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Existence and Multiplicity Results for a Class of Kirchhoff Type Problems Involving the $p(x)$-Biharmonic Operator

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Abstract: The aim of this paper is to establish the existence and multiplicity of solutions for a class of nonlocal problem involving the $p(x)$-biharmonic operator of the form

$$\begin{cases} M\left(\int_\Omega \frac{|\Delta u|^{p(x)}}{p(x)}dx\right)\Delta(|\Delta u|^{p(x)-2}\Delta u) = f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases}$$

Our technical approach is based on direct variational method and the theory of variable exponent Sobolev spaces.

Key Words: Variational method, Mountain pass theorem, $p(x)$-Biharmonic Operator, Kirchhoff type equation, Nonlocal problem, Navier boundary conditions, generalized Lebesgue-Sobolev spaces.

Contents

1. Introduction

In this paper we are concerned with the following problem

$$\begin{cases} M\left(\int_\Omega \frac{|\Delta u|^{p(x)}}{p(x)}dx\right)\Delta(|\Delta u|^{p(x)-2}\Delta u) = f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \in \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with a smooth boundary $\partial\Omega$, p is a continuous function on $\overline{\Omega}$ such that $1 < p^- := \inf_{x\in\overline{\Omega}} p(x) \leq p^+ := \sup_{x\in\overline{\Omega}} p(x) < +\infty$, $\Delta(|\Delta u|^{p(x)-2}\Delta u)$ is the $p(x)$-Biharmonic operator, $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathédory function.
Many authors consider the existence of solutions for some fourth order problems with variable exponent and Navier boundary condition, see for instance [13,14,22]. This is a generalization of the p-biharmonic operator $\Delta(|\Delta u|^{p-2}\Delta u)$ obtained when p is a constant.
Problem (1.1) is called a nonlocal one because of the presence of the term M, which

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implies that the equation in (1.1) is no longer pointwise identities. This make the study of such problem particularly interesting. Nonlocal differential equations are also called Kirchhoff type equations, introduced by Kirchhoff [20]. More precisely, Kirchhoff introduced a model given by the equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \Big(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$

where E is the Young modulus of the material, ρ is the mass density, L is the length of the string, h is the area of the cross-section, and ρ_0 is the initial tension. Equation (1.2) extends the classical D'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations. Lions [21] has proposed an abstract framework for the Kirchhoff type equations. After the work of Lions [21], various equations of Kirchhoff type equations have been studied extensively, see [2,5,6,7]. The study of Kirchhoff type equations has already been extended to the case involving the p-Laplacian (for details, see [6,7,8,9]) and $p(x)$-Laplacian (see [10,11,18]). The generalization of Kirchhoff equations to the case involving the $p(x)$-Biharmonic operator is a quite new topic, so there exists only a few papers (see [3,4]). In [3] the authors study the problem (1.1) when the Carathéodory function is of the particular form $f(x,u) = \lambda(x)|u|^{p(x)-2}u$ in which the weight function $\lambda(x) \in L^\infty(\Omega)$ does not change sign and when $f(x,u) = \lambda|u|^{p(x)-2}u$ where λ is a positive parameter. Moreover the Kirchhoff function M is non-degenerate, i.e.,

$$M(t) \geq m_0 > 0 \text{ for all } t \geq 0.$$

Motivated by the above references and some ideas in [10], we establish the existence and multiplicity of solutions for problem (1.1) using variational method and the theory of the variable exponent Sobolev spaces. Here the Kirchhoff function M may be degenerate at zero. More precisely, we assume that

(M_1) There exist $m_2 \geq m_1 > 0$ and $\alpha > 1$ such that for all $t \in \mathbb{R}^+$,

$$m_1 t^{\alpha-1} \leq M(t) \leq m_2 t^{\alpha-1}.$$

Throughout this paper, the nonlinear term f satisfy the following conditions:

(H_0) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathédory condition and there exist a constant $C_1 \geq 0$ such that

$$|f(x,t)| \leq C_1(1 + |t|^{\gamma(x)-1})$$

for all $(x,t) \in \Omega \times \mathbb{R}$ with $\gamma(x) \in C_+(\overline{\Omega})$ and $\gamma(x) < p_2^*(x)$ for all $x \in \overline{\Omega}$ where $p_2^*(x) := \frac{Np(x)}{N-2p(x)}$ if $p(x) < \frac{N}{2}$, $p_2^*(x) = \infty$ if $p(x) \geq \frac{N}{2}$.

