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A Note on Mathematical Structures

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ABSTRACT: In this paper we shall discuss the interrelations between generalizations of topology and mathematical structures. We also discuss the algebraic nature of generalizations of topology and mathematical structures.

Key Words: Topology, Generalized topology, Hereditary class, Filter, Semi-group.

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1. Introduction and Preliminaries

Generalized Topology:

The concept of Generalized Topology which is a generalization of topology was introduced by Á. Császár in 2002 although a generalization of topology is not a new concept in literature. A series of papers have been published using the same idea by Á. Császár and others. Formally a subcollection $\lambda \subset 2^X$ is called a generalized topology [3] (briefly GT) on X if $\emptyset \in \lambda$ and $\{G_i\} \subseteq \lambda$, for $i \in I \neq \emptyset$ implies $\bigcup_i G_i \in \lambda$. We will denote the collection of all GTs on a set X by $\bigwedge_X$. It is noted that topology is a particular case of GT. In 1982, Lugojan [12] introduced a generalization of topology as follows: a subcollection G_X of 2^X is called a generalized topology if $\emptyset,\ X \in G_X$ and G_X is closed under arbitrary union. The collection of all G_X on X is denoted by $\mathcal{G}_X$. One of the generalizations of GT has been introduced by Kim and Min in 2013 and this generalization is called a σ - structure. A subcollection $s \subset 2^X$ is called a σ - structure [10] on X if, for $i \in I \neq \emptyset$, $U_i \in s$ implies $\cup_{i\in I} \in s$. The collection of all σ - structures on X is denoted as Σ_X.

Supratopology and m - structure:

In 1983, the notion called the supratopology is introduced as a generalization of the topology. A subcollection $\tau^* \subset 2^X$ is called a supra topology [13] on X if $X \in \tau^*$, $\{V_i\}_{i\in I} \subseteq \tau^*$ implies $\bigcup_i V_i \in \tau^*$. The collection of all supra topologies on a set X will be denoted as $\mathcal{T}^*_X$. Al-Omari and Noiri [1] have introduced a mathematical structure which is called an m - structure in 2012. This structure has been defined by relinquishing the arbitrary union property from topology. In this view a subfamily $\Upsilon \subseteq 2^X$ is called an m - structure on X if $\emptyset,\ X \in \Upsilon$ and

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$V_1,\ V_2 \in \Upsilon$ implies $V_1 \cap V_2 \in \Upsilon$ and the collection of all m - structures on X is denoted as Γ_X.

Minimal Structure:

Popa and Noiri [15] made another generalization by the name of Minimal Structure and it is a generalization of m - structures. Ozbakir and Yildirim [14] have studied this field extensively. A subfamily $m_X \subseteq 2^X$ is called a minimal structure [15] on X if $\emptyset,\ X \in m_X$. We are here intimating that if $\emptyset \in m_X$ then it is called a weak structure [5] and it is denoted as WS. The collection of all minimal structures and weak structures on X is denoted as $\mathcal{M}_X$ and $\mathcal{W}_X$, respectively.

Hereditary class and Antihereditary class:

A subfamily $\mathcal{H} \subset 2^X$ is called a hereditary class [4] if $A \subset B,\ B \in \mathcal{H}$ implies $A \in \mathcal{H}$. This structure has been introduced by Á. Császár in 2007 for the purpose of parallel study of ideal topological spaces [11,17]. If we add a condition in hereditary class then we obtain the notion of ideals which are well known in literature. A subfamily $I \subset 2^X$ is called an ideal if I is a hereditary class and closed under finite additivity. A topological space with an ideal is called an ideal topological space. The collection of all hereditary classes on the set X is denoted as $\mathbb{H}_X$ and the collection of all ideals on the set X is denoted as $\mathbb{I}_X$.

Filter is also an another concept in the study of topological spaces and its formal definition is: A nonempty subfamily $\mathcal{F} \subset 2^X$ is called a filter on X if $\emptyset \notin \mathcal{F}$, $A \subset B,\ A \in \mathcal{F}$ implies $B \in \mathcal{F}$ and $A,\ B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$. The collection of all filters on X is denoted as $\mathbb{F}_X$.

