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Riesz Almost Lacunary Triple Sequence Spaces Of Γ^3 Defined By a Musielak-Orlicz Function

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abstract: In this paper we introduce a new concept for Riesz almost lacunary Γ^3 sequence spaces strong $P-$ convergent to zero with respect to an Musielak-Orlicz function and examine some properties of the resulting sequence spaces. We also introduce and study statistical convergence of Riesz almost lacunary Γ^3 sequence spaces and also some inclusion theorems are discussed.

Key Words: Analytic Sequence, Musielak-Orlicz Function, Multiple Triple Sequence Spaces, Entire Sequence, Riesz Space.

Contents

1. Introduction

A triple sequence (real or complex) can be defined as a function $x : \mathbb{N}\times\mathbb{N}\times\mathbb{N} \to \mathbb{R}(\mathbb{C})$, where $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequence was introduced and investigated at the initial by *Sahiner et al. [12,13]*, *Esi et al. [1-3]*, *Datta et al. [4]*,*Subramanian et al. [14]*, *Debnath et al. [5-8]*, *Tripathy et al. [16-29]* and many others.
A triple sequence $x = (x_{mnk})$ is said to be triple analytic if

$$sup_{m,n,k} |x_{mnk}|^{\frac{1}{m+n+k}} < \infty.$$

The space of all triple analytic sequences are usually denoted by Λ^3. A triple sequence $x = (x_{mnk})$ is called triple entire sequence if

$$|x_{mnk}|^{\frac{1}{m+n+k}} \to 0 \text{ as } m, n, k \to \infty.$$

The space of all triple entire sequences are usually denoted by Γ^3.

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2. Definitions and Preliminaries

Definition 2.1. *An Orlicz function ([see [9]) is a function $M : [0,\infty) \to [0,\infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by $M(x+y) \leq M(x) + M(y)$, then this function is called modulus function.*

Lindenstrauss and Tzafriri ([10]) used the idea of Orlicz function to construct Orlicz sequence space.

A sequence $g = (g_{mn})$ defined by

$$g_{mn}(v) = sup\left\{|v|\, u - (f_{mnk})(u) : u \geq 0\right\}, m, n, k = 1, 2, \cdots$$

is called the complementary function of a Musielak-Orlicz function f. For a given Musielak-Orlicz function f, [see [11]] the Musielak-Orlicz sequence space t_f is defined as follows

$$t_f = \left\{x \in w^3 : I_f\left(|x_{mnk}|\right)^{1/m+n+k} \to 0\, as\, m, n, k \to \infty\right\},$$

where I_f is a convex modular defined by

$$I_f(x) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} f_{mnk}\left(|x_{mnk}|\right)^{1/m+n+k}, x = (x_{mnk}) \in t_f.$$

We consider t_f equipped with the Luxemburg metric

$$d(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} f_{mnk}\left(\frac{|x_{mnk}|^{1/m+n+k}}{mnk}\right)$$

is an exteneded real number.

Definition 2.2. *Let X, Y be a real vector space of dimension w, where $n \leq m$. A real valued function $d_p(x_1, \ldots, x_n) = \|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p$ on X satisfying the following four conditions:*
(i) $\|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p = 0$ if and only if $d_1(x_1, 0), \ldots, d_n(x_n, 0)$ are linearly dependent,
(ii) $\|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p$ is invariant under permutation,
(iii) $\|(\alpha d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p = |\alpha|\, \|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p, \alpha \in \mathbb{R}$
(iv) $d_p((x_1, y_1), (x_2, y_2) \cdots (x_n, y_n)) = (d_X(x_1, x_2, \cdots x_n)^p + d_Y(y_1, y_2, \cdots y_n)^p)^{1/p}$ $for 1 \leq p < \infty$; (or)
(v) $d((x_1, y_1), (x_2, y_2), \cdots (x_n, y_n)) := \sup\left\{d_X(x_1, x_2, \cdots x_n), d_Y(y_1, y_2, \cdots y_n)\right\}$,
for $x_1, x_2, \cdots x_n \in X, y_1, y_2, \cdots y_n \in Y$ is called the p product metric of the Cartesian product of n metric spaces (see [15]) .