(H_1) There exist $K > 0$, $\theta > \frac{m_2\alpha(p^+)^\alpha}{m_1(p^-)^{\alpha-1}}$ such that for all $x \in \Omega$ and all $t \in \mathbb{R}$ with $|t| \geq K$,

$$0 < \theta F(x,t) \leq t f(x,t),$$

where $F(x,t) = \int_0^t f(x,s)ds$ and m_1, m_2, α come from (M_1).

(H_2) $f(x,t) = o(|t|^{\alpha p^+ -1})$ as $t \to 0$ uniformly with respect to $x \in \Omega$, with $\gamma^- >$

$\alpha(p^+)^\alpha$, where α comes from (M_1).
(H_3) $f(x,-t) = -f(x,t)$ for all $x \in \Omega$ and $t \in \mathbb{R}$.
This article is organized as follows. In section 2, we recall some basic results on the theory of Lebesgue-Sobolev spaces with variable exponent. In section 3, we state and prove our main results.

2. Preliminaries

In this section we recall some definitions and basic properties of the variable exponent Lebesgue and Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^N$. For more details, see [12,15,16,17,19].
Denote

$$C_+(\overline{\Omega}) = \left\{ h \in C(\overline{\Omega}) \quad \text{and} \quad \inf_{x\in\overline{\Omega}} h(x) > 1 \right\}.$$

For any $h \in C_+(\overline{\Omega})$, we define

$$h^+ := \max\{h(x), x \in \overline{\Omega}\},\ h^- := \min\{h(x), x \in \overline{\Omega}\}.$$

For any $p \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ is a measurable real-valued function} : \int_\Omega |u|^{p(x)} dx < +\infty \right\},$$

endowed with the Luxemburg norm

$$|u|_{p(x)} = \|u\|_{L^{p(x)}(\Omega)} = \inf \left\{ \tau > 0; \int_\Omega \left| \frac{u}{\tau} \right|^{p(x)} dx \le 1 \right\}.$$

Then $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ is a Banach space.
Denote by $p^- := \inf_{x\in\overline{\Omega}} p(x)$ and $p^+ := \sup_{x\in\overline{\Omega}} p(x)$.

Proposition 2.1. *(i) The space $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ is a separable, reflexive, uniformly convex Banach space and its conjugate space is $L^{q(x)}$ where $q(x)$ is the conjugate function of $p(x)$, that is*

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1 \text{ for all } x \in \Omega.$$

(ii) If $p_1(x), p_2(x) \in C_+(\overline{\Omega})$, $p_1(x) \le p_2(x)$ for all $x \in \overline{\Omega}$, then $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ and the embedding is continuous.

For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, we have

$$\left| \int_\Omega uv dx \right| \le \left(\frac{1}{p^-} + \frac{1}{q^-} \right) |u|_{p(x)} |v|_{q(x)} \le 2|u|_{p(x)} |v|_{q(x)}.$$

The Sobolev space with variable exponent $W^{k,p(x)}(\Omega)$ is defined by

$$W^{k,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \le k \right\},$$

where

$$D^\alpha u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_N^{\alpha_N}} u, \text{ with } \alpha = (\alpha_1, \alpha_2, ..., \alpha_N)$$

is a multi-index and $|\alpha| = \sum_{i=1}^{N} \alpha_i$. The space $W^{k,p(x)}(\Omega)$ equipped with the norm

$$\|u\|_{k,p(x)} = \sum_{|\alpha| \leq k} |D^\alpha u|_{p(x)},$$

becomes a separable and reflexive Banach space.
Denote for $x \in \overline{\Omega}$ and $k \geq 1$

$$p_k^*(x) = \begin{cases} \frac{Np(x)}{N-kp(x)}, & \text{if } kp(x) < N, \\ \infty, & \text{if } kp(x) \geq N. \end{cases}$$

