + (2n-1)B(W) - \tfrac{r}{2n+1}A(W)\\ &+A(W)\left(\tfrac{r}{2n(2n+1)} - (\alpha^2-\rho)\right) + \tfrac{dr(W)}{2n+1} - 2\alpha(W\alpha) - \tfrac{dr(W)}{2n(2n+1)}g(Y,Z)\\ &+\left[A(W)\left(\tfrac{r}{2n(2n+1)} - (\alpha^2-\rho)\right) - (2\alpha(W\alpha)) - \tfrac{dr(W)}{2n(2n+1)}\right]\eta(Y)\eta(Z)\\ &-(\alpha^2-\rho)\eta(Y)\,(g(Z,W) + \eta(Z)\eta(W))\end{aligned} \tag{3.19}$$

In view of (3.19), we can state the following:

Theorem 3.7. *An extended generalized concircularly ϕ-recurrent Lorentzian concircular structure manifold $M^{2n+1}(\phi,\xi,\eta,g), n>1$, is generalized Ricci-recurrent if and only if the following relation holds.*

$$\left[A(W)\left\{\frac{r}{2n(2n+1)}-(\alpha^2-\rho)\right\}-(2\alpha(W\alpha))-\frac{dr(W)}{2n(2n+1)}-B(W)\right]\eta(Y)\eta(Z)$$
$$-(\alpha^2-\rho)\eta(Y)\left\{g(Z,W)+\eta(Z)\eta(W)\right\}=0.$$

Setting $Y=Z=\xi$ in (3.19) and using (2.13), we get

$$\begin{aligned}&\left[\tfrac{2\,r\,(n-1)}{2\,n\,(2n+1)}+2\,(n-1)(\alpha^2-\rho)\right]A\,(W)-2(n-1)B(W)\\&=dr(W)\left\{\tfrac{2n-1}{2n(2n+1)}\right\}-4\alpha\,(W\alpha).\end{aligned}\tag{3.20}$$

This leads to the following:

Theorem 3.8. *In an extended generalized concircularly ϕ-recurrent Lorentzian concircular structures manifold $M^{2n+1}(\phi,\xi,\eta,g), n>1$, the 1-forms A and B are related by the relation (3.20).*

Corollary 3.9. *In an extended generalized concircularly ϕ-recurrent LP-Sasakian manifold $M^{2n+1}(\phi,\xi,\eta,g), n>1$, with constant scalar curvature, the associated 1-forms A and B are related by $A=cB$, where c is a nonzero constant.*

Finally, in view of Lemma 2.2 and (3.20), (3.13), can be reduces to

$$\begin{aligned}(\nabla_W C)(X,Y)Z&=\left\{\tfrac{d\,r(W)}{2n(2n+1)}-(2\alpha\rho-\beta)\eta(W)\right\}(g(Y,Z)\eta(X)\xi-g(X,Z)\eta(Y)\xi)\\&+\alpha\,(\alpha^2-\rho)\,\{\,g(Y,W)g(X,Z)\xi-g(X,W)\,g(Y,Z)\xi\,\}-\alpha\,g(R(X,Y)W,Z)\xi\\&+A\,(W)\,\{C(X,Y)Z+\eta(C(X,Y)Z)\xi\}+B\,(W)\,\{G(X,Y)Z+\eta(G(X,Y)Z)\xi\}\,.\end{aligned}\tag{3.21}$$

Conversely, applying ϕ^2 on both sides of (3.21), we get the relation (3.17). This leads to the following:

Theorem 3.10. *A Lorentzian concircular structure manifold $M^{2n+1}(\phi,\xi,\eta,g), n>1$, is an extended generalized concircularly ϕ -recurrent if and only if the relation (3.21)) holds.*

4. Example of generalized ϕ-recurrent$(LCS)_{2n+1}$-manifolds

We consider a 3-dimensional manifold $M=\left\{(x,y,z)\in\Re^3 : z\neq 0\right\}$, where (x,y,z) are the standard coordinates in $\Re^3$. Let$\{E_1,E_2,E_3\}$ be linearly independent global frame on M given by

$$E_1=e^z\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right),\ E_2=e^z\frac{\partial}{\partial\,y},\ E_3=e^{2z}\frac{\partial}{\partial\,z}.$$