Definition 2.3. *The four dimensional matrix A is said to be RH-regular if it maps every bounded $P-$ convergent sequence into a $P-$ convergent sequence with the same $P-$ limit. The assumption of boundedness was made because a triple sequence spaces which is $P-$ convergent is not necessarily bounded.*

Definition 2.4. *A triple sequence $x = (x_{mnk})$ of real numbers is called almost $P-$ convergent to a limit 0 if*

$$P - \lim_{p,q,u\to\infty} sup_{r,s,t\geq 0} \frac{1}{pqu} \sum_{m=r}^{r+p-1} \sum_{n=s}^{s+q-1} \sum_{k=t}^{t+u-1} |x_{mnk}|^{1/m+n+k} \to 0.$$

that is, the average value of (x_{mnk}) *taken over any rectangle* $\{(m,n,k) : r \leq m \leq r+p-1, s \leq n \leq s+q-1, t \leq k \leq t+u-1\}$ *tends to 0 as both* p,q *and* u *to* ∞, *and this* $P-$ *convergence is uniform in* i, ℓ *and* j. *Let us denote the set of sequences with this property as* $\left[\widehat{\Gamma^3}\right]$.

Definition 2.5. *Let* $(q_{rst}), (\overline{q_{rst}}), (\overline{\overline{q_{rst}}})$ *be sequences of positive numbers and*

$$Q_r = \begin{bmatrix} q_{111} & q_{122} & \cdots & q_{11s} & 0\ldots \\ q_{211} & q_{222} & \cdots & q_{22s} & 0\ldots \\ . \\ . \\ . \\ q_{r11} & q_{r22} & \cdots & q_{rst} & 0\ldots \\ 0 & 0 & \ldots 0 & 0 & 0\ldots \end{bmatrix} = q_{111} + q_{122} + \ldots + q_{rst} \neq 0,$$

$$\overline{Q}_s = \begin{bmatrix} \overline{q}_{111} & \overline{q}_{122} & \cdots & \overline{q}_{11s} & 0\ldots \\ \overline{q}_{211} & \overline{q}_{222} & \cdots & \overline{q}_{22s} & 0\ldots \\ . \\ . \\ . \\ \overline{q}_{r11} & \overline{q}_{r22} & \cdots & \overline{q}_{rst} & 0\ldots \\ 0 & 0 & \ldots 0 & 0 & 0\ldots \end{bmatrix} = \overline{q}_{112} + \overline{q}_{122} + \ldots + \overline{q}_{rst} \neq 0,$$

$$\overline{\overline{Q}}_t = \begin{bmatrix} \overline{\overline{q}}_{111} & \overline{\overline{q}}_{122} & \cdots & \overline{\overline{q}}_{11s} & 0\ldots \\ \overline{\overline{q}}_{211} & \overline{\overline{q}}_{222} & \cdots & \overline{\overline{q}}_{22s} & 0\ldots \\ . \\ . \\ . \\ \overline{\overline{q}}_{r11} & \overline{\overline{q}}_{r22} & \cdots & \overline{\overline{q}}_{rst} & 0\ldots \\ 0 & 0 & \ldots 0 & 0 & 0\ldots \end{bmatrix} = \overline{q}_{111} + \overline{q}_{122} + \ldots + \overline{q}_{rst} \neq 0.$$ *Then the transformation is given by*

$T_{rst} = \sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1}^{t} q_{mnk} \overline{q}_{mnk} \overline{\overline{q}}_{mnk} |x_{mnk}|^{1/m+n+k}$ *is called the Riesz mean of triple sequence* $x = (x_{mnk})$. *If* $P - \lim_{rst} T_{rst}(x) = 0, 0 \in \mathbb{R}$, *then the sequence* $x = (x_{mnk})$ *is said to be Riesz convergent to 0. If* $x = (x_{mnk})$ *is Riesz convergent to 0, then we write* $P_R - lim x = 0$.

Definition 2.6. *The triple sequence* $\theta_{i,\ell,j} = \{(m_i, n_\ell, k_j)\}$ *is called triple lacunary if there exist three increasing sequences of integers such that*

$$m_0 = 0, h_i = m_i - m_{i-1} \to \infty \text{ as } i \to \infty \text{ and}$$
$$n_0 = 0, \overline{h_\ell} = n_\ell - n_{\ell-1} \to \infty \text{ as } \ell \to \infty.$$
$$k_0 = 0, \overline{h_j} = k_j - k_{j-1} \to \infty \text{ as } j \to \infty.$$

Let $m_{i,\ell,j} = m_i n_\ell k_j, h_{i,\ell,j} = h_i \overline{h_\ell h_j}$, *and* $\theta_{i,\ell,j}$ *is determine by* $I_{i,\ell,j} = \{(m,n,k) : m_{i-1} < m < m_i \, and \, n_{\ell-1} < n \leq n_\ell \, and \, k_{j-1} < k \leq k_j\}$, $q_k = \frac{m_k}{m_{k-1}}, \overline{q_\ell} = \frac{n_\ell}{n_{\ell-1}}, \overline{q_j} = \frac{k_j}{k_{j-1}}$.

Using the notations of lacunary sequence and Riesz mean for triple sequences.