Proposition 2.2. *Let $p, r \in C_+(\overline{\Omega})$ such that $r(x) \leq p_k^*(x)$ for all $x \in \overline{\Omega}$. Then there is a continuous embedding $W^{k,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$. If we replace $\leq$ with $<$, the embedding is compact.*

We denote by $W_0^{k,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$. Note that the weak solutions of problem (1.1) are considered in the generalized Sobolev space $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ endowed with the norm

$$\|u\| = \inf \left\{ \tau > 0; \int_\Omega \left| \frac{\Delta u(x)}{\tau} \right|^{p(x)} dx \leq 1 \right\}.$$

Moreover, according to [24] , the norm $\|u\|_{2,p(x)}$ is equivalent to the norm $|\Delta u|_{p(x)}$ in the space $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$. Consequently, the norms $\|u\|_{2,p(x)}$, $|\Delta u|_{p(x)}$ and $\|u\|$ are equivalent.
Consider the functional $I(u) = \int_\Omega |\Delta u|^{p(x)} dx$. Then we have the following

Proposition 2.3. *(See [14]) If $u \in W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$, then*

- *(i)* $\|u\| < 1 (= 1, > 1) \Leftrightarrow I(u) < 1 (= 1, > 1)$;
- *(ii)* $\|u\| > 1 \Rightarrow \|u\|^{p^-} \leq I(u) \leq \|u\|^{p^+}$;
- *(iii)* $\|u\| < 1 \Rightarrow \|u\|^{p^+} \leq I(u) \leq \|u\|^{p^-}$;
- *(iv)* $\|u\| \to 0 (\to \infty) \Leftrightarrow I(u) \to 0 (\to \infty)$.

Let for any $u \in X = W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$, $G(u) = \int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)} dx$, and $L := G' : X \to X^*$, then

$$< L(u), v >= \int_\Omega |\Delta u|^{p(x)-2} \Delta u \Delta v dx, \text{ for all } u, v \in X.$$

Lemma 2.4. *(See [14])*

1. *(i) $L : X \to X^*$ is a continuous, bounded homeomorphism and strictly monotone operator.*
2. *(ii) L is a mapping of type (S_+), i.e. if $u_n \rightharpoonup u$ in X and $\limsup_{n\to\infty} < L(u_n) - L(u), u_n - u > \leq 0$, then $u_n \to u$ in X.*

In this paper, we denote by $X = W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$, X^* its dual space and $< .,. >$ denote the duality product. For simplicity, we use C to denote the general positive constants whose exact values may change from line to line.

3. Proof of the main result

Definition 3.1. *We say that $u \in X$ is a weak solution of problem (1.1) if*

$$M\left(\int_\Omega \frac{|\Delta u|^{p(x)}}{p(x)}dx\right)\int_\Omega |\Delta u|^{p(x)-2}\Delta u \Delta v dx = \int_\Omega f(x,u)v dx, \quad \forall v \in X.$$

The Euler-Lagrange functional associated to problem (1.1) is given by

$$J(u) = \widehat{M}\left(\int_\Omega \frac{|\Delta u|^{p(x)}}{p(x)}dx\right) - \int_\Omega F(x,u)dx, \tag{3.1}$$

where $\widehat{M}(t) = \int_0^t M(s)ds$. Then it is easy to verify that $J \in C^1(X, \mathbb{R})$ and weakly lower semi-continuous with the derivative given by

$$< J'(u), v >= M\left(\int_\Omega \frac{|\Delta u|^{p(x)}}{p(x)}dx\right)\int_\Omega |\Delta u|^{p(x)-2}\Delta u \Delta v dx - \int_\Omega f(x,u)v dx$$

for all $u, v \in X$. Thus, we can infer that critical points of functional J are exactly the weak solutions of problem (1.1).

Theorem 3.2. *If (M_1) holds and f satisfies*

$$|f(x,t)| \leq C(1 + |t|^{\eta-1}), \tag{3.2}$$

where $1 \leq \eta < \alpha p^-$, then problem (1.1) has a weak solution.