A mathematical tool grill [2,17,16] has been introduced for the purpose of the study of proximity spaces. A nonempty subcollection $\mathcal{G} \subset 2^X$ on a set X is called a grill on X if $\emptyset \notin \mathcal{G}$, $A \in \mathcal{G}$ and $A \subset B$ implies $B \in \mathcal{G}$ and $A, B \subseteq X$ and $A \cup B \in \mathcal{G}$ implies $A \in \mathcal{G}$ or $B \in \mathcal{G}$. The collection of all grills on a set X is denoted as $\mathbb{G}_X$.

A nonempty subfamily $\mathcal{F} \subset 2^X$ is called a stack [6,16] on X if $\emptyset \notin \mathcal{S}$ and $A \subseteq B,\ A \in \mathcal{S}$ implies $B \in \mathcal{S}$. The collection of all stacks on X is denoted as $\mathbb{S}_X$.

The collections of all topologies on a set X is denoted as $\mathcal{T}_X$.

Through this paper, we shall show that generalizations of topology and mathematical structures are not mutually exclusive between them. We also try to show that generalizations of topology and mathematical structures have algebraic nature.

2. Operations on mathematical structures

We have the following diagrams from the above discussion:

$$
\begin{tikzcd}
\text{Topology} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \begin{array}{c}\text{Generalized Topolgy}\\ \text{(Lugojan)}\end{array} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \begin{array}{c}\text{Generalized Topology}\\ \text{(Csaszar)}\end{array} \arrow[d, Rightarrow] \\
\text{m-Structure} \arrow[r, Rightarrow] & \text{Minimal structure} \arrow[r, Rightarrow] & \text{Weak Structure}
\end{tikzcd}
$$

DIAGRAM - I

$$\begin{tikzcd}
\text{Generalized Topology (Lugojan)} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{Generalized Topology (Csaszar)} \arrow[d, Rightarrow] \\
\text{Supratopology} \arrow[r, Rightarrow] & \sigma\text{- structure}
\end{tikzcd}$$

DIAGRAM - II

Converse relations need not hold in general.
$IDEAL \Longrightarrow HEREDITARYCLASS$, $GRILL \Longrightarrow STACK$.

DIAGRAM - III

Theorem 2.1. *Let $\mathcal{T}_X$ (resp. $\mathcal{G}_X$, $\mathcal{T}^*_X$, $\bigwedge_X$, Σ_X, $\mathcal{M}_X$, $\mathcal{W}_X$, Γ_X) be the collection of all topologies (resp. generalized topologies (Lugojan), supratopologies, generalized topologies (Császár), σ-structures, minimal structures, weak structures, m-structures) on X, then $\bigcap \mathcal{T}_X$ (resp. $\bigcap \mathcal{G}_X$, $\bigcap \mathcal{T}^*_X$, $\bigcap \bigwedge_X$, $\bigcap \Sigma_X$, $\bigcap \mathcal{M}_X$, $\bigcap \mathcal{W}_X$, $\bigcap \Gamma_X$) is a topology (resp. generalized topology (Lugojan), supratopology, generalized topology (Császár), σ-structure, minimal structure, weak structure, m - structure) on X.*

This topology (resp. generalized topology (Lugojan), supratopology, generalized topology (Császár), σ-structure, minimal structure, weak structure, m - structure) is called the smallest topology (resp. generalized topology (Lugojan), supratopology, generalized topology (Császár), σ-structure, minimal structure, weak structure, m - structure) on X contained in all topologies (resp. generalized topologies (Lugojan), supratopologies, generalized topologies (Császár), σ - structures, minimal structures, weak structures, m - structures) on X.

Since all of the mathematical structures $\mathcal{T}_X$, $\mathcal{G}_X$, $\mathcal{T}^*_X$, $\bigwedge_X$, Σ_X, $\mathcal{M}_X$, $\mathcal{W}_X$ and Γ_X are closed under intersection, we have the following theorem:

Theorem 2.2. *For the mathematical structures $\mathcal{T}_X$, $\mathcal{G}_X$, $\mathcal{T}^*_X$, $\bigwedge_X$, Σ_X, $\mathcal{M}_X$, $\mathcal{W}_X$ and Γ_X, the following hold:*