$\theta_{i,\ell,j} = \{(m_i, n_\ell, k_j)\}$ be a triple lacunary sequence and $q_{mnk}\overline{q}_{mnk}\overline{\overline{q}}_{mnk}$ be sequences of positive real numbers such that $Q_{m_i} = \sum_{m\in(0,m_i]} p_{m_i}, Q_{n_\ell} = \sum_{n\in(0,n_\ell]} p_{n_\ell}, Q_{n_j} = \sum_{k\in(0,k_j]} p_{k_j}$ and $H_i = \sum_{m\in(0,m_i]} p_{m_i}, \overline{H} = \sum_{n\in(0,n_\ell]} p_{n_\ell}, \overline{\overline{H}} = \sum_{k\in(0,k_j]} p_{k_j}$. Clearly, $H_i = Q_{m_i} - Q_{m_{i-1}}, \overline{H}_\ell = Q_{n_\ell} - Q_{n_{\ell-1}}, \overline{\overline{H}}_j = Q_{k_j} - Q_{k_{j-1}}$. If the Riesz transformation of triple sequences is RH-regular, and $H_i = Q_{m_i} - Q_{m_{i-1}} \to \infty$ as $i \to \infty, \overline{H} = \sum_{n\in(0,n_\ell]} p_{n_\ell} \to \infty$ as $\ell \to \infty, \overline{\overline{H}} = \sum_{k\in(0,k_j]} p_{k_j} \to \infty$ as $j \to \infty$, then $\theta'_{i,\ell,j} = \{(m_i, n_\ell, k_j)\} = \left\{\left(Q_{m_i} Q_{n_j} Q_{k_k}\right)\right\}$ is a triple lacunary sequence. If the assumptions $Q_r \to \infty$ as $r \to \infty$, $\overline{Q}_s \to \infty$ as $s \to \infty$ and $\overline{\overline{Q}}_t \to \infty$ as $t \to \infty$ may be not enough to obtain the conditions $H_i \to \infty$ as $i \to \infty, \overline{H}_\ell \to \infty$ as $\ell \to \infty$ and $\overline{\overline{H}}_j \to \infty$ as $j \to \infty$ respectively. For any lacunary sequences $(m_i), (n_\ell)$ and (k_j) are integers.

Throughout the paper, we assume that $Q_r = q_{111} + q_{122} + \ldots + q_{rst} \to \infty\, (r \to \infty)$, $\overline{Q}_s = \overline{q}_{111} + \overline{q}_{122} + \ldots + \overline{q}_{rst} \to \infty\, (s \to \infty), \overline{\overline{Q}}_t = \overline{\overline{q}}_{111} + \overline{\overline{q}}_{122} + \ldots + \overline{\overline{q}}_{rst} \to \infty\, (t \to \infty)$, such that $H_i = Q_{m_i} - Q_{m_{i-1}} \to \infty$ as $i \to \infty, \overline{H}_\ell = Q_{n_\ell} - Q_{n_{\ell-1}} \to \infty$ as $\ell \to \infty$ and $\overline{\overline{H}}_j = Q_{k_j} - Q_{k_{j-1}} \to \infty$ as $j \to \infty$.

Let $Q_{m_i,n_\ell,k_j} = Q_{m_i}\overline{Q}_{n_\ell}\overline{\overline{Q}}_{k_j}, H_{i\ell j} = H_i\overline{H}_\ell\overline{\overline{H}}_j$,

$I'_{i\ell j} = \left\{(m,n,k) : Q_{m_{i-1}} < m < Q_{m_i}, \overline{Q}_{n_{\ell-1}} < n < Q_{n_\ell} \text{ and } \overline{Q}_{k_{j-1}} < k < \overline{Q}_{k_j}\right\}$,

$V_i = \frac{Q_{m_i}}{Q_{m_{i-1}}}, \overline{V}_\ell = \frac{Q_{n_\ell}}{Q_{n_{\ell-1}}}$ *and* $\overline{\overline{V}}_j = \frac{Q_{k_j}}{Q_{k_{j-1}}}$. *and* $V_{i\ell j} = V_i\overline{V}_\ell\overline{\overline{V}}_j$.

If we take $q_{mnk} = 1, \overline{q}_{mnk} = 1$ and $\overline{\overline{q}}_{mnk} = 1$ for all m, n and k then $H_{i\ell j}, Q_{i\ell j}, V_{i\ell j}$ and $I'_{i\ell j}$ reduce to $h_{i\ell j}, q_{i\ell j}, v_{i\ell j}$ and $I_{i\ell j}$.