Proof: From (3.2) we have $|F(x,t)| \leq C(|t| + |t|^\eta)$. Then

$$\begin{aligned} J(u) &= \widehat{M}\left(\int_\Omega \frac{|\Delta u|^{p(x)}}{p(x)}dx\right) - \int_\Omega F(x,u)dx \\ &\geq \frac{m_1}{\alpha}\left(\int_\Omega \frac{|\Delta u|^{p(x)}}{p(x)}dx\right)^\alpha - C\int_\Omega |u|dx - C\int_\Omega |u|^\eta dx \\ &\geq \frac{m_1}{\alpha(p^+)^\alpha}\|u\|^{\alpha p^-} - C\|u\| - C\|u\|^\eta \to +\infty \quad \text{as } \|u\| \to +\infty. \end{aligned}$$

By the condition (M_1) and Proposition 2.2 , it is easy to verify that J is weakly lower semi continuous. So J has a minimum point $u \in X$ and then u is a weak solution of problem (1.1). □

Definition 3.3. *We say that J satisfies the Palais-Smale condition (PS) in X if any sequence $(u_n) \subset X$ such that $J(u_n)$ is bounded and $J'(u_n) \to 0$ as $n \to +\infty$, has a convergent subsequence.*

Lemma 3.4. *Assume that (M_1), (H_0) and (H_1) hold. Then J satisfies the (PS) condition.*

Proof: Suppose that $(u_n) \subset X$ such that $|J(u_n)| \leq C$ and $J'(u_n) \to 0$. Arguing by contradiction. We assume that, passing eventually to a subsequence, still denote by (u_n), $\|u_n\| \to \infty$ and $\|u_n\| > 1$ for all n. Then for n large enough, we have

$$\begin{aligned}
C + \|u_n\| &\geq J(u_n) - \frac{1}{\theta} < J'(u_n), u_n > \\
&= \widehat{M}\left(\int_\Omega \frac{|\Delta u_n|^{p(x)}}{p(x)} dx\right) - \int_\Omega F(x, u_n) dx \\
&- \frac{1}{\theta}\left[M\left(\int_\Omega \frac{|\Delta u_n|^{p(x)}}{p(x)} dx\right) \int_\Omega |\Delta u_n|^{p(x)} dx - \int_\Omega f(x, u_n) u_n dx\right] \\
&\geq \frac{m_1}{\alpha}\left(\int_\Omega \frac{|\Delta u_n|^{p(x)}}{p(x)} dx\right)^\alpha \\
&- \frac{m_2}{\theta}\left(\int_\Omega \frac{|\Delta u_n|^{p(x)}}{p(x)} dx\right)^{\alpha-1} \int_\Omega |\Delta u_n|^{p(x)} dx \\
&+ \int_\Omega \left(\frac{1}{\theta} f(x, u_n) u_n - F(x, u_n)\right) dx \\
&\geq \left(\frac{m_1}{\alpha(p^+)^\alpha} - \frac{m_2}{\theta(p^-)^{\alpha-1}}\right)\left(\int_\Omega |\Delta u_n|^{p(x)} dx\right)^\alpha \\
&+ \int_\Omega \left(\frac{1}{\theta} f(x, u_n) u_n - F(x, u_n)\right) dx \\
&\geq \left(\frac{m_1}{\alpha(p^+)^\alpha} - \frac{m_2}{\theta(p^-)^{\alpha-1}}\right) \|u_n\|^{\alpha p^-} - C,
\end{aligned}$$