1. *$(\mathcal{T}_X, \bigcap)$ (resp. $(\mathcal{G}_X, \bigcap)$, $(\mathcal{T}^*_X, \bigcap)$, $(\bigwedge_X, \bigcap)$, $(\Sigma_X, \bigcap)$, $(\mathcal{M}_X, \bigcap)$, $(\mathcal{W}_X, \bigcap)$ and $(\Gamma_X, \bigcap)$) is a semigroup [7].*
2. *$(\mathcal{T}_X, \bigcap)$ (resp. $(\mathcal{G}_X, \bigcap)$, $(\mathcal{T}^*_X, \bigcap)$, $(\bigwedge_X, \bigcap)$, $(\Sigma_X, \bigcap)$, $(\mathcal{M}_X, \bigcap)$, $(\mathcal{W}_X, \bigcap)$ and $(\Gamma_X, \bigcap)$) is a commutative semigroup [7].*
3. *Each elements of $(\mathcal{T}_X, \bigcap)$ (resp. $(\mathcal{G}_X, \bigcap)$, $(\mathcal{T}^*_X, \bigcap)$, $(\bigwedge_X, \bigcap)$, $(\Sigma_X, \bigcap)$, $(\mathcal{M}_X, \bigcap)$, $(\mathcal{W}_X, \bigcap)$ and $(\Gamma_X, \bigcap)$) is idempotent [7].*
4. *2^X is the identity [7] of $(\mathcal{T}_X, \bigcap)$ (resp. $(\mathcal{G}_X, \bigcap)$, $(\mathcal{T}^*_X, \bigcap)$, $(\bigwedge_X, \bigcap)$, $(\Sigma_X, \bigcap)$, $(\mathcal{M}_X, \bigcap)$, $(\mathcal{W}_X, \bigcap)$ and $(\Gamma_X, \bigcap)$).*

It is difficult to determine the inverse element of the above mathematical structures under the operation of intersection.

Theorem 2.3. *Let $\mathcal{M}_X$ (resp. $\mathcal{W}_X$) be the collection of all minimal structures (resp. weak structures) on X, then $\bigcup \mathcal{M}_X$ (resp. $\bigcup \mathcal{W}_X$) is a minimal structure (resp. weak structure) on X.*

Remark 2.4. *For the mathematical structures $\mathcal{M}_X$ and $\mathcal{W}_X$, the following hold:*

1. $(\mathcal{M}_X, \bigcup)$ *(resp.* $(\mathcal{W}_X, \bigcup)$*) is a semigroup.*
2. $(\mathcal{M}_X, \bigcup)$ *(resp.* $(\mathcal{W}_X, \bigcup)$*) is a commutative semigroup.*
3. *Each elements of $\mathcal{M}_X$ (resp. $\mathcal{W}_X$) is idempotent.*
4. $\{\emptyset, X\}$ *and* $\{\emptyset\}$ *is the identity of $\mathcal{M}_X$ and $\mathcal{W}_X$, respectively.*

It is difficult to determine the inverse element of the above mathematical structures under the operation of union.

The union of two m - structures need not be an m - structure in general.

Example 2.5. *Let $X = \{a, b, c\}$, $\Upsilon_1 = \{\emptyset, X, \{a, b\}\}$ and $\Upsilon_2 = \{\emptyset, X, \{a, c\}\}$. Then Υ_1 and Υ_2 are m - structures but $\Upsilon_1 \cup \Upsilon_2 = \{\emptyset, X, \{a, c\}, \{a, b\}\}$ is not an m-structure on X because $\{a\} \notin \Upsilon_1 \cup \Upsilon_2$.*

Arbitrary union of the classes of topologies (resp. generalized topologies (Lugojan), supratopologies, σ-structures, generalized topologies (Császár)) on X may not be a topology (resp. generalized topology (Lugojan), supratopology, σ - structure, generalized topology (Császár)) again on X.

It is sufficient that if we give an example on the class of topologies then other classes follow from this example.

Example 2.6. *Let $\mathbb{R}$ be the set of reals and $\mathbb{N}$ be the set of naturals. Let $n \in \mathbb{N}$. Consider*

$$\mathcal{T}_n = \{\emptyset, \mathbb{R}, \{1\}, \{1, 2\}, \{1, 2, 3\}, ..., \{1, 2, 3, ..., n\}\}$$

then $\{\mathcal{T}_n\}$ is a class of topologies on $\mathbb{R}$; if $U_n = \{1, 2, 3, ..., n\}$. then $U_n \in \mathcal{T}_n$ and hence $U_n \in \mathcal{T} = \bigcup \mathcal{T}_k$; but $\bigcup_{n=1}^{\infty} U_n \notin \mathcal{T}$; hence $\mathcal{T}$ is not a topology. Hence we have arbitrary union of a class of topologies on a set need not be a topology.