Let f be an Musielak-Orlicz function and $p = (p_{mnk})$ be any factorable triple sequence of positive real numbers. We define the following sequence spaces:

$\left[\Gamma^3_R, \theta_{i\ell j}, q, f, p, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] =$
$P - \mathbf{lim}_{i,\ell,j\to\infty} \frac{1}{H_{i,\ell j}} \sum_{i\in I_{i\ell j}} \sum_{\ell\in I_{i\ell j}} \sum_{j\in I_{i\ell j}} q_{mnk}\overline{q}_{mnk}\overline{\overline{q}}_{mnk}$
$\left[f\left(|x_{m+i,n+\ell,k+j}|\right)^{p_{mnk}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] = 0$, *uniformly, in i, ℓ and j.*

$\left[\Lambda^3_R, \theta_{i\ell j}, q, f, p, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] =$
$P - \mathbf{sup}_{i,\ell,j} \frac{1}{H_{i,\ell j}} \sum_{i\in I_{i\ell j}} \sum_{\ell\in I_{i\ell j}} \sum_{j\in I_{i\ell j}} q_{mnk}\overline{q}_{mnk}\overline{\overline{q}}_{mnk}$
$\left[f\,|x_{m+i,n+\ell,k+j}|^{p_{mnk}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] < \infty$, *uniformly, in i, ℓ and j.*

Let f be an Musielak-Orlicz function, $p = p_{mnk}$ be any factorable triple sequence of positive real numbers and and $q_{mnk}, \overline{q}_{mnk}$ and $\overline{\overline{q}}_{mnk}$ be sequences of positive numbers and $Q_r = q_{111} + \cdots q_{rst}$, $\overline{Q}_s = \overline{q}_{111} \cdots \overline{q}_{rst}$ and $\overline{\overline{Q}}_t = \overline{\overline{q}}_{112} \cdots \overline{\overline{q}}_{rst}$,

If we choose $q_{mnk} = 1, \overline{q}_{mnk} = 1$ and $\overline{\overline{q}}_{mnk} = 1$ for all m, n and k, then we obtain the following sequence spaces.

$\left[\Gamma^3_R, q, f, p, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] =$
$P - \mathbf{lim}_{r,s,t\to\infty} \sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1}^{t} q_{mnk}\overline{q}_{mnk}\overline{\overline{q}}_{mnk}$

$\left[f\left(|x_{m+i,n+\ell,k+j}|^{p_{mnk}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right)\right] = 0$, *uniformly, in* i, ℓ *and* j.

$\left[\Lambda^3_R, q, f, p, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] =$
$P - \boldsymbol{sup}_{r,s,t} \sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1}^{t} q_{mnk}\overline{q}_{mnk}\overline{\overline{q}}_{mnk}$
$\left[f(|x_{m+i,n+\ell,k+j}|)^{p_{mnk}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] < \infty$, *uniformly, in* i, ℓ *and* j.

Definition 2.7. *Let* f *be an Orlicz function and* $p = (p_{mnk})$ *be any factorable triple sequence of strictly positive real numbers, we define the following sequence space:*
$\theta_{i,\ell,j} = \{(m_i, n_\ell, k_j)\}$ *be a triple lacunary sequence*
$\Gamma^3_f\left[AC_{\theta_{i,\ell,j}}, P, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] =$
$P - \boldsymbol{lim}_{i,\ell,j} \frac{1}{h_{i\ell j}} \sum_{m\in I_{i,\ell,j}} \sum_{n\in I_{i,\ell,j}} \sum_{k\in I_{i,\ell,j}}$
$\left[f(|x_{m+i,n+\ell,k+j}|)^{p_{mnk}/m+n+k}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] = 0$, *uniformly in* i, ℓ *and* j.

We shall denote $\Gamma^3_f\left[AC_{\theta_{i,\ell,j}}, P, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ *as* $\Gamma^3\left[AC_{\theta_{i,\ell,j}}, P, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ *respectively when* $p_{mnk} = 1$ *for all* m, n *and* k *If* x *is in* $\Gamma^3\left[AC_{\theta_{i,\ell,j}}, P, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$, *we shall say that* x *is almost lacunary* Γ^3 *strongly* $p-$*convergent with respect to the Musielak-Orlicz function* f. *Also note that if* $f(x) = x, p_{mnk} = 1$ *for all* m, n *and* k *then* $\Gamma^3_f\left[AC_{\theta_{i,\ell,j}}, p, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] =$
$\Gamma^3\left[AC_{\theta_{i,\ell,j}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ *which are defined as follows:*
$\Gamma^3\left[AC_{\theta_{i,\ell,j}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] =$
$P - \boldsymbol{lim}_{i,\ell,j} \frac{1}{h_{i\ell j}} \sum_{m\in I_{i,\ell,j}} \sum_{n\in I_{i,\ell,j}} \sum_{k\in I_{i,\ell,j}}$
$f\left[|x_{m+i,n+\ell,k+j}|^{1/m+n+k}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] = 0$, *uniformly in* i, ℓ *and* j.
Again note if $p_{mnk} = 1$ *for all* m, n *and* k *then*
$\Gamma^3_f\left[AC_{\theta_{i,\ell,j}}, p, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] =$
$\Gamma^3_f\left[AC_{\theta_{i,\ell,j}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$. *we define*
$\Gamma^3_f\left[AC_{\theta_{i,\ell,j}}, p, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] =$
$P - \boldsymbol{lim}_{i,\ell,j} \frac{1}{h_{i\ell j}} \sum_{m\in I_{k,\ell,j}} \sum_{n\in I_{i,\ell,j}} \sum_{k\in I_{i,\ell,j}}$
$f\left[|x_{m+i,n+\ell,k+j}|^{p_{mnk}1/m+n+k}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] = 0$, *uniformly, in* i, ℓ *and* j.