From (H_1), we know that $\theta > \frac{m_2 \alpha (p^+)^\alpha}{m_1 (p^-)^{\alpha-1}}$, that is

$$\frac{m_1}{\alpha(p^+)^\alpha} - \frac{m_2}{\theta(p^-)^{\alpha-1}} > 0.$$

Dividing the above inequality by $\|u_n\|^{\alpha p^-}$, and passing to the limit as $n \to \infty$, we obtain a contradiction. Therefore (u_n) is bounded in X. Without loss of generality, we assume that $u_n \rightharpoonup u$, then $J'(u_n)(u_n - u) \to 0$ as $n \to \infty$. Thus we obtain

$$\begin{aligned}
< J'(u_n), u_n - u > &= M\left(\int_\Omega \frac{|\Delta u_n|^{p(x)}}{p(x)} dx\right) \int_\Omega |\Delta u_n|^{p(x)-2} \Delta u_n (\Delta u_n - \Delta u) dx \\
&- \int_\Omega f(x, u_n)(u_n - u) dx \to 0.
\end{aligned}$$

From (H_0), Proposition 2.1 and Proposition 2.2, we deduce easily that

$$\int_\Omega f(x,u_n)(u_n-u)dx \to 0 \quad \text{as } n\to\infty.$$

Hence, we have as $n\to\infty$,

$$M\left(\int_\Omega \frac{|\Delta u_n|^{p(x)}}{p(x)}dx\right)\int_\Omega |\Delta u_n|^{p(x)-2}\Delta u_n(\Delta u_n-\Delta u)dx \to 0 \tag{3.3}$$

Since (u_n) is bounded in X, passing to a subsequence, if necessary, we can assume that

$$\int_\Omega \frac{|\Delta u_n|^{p(x)}}{p(x)}dx \to t_0\geq 0 \quad \text{as} \quad n\to\infty.$$

If $t_0=0$ then (u_n) converge strongly to $u=0$ in X and the proof is finished. Otherwise, since the function M is continuous, we have

$$M\left(\int_\Omega \frac{|\Delta u_n|^{p(x)}}{p(x)}dx\right) \to M(t_0)\geq 0 \quad \text{as} \quad n\to\infty.$$

Therefore, in view of (M_1), for n sufficiently large, we get

$$0<C_1\leq M\left(\int_\Omega \frac{|\Delta u_n|^{p(x)}}{p(x)}dx\right)\leq C_2, \tag{3.4}$$

where C_1, C_2 are positive constants. In view of (3.3) and (3.4), we conclude that

$$\lim_{n\to\infty}\int_\Omega |\Delta u_n|^{p(x)-2}\Delta u_n(\Delta u_n-\Delta u)dx = 0. \tag{3.5}$$

Thus, from Lemma 2.4 (ii), it follows that $u_n\to u$ strongly in X as $n\to\infty$ and then functional J satisfies the (PS) condition. □

Theorem 3.5. *If (M_1), (H_0), (H_1) and (H_2) hold, then problem (1.1) has nontrivial weak solution.*

Proof: Let us show that J satisfies the conditions of mountain pass theorem (see [1]). From Lemma 3.4, we know that J satisfies the (PS) condition in X. Since $\alpha p^+ < \alpha(p^+)^\alpha < \gamma^- \leq \gamma(x) < p_2^*$, $X \hookrightarrow L^{\alpha p^+}(\Omega)$ and then there exists $C>0$ such that

$$|u|_{\alpha p^+}\leq C\|u\|, \quad \text{for all } u\in X.$$

Let $\varepsilon>0$ be small enough such that $\varepsilon C^{\alpha p^+} < \frac{m_1}{2\alpha(p^+)^\alpha}$. By the assumptions (H_0) and (H_2), we have

$$F(x,t)\leq \varepsilon|t|^{\alpha p^+} + C(\varepsilon)|t|^{\gamma(x)} \quad \text{for all } (x,t)\in\Omega\times\mathbb{R}. \tag{3.6}$$

In view of (M_1) and (3.6), we have

$$\begin{aligned} J(u) &\geq \frac{m_1}{\alpha}\left(\int_\Omega \frac{|\Delta u|^{p(x)}}{p(x)}dx\right)^\alpha - \varepsilon \int_\Omega |u|^{\alpha p^+}dx - C(\varepsilon)\int_\Omega |u|^{\gamma(x)}dx \\ &\geq \frac{m_1}{\alpha(p^+)^\alpha}\|u\|^{\alpha p^+} - \varepsilon C^{\alpha p^+}\|u\|^{\alpha p^+} - C\|u\|^{\gamma^-} \\ &\geq \frac{m_1}{2\alpha(p^+)^\alpha}\|u\|^{\alpha p^+} - C\|u\|^{\gamma^-} \quad \text{when } \|u\| \leq 1. \end{aligned}$$