Let g be the Lorentzian metric defined by

$$g(E_1, E_3) = g(E_2, E_3) = g(E_1, E_2) = 0,$$

$$g(E_1, E_1) = g(E_2, E_2) = 1, \quad g(E_3, E_3) = -1.$$

Let η be the 1-form defined by $\eta(V) = g(V, E_3)$ for any $V \in \chi(M)$.
Let ϕ be the $(1,1)$-tensor field defined by $\phi E_1 = E_1,\ \phi E_2 = E_2,\ \phi E_3 = 0$. Then using the linearity of ϕ and g we have

$$\eta(E_3) = -1, \quad \phi V = V + \eta(V) E_3, \quad g(\phi V, \phi W) = g(V, W) + \eta(V)\eta(W),$$

for any $V, W \in \chi(M)$.
Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g and R be the curvature tensor of g. Then we have

$$[E_1, E_2] = -e^z E_2,\ [E_1, E_3] = -e^{2z} E_1,\ [E_2, E_3] = -e^{2z} E_2.$$

Taking $E_3 = \xi$ and using Koszula formula for the Lorentzian metric g, we can easily calculate

$$\nabla_{E_1} E_3 = -\frac{1}{z} E_1,\ \nabla_{E_1} E_1 = -e^{2z} E_3,\ \nabla_{E_1} E_2 = 0,$$

$$\nabla_{E_2} E_3 = -e^{2z} E_2,\ \nabla_{E_3} E_2 = 0,\ \nabla_{E_2} E_1 = -e^{2z} E_2,$$

$$\nabla_{E_3} E_3 = 0,\ \nabla_{E_2} E_2 = -e^{2z} E_3 - e^z E_1,\ \nabla_{E_3} E_1 = 0.$$

From the above it can be easily seen that $E_3 = \xi$ is a unit timelike concircular vector field and hence (ϕ, ξ, η, g) is a $(LCS)_3$-structure on M. Consequently $M^3(\phi, \xi, \eta, g)$ is a $(LCS)_3$-manifolds with $\alpha = -e^{2z} \neq 0$ such that $(X\alpha) = \rho\eta(X)$ where $\rho = 2e^{4z}$ Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor R as follows:

$$R(E_2, E_3)E_3 = e^{4z} E_2,\ R(E_1, E_3)E_3 = e^{4z} E_1,\ R(E_1, E_2)E_2 = \{e^{4z} - e^{2z}\} E_1,$$

$$[R(E_2, E_3)E_2 = e^{4z} E_3,\ R(E_1, E_3)E_1 = e^{4z} E_3,\ R(E_1, E_2)E_1 = \{-e^{4z} - e^{2z}\} E_2.$$

and the components which can be obtained from these by the symmetric properties. Since $\{E_1, E_2, E_3\}$ forms a basic of the 3-dimensional (LCS)-manifolds, any vector field $X, Y, Z \in \chi(M)$ can be written as

$$X = a_1 E_1 + b_1 E_2 + c_1 E_3,\ Y = a_2 E_1 + b_2 E_2 + c_2 E_3,\ Z = a_3 E_1 + b_3 E_2 + c_3 E_3,$$

where $a_i, b_i, c_i \in \Re^+$ (the set of all positive real numbers) , $i = 1, 2, 3$. Then

$$\begin{aligned} R(X,Y)Z = & \begin{bmatrix} e^{4z}\{(b_2 b_3 - c_2 c_3) a_1 + b_1 b_2 (c_3 - a_3)\} + e^{3z}\{(2c_3 - c_2) b_1 b_2 \\ -e^{z}(b_1 b_2 b_3) \end{bmatrix} E_1 \\ & + \begin{bmatrix} e^{4z}\{(a_3 + b_3) b_1 b_2 + (b_1 c_3 - a_2 b_1) a_3 + b_1 (a_2 c_3 - a_3 c_2 \\ + b_2 (c_2 c_3 - a_1 a_3)\} + e^{3z}(a_1 b_2 b_3) \end{bmatrix} E_2 \\ & + \begin{bmatrix} e^{4z}\{(c_3 - a_3) b_1 b_2 - a_1 (a_2 c_3 - a_3 c_2) - (a_3 b_1 b_3)\} + e^{3z}(b_3 - a_3) b_1 b_2 \\ -e^{2z}(b_1 b_2 a_3) \end{bmatrix} E_3, \end{aligned} \tag{4.1}$$