Recall that a partial ordered set which is also a linear order then it is called a chain.

Hence we have arbitrary union of a class of topologies on a set need not be a topology. Even if the collection of topologies is a chain (increasing or decreasing), the union need not be a topology. For the above $\{\tau_n\}$ is an increasing chain of topology but the union is not a topology. Then we have the following remark:

Remark 2.7. *If $\{\tau_i\}$ is a chain of topologies (resp. supratopologies, generalized topologies) on a set X, then $\bigcup \tau_i$ is not necessarily a topology (resp. supratopology, generalized topology) on the set X.*

Hence from the above remark, in Theorem 2.2, if we replace the operation intersection by union then the structures $\mathcal{T}_X$, $\mathcal{G}_X$, $\mathcal{T}^*_X$, $\bigwedge_X$, Σ_X and Γ_X are not the semi group.

Theorem 2.8. *Let $\mathbb{I}_X$ (resp. $\mathbb{G}_X$, $\mathbb{H}_X$, $\mathbb{F}_X$, $\mathbb{S}_X$) be the set of all ideals (resp. grills, hereditary classes, filters, stacks) on X. Then $\bigcap \mathbb{I}_X$ (resp. $\bigcap \mathbb{G}_X$, $\bigcap \mathbb{H}_X$, $\bigcap \mathbb{F}_X$, $\bigcap \mathbb{S}_X$) is a ideal (resp. grill, hereditary class, filter, stack) on X if $\bigcap \mathbb{I}_X \neq \{\emptyset\}$ (resp. $\bigcap \mathbb{G}_X \neq \{\emptyset\}$, $\bigcap \mathbb{H}_X \neq \{\emptyset\}$, $\bigcap \mathbb{F}_X \neq \{\emptyset\}$, $\bigcap \mathbb{S}_X \neq \{\emptyset\}$).*

This ideal (resp. grill, hereditary class, filter, stack) is called the smallest ideal (resp. grill, hereditary class, filter, stack) on X contained in all ideals (resp. grills, hereditary classes, filters, stacks) on X.

Union of two filters (resp. grills, hereditary class, ideals, stacks) need not be a filter (resp. grill, hereditary class, ideals, stack) again.

Example 2.9. *(i) Let $X = \{a, b, c\}$. Then $\mathcal{F}_1 = \{\{a, b\}, X\}$ and $\mathcal{F}_2 = \{\{a, c\}, X\}$ be two filters on X. But their union $\mathcal{F}_1 \cup \mathcal{F}_2 = \{\{a, b\}, \{a, c\}, X\}$ is not a filter on X.*

(ii). Let $X = \{a, b, c\}$. Then $\mathcal{I}_1 = \{\emptyset, \{a\}\}$ and $\mathcal{I}_2 = \{\emptyset, \{c\}\}$ be two ideal on X. But their union $\mathcal{I}_1 \cup \mathcal{I}_2 = \{\emptyset, \{a\}, \{c\}\}$ is not an ideal on X.

Theorem 2.10. *Let $\mathcal{S}_1$, $\mathcal{S}_2 \in \mathbb{S}_X$. Then $\mathcal{S}_1 \cup \mathcal{S}_2 \in \mathbb{S}_X$.*

Proof: Suppose $\mathcal{S}_1 \cup \mathcal{S}_2 \notin \mathbb{S}_X$. Then for A, $\exists\ B \supseteq A$ such that $A \in \mathcal{S}_1 \cup \mathcal{S}_2$ but $B \notin \mathcal{S}_1 \cup \mathcal{S}_2$. Then $A \in \mathcal{S}_1$ or $A \in \mathcal{S}_2$. If $A \in \mathcal{S}_1$, then $B \in \mathcal{S}_1$ by definition of stack, a contradiction. Again if $A \in \mathcal{S}_2$, then $B \in \mathcal{S}_2$ by definition of stack, a contradiction. Hence $\mathcal{S}_1 \cup \mathcal{S}_2 \in \mathbb{S}_X$. $\square$

Theorem 2.11. *Let $\mathcal{G}_1$, $\mathcal{G}_2 \in \mathbb{G}_X$. Then $\mathcal{G}_1 \cup \mathcal{G}_2 \in \mathbb{G}_X$.*

Proof: (1) If $A \in \mathcal{G}_1 \cup \mathcal{G}_2$ and $A \subset B$, then by Theorem 2.10 we obtain $B \in \mathcal{G}_1 \cup \mathcal{G}_2$.