Therefore, there exist $r > 0$ and $\delta > 0$ such that $J(u) \geq \delta > 0$ for every $\|u\| = r$. From (H_1) it follows that

$$F(x,t) \geq C|t|^\theta - C, \quad \text{for all } x \in \Omega \quad \text{and } |t| \geq K. \tag{3.7}$$

For $w \in X\backslash\{0\}$ and $t > 1$, in view of (M_1) and (3.7), we have

$$\begin{aligned} J(tw) &= \widehat{M}\left(\int_\Omega \frac{|\Delta tw|^{p(x)}}{p(x)}dx\right) - \int_\Omega F(x,tw)dx \\ &\leq \frac{m_2}{\alpha(p^-)^\alpha}t^{\alpha p^+}\left(\int_\Omega |\Delta w|^{p(x)}dx\right)^\alpha - Ct^\theta \int_\Omega |w|^\theta dx - C|\Omega|. \end{aligned}$$

In view of (H_1), we have $\theta > \frac{m_2\alpha(p^+)^\alpha}{m_1(p^-)^{\alpha-1}}$, which imply that $\theta > \alpha p^+$. Therefore $J(tw) \to -\infty$ as $t \to +\infty$. J satisfies the conditions of the mountain pass theorem, since $J(0) = 0$. So J admits at least one nontrivial critical point. □

Theorem 3.6. *If $(M_1), (H_0), (H_1)$, (H_3) hold and $\gamma^- > \alpha(p^+)^\alpha > \alpha p^+$, then problem (1.1) has a sequence of weak solutions $(\pm u_k)$ such that $J(\pm u_k) \to +\infty$ as $k \to \infty$.*

To prove Theorem 3.6, we will use the following "Fountain theorem".
Since X is a separable and reflexive Banach space (see [15]), there exist $\{e_j\}_{j=1}^\infty \subset X$ and $\{e_j^*\}_{j=1}^\infty \subset X^*$ such that

$$< e_i, e_j^* >= \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases}$$

$$X = \overline{\mathrm{span}\{e_j : j = 1, 2, \dots\}}, \quad X^* = \overline{\mathrm{span}\{e_j^* : j = 1, 2, \dots\}}.$$

For $k = 1, 2, \dots$. Put

$$X_j = \mathrm{span}\{e_j\}, \quad Y_k = \oplus_{j=1}^k X_j, \quad Z_k = \overline{\oplus_{j=k}^\infty X_j}.$$

Lemma 3.7. *(see [14]) If $\gamma(x) \in C_+(\overline{\Omega})$ and $\gamma(x) < p_2^*(x)$ for all $x \in \Omega$, denote*

$$\gamma_k = \sup\{|u|_{\gamma(x)} : \|u\| = 1, u \in Z_k\},$$

then $\lim_{k\to+\infty} \gamma_k = 0$.

Lemma 3.8. *(Fountain Theorem, See [23]) Assume*

(A_1) *X is a Banach space, $J \in C^1(X,\mathbb{R})$ is an even functional. If for each $k = 1, 2, ...$ there exist $\rho_k > r_k > 0$ such that*

(A_2) $\inf\{J(u) : u \in Z_k, \|u\| = r_k\} \to +\infty$ *as* $k \to +\infty$.

(A_3) $\max\{J(u) : u \in Y_k, \|u\| = \rho_k\} \le 0$.

(A_4) *J satisfies the (PS) condition for every $c > 0$.*

Then J has an unbounded sequence of critical points.