$$\begin{aligned} G(X,Y)Z &= (a_2a_3 + b_2b_3 - c_2c_3)(a_1E_1 + b_1E_2 + c_1E_3) \\ &-(a_1a_3 + b_1b_3 - c_1c_3)(a_2E_1 + b_2E_2 + c_2E_3) \end{aligned} \tag{4.2}$$

By virtue of (4.1) we have the following

$$\begin{aligned} (\nabla_{E_1}R)(X,Y)Z &= \begin{bmatrix} 4e^{6z}\{(b_1b_3 - c_1c_3)a_2 + b_1b_2(c_2 - a_3)\} \\ +3e^{5z}\{(2c_3 - c_2)b_1b_2\} - e^{3z}(b_1b_2b_3) \end{bmatrix} E_1 \\ &+ \begin{bmatrix} 4e^{6z}\{(5a_3 + b_3)b_1b_2 + 3(b_3c_3 - a_2b_1)a_3 \\ +b_2(a_2c_3 - a_3c_2) + b_3(c_2c_1 - a_1a_3)\} + 3e^{5z}(a_2b_1b_3) \end{bmatrix} E_2 \\ &+ \begin{bmatrix} 4e^{6z}\{(c_3 - a_3)b_3b_1 - a_1(a_2c_3 - a_3c_2) - (a_3b_1b_3)\} \\ +4e^{5z}(b_3 - a_3)b_1b_2 \end{bmatrix} E_3, \end{aligned} \tag{4.3}$$

$$\begin{aligned} (\nabla_{E_2}R)(X,Y)Z &= \begin{bmatrix} -e^{5z}\{ b_1b_2(c_3 + a_3) + (b_1c_3 - b_1b_2)a_3 \\ +b_2(c_2c_3 - a_1a_3) + b_1a_3(c_3 - c_2)\} - e^{4z}(a_1b_2b_3 \end{bmatrix} E_1 \\ &+ \begin{bmatrix} -e^{6z}\{(c_3 - a_3)b_1b_2 + (b_2b_3 - c_2c_3)a_1\} \\ -e^{-5z}(b_1b_2c_3) + e^{3z}(b_1b_2b_3) \end{bmatrix} E_2 \\ &+ \begin{bmatrix} -e^{6z}\{(a_3 + b_3)b_1b_2 + (c_3 - a_2)b_1a_3 \\ +b_2(c_2c_3 - a_1a_3) + a_3b_1(c_3 - c_2)\} - e^{5z}(a_1b_2b_3) \end{bmatrix} E_3, \end{aligned} \tag{4.4}$$

$$\begin{aligned} (\nabla_{E_3}R)(X,Y)Z &= \begin{bmatrix} 4e^{64z}\{(b_2b_3 - c_2c_3)a_1 + b_1b_2(c_3 - a_3)\} \\ +3e^{5z}\{(2c_3 - c_2)b_1b_2\} - e^{3z}(b_1b_2b_3) \end{bmatrix} E_1 \\ &+ \begin{bmatrix} 4e^{6z}\{(a_3 + b_3)b_1b_2 + (b_1c_3 - a_2b_1)a_3 \\ +b_1(a_2c_3 - a_3c_2) + b_2(c_2c_3 - a_1a_3)\} + 3e^{5z}(a_1b_2b_3) \end{bmatrix} E_2 \\ &+ \begin{bmatrix} 4e^{6z}\{(c_3 - a_3)b_1b_2 - a_1(a_2c_3 - a_3c_2) \\ -(a_3b_1b_3)\} + 3e^{5z}(b_3 - a_3)b_1b_2 - 2e^{4z}(b_1b_2a_3) \end{bmatrix} E_3, \end{aligned} \tag{4.5}$$