Definition 2.8. *Let* f *be an Musielak-Orlicz function* $p = (p_{mnk})$ *be any factorable triple sequence of strictly positive real numbers. We define the following sequence*

space: $\Gamma^3_f\left[p, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] =$
$P - \mathit{lim}_{r,s,t\to\infty} \frac{1}{rst}\sum_{m=1}^{r}\sum_{n=1}^{s}\sum_{k=1}^{t}$
$f\left[|x_{m+i,n+\ell,k+j}|^{p_{mnk}/m+n+k}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] = 0,$ *uniformly, in* i, ℓ *and* j.

If we take $f(x) = x, p_{mnk} = 1$ *for all* m, n *and* k *then*
$\Gamma^3_f\left[p, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] = \Gamma^3.$

Definition 2.9. *Let* $\theta_{i,\ell,j}$ *be a triple lacunary sequence; the triple number sequence* x *is* $\widehat{S_{\theta i,\ell,j}} - p-$ *convergent to 0 then*
$P - \mathit{lim}_{i,\ell,j} \frac{1}{h_{i,\ell,j}} max_{i,\ell,j}\left|\left\{(m,n,k) \in I_{i,\ell,j} : f\left(|x_{m+i,n+\ell,k+j} - 0|\right)^{1/m+n+k}\right\}\right| = 0.$
In this case we write $\widehat{S_{\theta i,\ell,j}} - \mathit{lim}(f|x_{m+i,n+\ell,k+j} - 0|)^{1/m+n+k} = 0.$

3. Main Results

Theorem 3.1. *Let* f *be any Musielak-Orlicz function and a bounded factorable positive triple number sequence* p_{mnk} *then*

$$\Gamma^3_f\left[AC_{\theta_{i,\ell,j}}, P, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$$

is linear space

Proof. The proof is easy. Theorefore omit the proof. □

Theorem 3.2. *Let* f *be an Musielak-Orlicz, then*
$\Gamma^3\left[AC_{\theta_{i,\ell,j}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] \subset$
$\Gamma^3_f\left[AC_{\theta_{i,\ell,j}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$

Proof. Let $x \in \Gamma^3\left[AC_{\theta_{i,\ell,j}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$. For each i, ℓ and j
$\Gamma^3\left[AC_{\theta_{i,\ell,j}}\right] = \mathtt{lim}_{i,\ell,j} \frac{1}{h_{i\ell j}}\sum_{m\in I_{i,\ell,j}}\sum_{n\in I_{i,\ell,j}}\sum_{k\in I_{i,\ell,j}}$
$\left[|x_{m+i,n+\ell,k+j}|^{1/m+n+k}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] = 0.$
Since f is continuous at zero, for $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \epsilon$ for every t with $0 \le t \le \delta$. We obtain the following,
$\frac{1}{h_{i\ell j}}(h_{i\ell j}\epsilon) + \frac{1}{h_{i\ell j}}\sum_{m\in I_{i,\ell,j}}\sum_{n\in I_{i,\ell,j}}\sum_{k\in I_{i,\ell,j}\, and\, |x_{m+i,n+\ell,k+j}-0|>\delta}$
$f\left[|x_{m+i,n+\ell,k+j}|^{1/m+n+k}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$
$\frac{1}{h_{i\ell j}}(h_{i\ell j}\epsilon) + \frac{1}{h_{i\ell j}}K\delta^{-1}f(2)\, h_{i\ell j}\,\Gamma^3\left[AC_{\theta_{i,\ell,j}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right].$
Hence i, ℓ and j goes to infinity, we are granted
$x \in \Gamma^3_f\left[AC_{\theta_{i,\ell,j}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right].$ □