Definition 3.9. *We say that J satisfies the $(PS)_c^*$ condition (with respect to (Y_n)), if any sequence $(u_{n_j}) \subset X$ such that $n_j \to +\infty$, $u_{n_j} \in Y_{n_j}$, $J(u_{n_j}) \to c$ and $(J|_{Y_{n_j}})'(u_{n_j}) \to 0$ contain a subsequence converging to a critical point of J.*

Proof of Theorem 3.6. According to (H_3) and Lemma 3.4, J is an even functional and satisfies the (PS) condition. We will prove that if k is large enough, then there exist $\rho_k > r_k > 0$ such that (A_2) and (A_3) hold. Thus, the conclusion of Theorem 3.6 can be reached by the Fountain theorem.
(A_2) For any $u \in Z_k$, $\|u\| = r_k > 1$, (r_k will be specified later), we have

$$\begin{aligned}
J(u) &= \widehat{M}\left(\int_\Omega \frac{|\Delta u|^{p(x)}}{p(x)}dx\right) - \int_\Omega F(x,u)dx \\
&\ge \frac{m_1}{\alpha}\left(\int_\Omega \frac{|\Delta u|^{p(x)}}{p(x)}dx\right)^\alpha - C_1\int_\Omega |u|^{\gamma(x)}dx - C\int_\Omega |u|dx \\
&\ge \frac{m_1}{\alpha(p^+)^\alpha}\|u\|^{\alpha p^-} - C_1\int_\Omega |u|^{\gamma(x)}dx - C\int_\Omega |u|dx \\
&\ge \begin{cases} \frac{m_1}{\alpha(p^+)^\alpha}\|u\|^{\alpha p^-} - C_1 - C\|u\| & \text{if } |u|_{\alpha(x)} \le 1 \\ \frac{m_1}{\alpha(p^+)^\alpha}\|u\|^{\alpha p^-} - C_1\gamma_k^{\gamma^+}\|u\|^{\gamma^+} - C\|u\| & \text{if } |u|_{\alpha(x)} > 1 \end{cases} \\
&\ge \frac{m_1}{\alpha(p^+)^\alpha}\|u\|^{\alpha p^-} - C_1\gamma_k^{\gamma^+}\|u\|^{\gamma^+} - C\|u\| - C \\
&= m_1 r_k^{\alpha p^-}\left(\frac{1}{\alpha(p^+)^\alpha} - C_1\gamma_k^{\gamma^+} m_1^{-1} r_k^{\gamma^+ - \alpha p^-}\right) - Cr_k - C.
\end{aligned}$$

Now we choose r_k as follows

$$r_k = (C_1\gamma^+\gamma_k^{\gamma^+} m_1^{-1})^{\frac{1}{\alpha p^- - \gamma^+}}.$$

Then

$$J(u) \ge m_1 r_k^{\alpha p^-}\left(\frac{1}{\alpha(p^+)^\alpha} - \frac{1}{\gamma^+}\right) - Cr_k - C \to +\infty \quad \text{as } k \to +\infty.$$

because of $\alpha(p^+)^\alpha < \gamma^+$, $\alpha p^- > 1$ and $\gamma_k \to 0$.
(A_3) From (H_1), we know that $F(x,t) \ge C|t|^\theta - C$. Therefore, for any $w \in Y_k$ with

$\|w\| = 1$ and $1 < t = \rho_k$, we have

$$\begin{aligned} J(tw) &= \widehat{M}\left(\int_\Omega \frac{|\Delta tw|^{p(x)}}{p(x)}dx\right) - \int_\Omega F(x,tw)dx \\ &\leq \frac{m_2}{\alpha}\left(\int_\Omega \frac{|\Delta tw|^{p(x)}}{p(x)}dx\right)^\alpha - C\int_\Omega |tw|^\theta dx - C \\ &\leq \frac{m_2}{\beta(p^-)^\alpha} t^{\alpha p^+}\left(\int_\Omega |\Delta w|^{p(x)}dx\right)^\alpha - Ct^\theta \int_\Omega |w|^\theta dx - C. \end{aligned}$$

By $\theta > \alpha p^+$ and $\dim Y_k < \infty$, it is easy to see that $J(u) \to -\infty$ as $\|u\| \to +\infty$ for $u \in Y_k$. So the assertion (A_3) holds and the Conclusion of Theorem is reached by using the Fountain theorem.

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