In view of (4.1) and (4.2), we get

$$\phi^2(R(X,Y)Z) = l\,E_1 + m\,E_2, \;\; \phi^2(G(X,Y)Z) = n\,E_1 + p\,E_2, \tag{4.6}$$

where $l = e^{4z}\{(b_2b_3 - c_2c_3)a_1 + b_1b_2(c_3 - a_3)\} + e^{3z}\{(2c_3 - c_2)b_1b_2\} - e^{z}(b_1b_2b_3)$,

$$\begin{aligned} m &= e^{4z}\{(a_3 + b_3)b_1b_2 + (b_1c_3 - a_2b_1)a_3 + b_1(a_2c_3 - a_3c_2) \\ &+b_2(c_2c_3 - a_1a_3)\} + e^{3z}(a_1b_2b_3), \end{aligned}$$

$$\begin{aligned} n &= (a_1b_2 - a_2b_1)b_3 - (a_1c_2 - a_3c_1)c_3, \\ p &= (a_2b_1 - a_1b_2)a_3 - (b_1c_2 - b_2c_1)c_3. \end{aligned}$$

Thus from (4.3)-(4.5), we have following

$$\phi^2\left((\nabla_{E_i}R)\,(X,Y)Z\right) = q_i\,E_1 + r_i\,E_2, \;\; \forall\, i = 1,2,3 \tag{4.7}$$

where

$$\begin{aligned} q_1 &= 4e^{6z}\{(b_1b_3 - c_1c_3)a_2 + b_1b_2(c_2 - a_3)\} + 3e^{5z}\{(2c_3 - c_2)b_1b_2\} - e^{3z}(b_1b_2b_3), \\ q_2 &= -e^{5z}\{ b_1b_2(c_3 + a_3) + (b_1c_3 - b_1b_2)a_3 \; + b_2(c_2c_3 - a_1a_3) + b_1a_3(c_3 - c_2)\} \\ &\;\; -e^{4z}(a_1b_2b_3), \\ q_3 &= 4e^{6z}\{(b_2b_3 - c_2c_3)a_1 + b_1b_2(c_3 - a_3)\} + 3e^{5z}\{(2c_3 - c_2)b_1b_2\} - e^{3z}(b_1b_2b_3), \end{aligned}$$

$$r_1 = 4e^{6z}\{(5a_3+b_3)b_1b_2+3(b_3c_3-a_2b_1)a_3+b_2(a_2c_3-a_3c_2)+b_3(c_2c_1-a_1a_3)\}$$
$$+3e^{5z}(a_2b_1b_3),$$
$$r_2 = -e^{6z}\{(c_3-a_3)b_1b_2+(b_2b_3-c_2c_3)a_1\}-e^{-5z}(b_1b_2c_3)+e^{3z}(b_1b_2b_3),$$
$$r_3 = 4e^{6z}\{(a_3+b_3)b_1b_2+(b_1c_3-a_2b_1)a_3+b_1(a_2c_3-a_3c_2)+b_2(c_2c_3-a_1a_3)\}$$
$$+3e^{5z}(a_1b_2b_3).$$

Now we consider the $1-$form as follows

$$A(E_i) = \frac{pq_i - nr_i}{lp - mn} \; B(E_i) = \frac{lr_i - mq_i}{lp - mn} \tag{4.8}$$

for $i = 1, 2, 3$ such that $lp - mn \neq 0, pq_i - nr_i \neq 0$ and $lr_i - mq_i \neq 0, i = 1, 2, 3$. From (3.1), we have

$$\phi^2((\nabla_{E_i} R)(X,Y)Z) = A(E_i)\,\phi^2(R(X,Y)Z) + B(E_i)\,\phi^2(G(X,Y)Z), i = 1, 2, 3. \tag{4.9}$$

From (4.6)-(4.8), it can be easily shown that the manifold satisfies the relation (4.9). Hence the manifold under consideration is an extended generalized ϕ-recurrent $(LCS)_{2n+1}$-manifolds, which is neither ϕ-recurrent nor generalized ϕ-recurrent. Therefore we have the following:

Theorem 4.1. *There exists an extended generalized ϕ-recurrent $(LCS)_{2n+1}$- manifolds $M^3(\phi, \xi, \eta, g)$, which is neither ϕ-recurrent nor generalized ϕ-recurrent.*

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