Theorem 3.3. *Let* $\theta_{i,\ell,j} = \{m_i, n_\ell, k_j\}$ *be a triple lacunary sequence with* $\textbf{lim inf}_i q_i > 1$, $\textbf{lim inf}_\ell \overline{q_\ell} > 1$ *and* $\textbf{lim inf}_j q_j > 1$ *then for any Orlicz function* f, $\Gamma^3_f(P) \subset \Gamma^3_f\left(AC_{\theta_{i,\ell,j}}, P\right)$

Proof. Suppose $\texttt{lim inf}_i q_i > 1$, $\texttt{lim inf}_\ell \overline{q_\ell} > 1$ and $\texttt{lim inf}_j q_j > 1$ then there exists $\delta > 0$ such that $q_i > 1+\delta$, $\overline{q_\ell} > 1+\delta$ and $q_j > 1+\delta$ This implies $\frac{h_i}{m_i} \geq \frac{\delta}{1+\delta}$, $\frac{h_\ell}{n_\ell} \geq \frac{\delta}{1+\delta}$ and $\frac{h_j}{k_j} \geq \frac{\delta}{1+\delta}$ Then for $x \in \Gamma^3_f(P)$, we can write for each r, s and u.

$$B_{i,\ell,j} = \frac{1}{h_{i\ell j}} \sum_{m\in I_{i,\ell,j}} \sum_{n\in I_{i,\ell,j}} \sum_{k\in I_{i,\ell,j}}$$
$$f\left[\left(|x_{m+i,n+\ell,k+j}|\right)^{p_{mnk}/m+n+k}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] =$$
$$\frac{1}{h_{i\ell j}} \sum_{m=1}^{m_i} \sum_{n=1}^{n_\ell} \sum_{k=1}^{k_j}$$
$$f\left[|x_{m+i,n+\ell,k+j}|^{p_{mnk}/m+n+k}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] -$$
$$\frac{1}{h_{i\ell j}} \sum_{m=1}^{m_{i-1}} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_{i-1}}$$
$$f\left[|x_{m+i,n+\ell,k+j}|^{p_{mnk}1/m+n+k}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] -$$
$$\frac{1}{h_{i\ell j}} \sum_{m=m_{i-1}+1}^{m_i} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_{j-1}}$$
$$f\left[|x_{m+i,n+\ell,k+j}|^{p_{mnk}/m+n+k}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] -$$
$$\frac{1}{h_{i\ell j}} \sum_{k=k_j+1}^{k_j} \sum_{n=n_{\ell-1}+1}^{n_\ell} \sum_{m=1}^{m_{k-1}}$$
$$f\left[|x_{m+i,n+\ell,k+j}|^{p_{mnk}/m+n+k}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$$
$$= \frac{m_i n_\ell k_j}{h_{i\ell j}} \frac{1}{m_i n_\ell k_j} \sum_{m=1}^{m_i} \sum_{n=1}^{n_\ell} \sum_{k=1}^{k_j}$$
$$f\left[|x_{m+i,n+\ell,k+j}|^{p_{mnk}/m+n+k}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] -$$
$$\frac{m_{k-1}n_{\ell-1}k_{j-1}}{h_{i\ell j}} \frac{1}{m_{i-1}n_{\ell-1}kj-1} \sum_{m=1}^{m_{i-1}} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_{j-1}}$$
$$f\left[|x_{m+i,n+\ell,k+j}|^{p_{mnk}/m+n+k}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$$
$$- \frac{k_{j-1}}{h_{i\ell j}} \frac{1}{k_{j-1}} \sum_{m=m_{i-1}+1}^{m_i} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_j}$$
$$f\left[|x_{m+i,n+\ell,k+j}|^{p_{mnk}/m+n+k}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$$
$$- \frac{n_{\ell-1}}{h_{i\ell j}} \frac{1}{n_{\ell-1}} \sum_{m=m_{k-1}+1}^{m_k} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_j}$$
$$f\left[|x_{m+i,n+\ell,k+j}|^{p_{mnk}/m+n+k}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] -$$
$$\frac{m_{k-1}}{h_{i\ell j}} \frac{1}{m_{k-1}} \sum_{k=1}^{k_j} \sum_{n=n_{\ell-1}+1}^{n_\ell} \sum_{m=1}^{m_{k-1}}$$
$$f\left[|x_{m+i,n+\ell,k+j}|^{p_{mnk}/m+n+k}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right].$$