(2) Let $A \cup B \in \mathcal{G}_1 \cup \mathcal{G}_2$. In case $A \cup B \in \mathcal{G}_1$, $A \in \mathcal{G}_1$ or $B \in \mathcal{G}_1$ and $A \in \mathcal{G}_1 \cup \mathcal{G}_2$ or $B \in \mathcal{G}_1 \cup \mathcal{G}_2$. Similarly, in case $A \cup B \in \mathcal{G}_2$, we have $A \in \mathcal{G}_1 \cup \mathcal{G}_2$ or $B \in \mathcal{G}_1 \cup \mathcal{G}_2$. Consequently, we obtain $A \in \mathcal{G}_1 \cup \mathcal{G}_2$ or $B \in \mathcal{G}_1 \cup \mathcal{G}_2$. $\square$

We know from Example 2.9, the union of two filters is not a filter. But if we define the union by the following manner then it is known that the union of two filters is a filter.

Theorem 2.12. *[8] Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two filters on the set X. Then the family $\mathcal{F}_1 \cup_1 \mathcal{F}_2 = \{F \cup G : \ F \in \mathcal{F}_1$ and $G \in \mathcal{F}_2\}$ forms a filter.*

Remark 2.13. *Let I_1 and I_2 be two ideals on the set X. Then $I_1 \cup_1 I_2$ is an ideal on X.*

Proof: (1) Let $F_1 \cup G_1,\ F_2 \cup G_2 \in I_1 \cup_1 I_2$. Then $(F_1 \cup G_1) \cup (F_2 \cup G_2) = (F_1 \cup F_2) \cup (G_1 \cup G_2)$. Since $F_1,\ F_2 \in I_1$ and $G_1,\ G_2 \in I_2$, then $(F_1 \cup G_1) \cup (F_2 \cup G_2) \in I_1 \cup_1 I_2$.

(2) Let $H \subset F \cup G \in I_1 \cup_1 I_2$.

Case (i): If $F \cap G = \emptyset$. Then either $H \subset F$ or $H \subset G$. Both the cases $H \in I_1 \cup_1 I_2$.

Case (ii): If $F \cap G \neq \emptyset$. Then $\exists\ H_1, H_2$ subsets of H such that $H = H_1 \cup H_2,\ H_1 \subset F$ and $H_2 \subset G$. Since $F \in I_1$ and $G \in I_2$, then $H \in I_1 \cup_1 I_2$. □

Now our question is that Theorem 2.12 and Remark 2.13 can be extended up to arbitrary unions? The answer of this question is as follows:

Remark 2.14. *Let $\{\mathcal{F}_\alpha : \alpha \in \bigvee\}$ be a family of filters (resp. ideals) such that $\bigvee$ is a linear ordered set and $\mathcal{F}_\alpha \subset \mathcal{F}_\beta$ (resp. $\mathcal{F}_\alpha \supset \mathcal{F}_\beta$) for $\alpha \leq \beta$. Then $\bigcup_{\alpha \in \bigvee} \mathcal{F}_\alpha = \{F : F \in \mathcal{F}_\alpha$ for some $\alpha \in \bigvee\}$ is also a filter (resp. ideal).*

Proof: The author Husain [8] has proved this Remark for filters. The proof of this remark for ideals has been done by the following fact: for a filter $\mathcal{F}$, $I = \{A : X - A \in \mathcal{F}\}$ is an ideal. □

Conclusion: Why we made the above relations? By the above relation we can determine who is finer and who is weaker structure. This relation help us by following: suppose a mathematical structure is separated by two elements of this structure, for two distinct elements. Then its weaker structure is also separated by the similar thing but its finer is not.

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