Since $x \in \Gamma^3_f(P)$ the last three terms tend to zero uniformly in m, n, k in the sense, thus, for each r, s and u

$$B_{i,\ell,j} = \frac{m_i n_\ell k_j}{h_{i\ell j}} \frac{1}{m_i n_\ell k_j} \sum_{m=1}^{m_i} \sum_{n=1}^{n_\ell} \sum_{k=1}^{k_j}$$
$$f\left[|x_{m+i,n+\ell,k+j}|^{p_{mnk}/m+n+k}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] -$$
$$\frac{m_{i-1}n_{\ell-1}k_{j-1}}{h_{i\ell j}} \frac{1}{m_{i-1}n_{\ell-1}k_{j-1}} \sum_{m=1}^{m_{i-1}} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_{j-1}}$$
$$f\left[|x_{m+i,n+\ell,k+j}|^{p_{mnk}/m+n+k}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] + O(1).$$

Since $h_{i\ell j} = m_i n_\ell k_j - m_{i-1} n_{\ell-1} k_{j-1}$ we are granted for each i, ℓ and j the following

$$\frac{m_i n_\ell kj}{h_{i\ell j}} \leq \frac{1+\delta}{\delta} \text{ and } \frac{m_{i-1} n_{\ell-1} k_{j-1}}{h_{i\ell j}} \leq \frac{1}{\delta}.$$

The terms
$\frac{1}{m_i n_\ell k_j} \sum_{m=1}^{m_i} \sum_{n=1}^{n_\ell} \sum_{k=1}^{k_j}$
$f\left[\left|x_{m+i,n+\ell,k+j}\right|^{p_{mnk}/m+n+k}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ and
$\frac{1}{m_{i-1} n_{\ell-1} k_{j-1}} \sum_{m=1}^{m_{i-1}} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_{j-1}}$
$f\left[\left|x_{m+i,n+\ell,k+j}\right|^{p_{mnk}/m+n+k}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ are both gai sequences for all i, ℓ and j. Thus $B_{i\ell j}$ is a gai sequence for each i, ℓ and j. Hence $x \in \Gamma_f^3\left(AC_{\theta_{i,\ell,j}}, P, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right)$. □

Theorem 3.4. *Let $\theta_{i,\ell,j} = \{m,n,k\}$ be a triple lacunary sequence with $\lim\sup_\eta q_\eta < \infty$ and $\lim\sup_i \overline{q_i} < \infty$ then for any Musielak-Orlicz function f,*
$\Gamma_f^3\left(AC_{\theta_{i,\ell,j}}, P, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right) \subset$
$\Gamma_f^3\left(P, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right)$.

Proof. Since $\lim\sup_i q_i < \infty$ and $\lim\sup_i \overline{q_i} < \infty$ there exists $H > 0$ such that $q_i < H$, $\overline{q_\ell} < H$ and $q_j < H$ for all i, ℓ and j.
Let $x \in \Gamma_f^3\left(AC_{\theta_{i,\ell,j}}, P, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right)$. Also there exist $i_0 > 0, \ell_0 > 0$ and $j_0 > 0$ such that for every $a \geq i_0$ $b \geq \ell_0$ and $c \geq j_0$ and i, ℓ and j.
$A'_{abc} = \frac{1}{h_{abc}} \sum_{m \in I_{a,b,c}} \sum_{n \in I_{a,b,c}} \sum_{k \in I_{a,b,c}}$
$f\left[\left|x_{m+i,n+\ell,k+j}\right|^{p_{mnk}/m+n+k}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] \to 0$ as $m,n,k \to \infty$. Let $G' = max\left\{A'_{a,b,c} : 1 \leq a \leq i_0,\ 1 \leq b \leq \ell_0 \ and \ 1 \leq c \leq j_0\right\}$ and p, q and t be such that $m_{i-1} < p \leq m_i$, $n_{\ell-1} < q \leq n_\ell$ and $m_{j-1} < t \leq m_j$. Thus we obtain the following:
$\frac{1}{pqt} \sum_{m=1}^{p} \sum_{n=1}^{q} \sum_{k=1}^{t}$
$f\left[\left|x_{m+i,n+\ell,k+j}\right|^{p_{mnk}/m+n+k}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$
$\leq \frac{1}{m_{i-1} n_{\ell-1} k_{j-1}} \sum_{m=1}^{m_i} \sum_{n=1}^{n_\ell} \sum_{k=1}^{k_j}$
$f\left[\left|x_{m+i,n+\ell,k+j}\right|^{p_{mnk}/m+n+k}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$
$\leq \frac{1}{m_{i-1} n_{\ell-1} k_{j-1}} \sum_{a=1}^{i} \sum_{b=1}^{\ell} \sum_{c=1}^{j}$
$\left(\sum_{m \in I_{a,b,c}} \sum_{n \in I_{a,b,c}} \sum_{k \in I_{a,b,c}} f\left[\left|x_{m+i,n+\ell,k+j}\right|\right)^{p_{mnk}/m+n+k}, \|d(x)\|_p\right]$
$= \frac{1}{m_{i-1} n_{\ell-1} k_{j-1}} \sum_{a=1}^{i_0} \sum_{b=1}^{\ell_0} \sum_{c=1}^{j_0} h_{a,b,c} A'_{a,b,c}$
$+ \frac{1}{m_{k-1} n_{\ell-1} k_{j-1}} \sum_{(i_0 < a \leq i) \bigcup (\ell_0 < b \leq \ell) \bigcup (j_0 < c \leq j)} h_{a,b,c} A'_{a,b,c}$
$\leq \frac{G'}{m_{i-1} n_{\ell-1} k_{j-1}} \sum_{a=1}^{i_0} \sum_{b=1}^{\ell_0} \sum_{c=1}^{j_0} h_{a,b,c}$
$+ \frac{1}{m_{i-1} n_{\ell-1} k_{j-1}} \sum_{(i_0 < a \leq i) \bigcup (\ell_0 < b \leq \ell) \bigcup (j_0 < c \leq \mathrm{J})} h_{a,b,c} A'_{a,b,c}$

$\leq \frac{G' m_{i_0} n_{\ell_0} k_{j_0} i_0 \ell_0 j_0}{m_{i-1} n_{\ell-1} k_{j-1}} + \frac{1}{m_{i-1} n_{\ell-1} j_{j-1}} \sum_{(i_0<a\leq i) \bigcup (\ell_0<b\leq \ell) \bigcup (j_0<c\leq j)} h_{a,b,c} A'_{a,b,c}$
$\leq \frac{G' m_{i_0} n_{\ell_0 k_{j_0}} i_0 \ell_0 j_0}{m_{i-1} n_{\ell-1} k_{j-1}}$
$+ \left(sup_{a\geq i_0 \bigcup b\geq \ell_0 \bigcup c\geq j_0} A'_{a,b,c} \right) \frac{1}{m_{i-1} n_{\ell-1} k_{j-1}} \sum_{(i_0<a\leq i) \bigcup (\ell_0<b\leq \ell) \bigcup (j_0<c\leq j)} h_{a,b,c}$
$\leq \frac{G' m_{i_0} n_{\ell_0 k_{j_0}} i_0 \ell_0 j_0}{m_{i-1} n_{\ell-1} k_{j-1}} + \frac{\epsilon}{m_{i-1} n_{\ell-1} k_{j-1}} \sum_{(i_0<a\leq i) \bigcup (\ell_0<b\leq \ell) \bigcup (j_0<c\leq j)} h_{a,b,c}$
$\leq \frac{G' m_{i_0} n_{\ell_0 k_{j_0}} i_0 \ell_0 j_0}{m_{i-1} n_{\ell-1} k_{j-1}} + \epsilon H^3.$
Since m_i, n_ℓ and k_j both approaches infinity as both p, q and t approaches infinity, it follows that

$$\frac{1}{pqt} \sum_{m=1}^{p} \sum_{n=1}^{q} \sum_{k=1}^{t} f \left[|x_{m+i,n+\ell,k+j}|^{p_{mnk}/m+n+k}, \|d(x)\|_p \right] = 0,$$

uniformly, in i, ℓ and j, with $d(x) = (d(x_1, 0), d(x_2, 0), \cdots, d(x_{n-1}, 0))$. Hence $x \in \Gamma^3_f \left(P, \|(d(x_1, 0), d(x_2, 0), \cdots, d(x_{n-1}, 0))\|_p \right)$. □

Theorem 3.5. *Let $\theta_{i,\ell,j}$ be a triple lacunary sequence then*
(i) $(x_{mnk}) \xrightarrow{P} \Gamma^3 \left(\widehat{S_{\theta_{i,\ell,j}}} \right)$
(ii)$\left(AC_{\theta_{i,\ell,j}} \right)$ *is a proper subset of* $\left(\widehat{S_{\theta_{i,\ell,j}}} \right)$
(iii) If $x \in \Lambda^3$ *and* $(x_{mnk}) \xrightarrow{P} \Gamma^3 \left(\widehat{S_{\theta_{i,\ell,j}}} \right)$ *then* $(x_{mnk}) \xrightarrow{P} \Gamma^3 \left(AC_{\theta_{i,\ell,j}} \right)$
(iv) $\Gamma^3 \left(\widehat{S_{\theta_{i,\ell,j}}} \right) \bigcap \Lambda^3 = \Gamma^3 \left[AC_{\theta_{i,,\ell,j}} \right] \bigcap \Lambda^3$.

Proof. (i) Since for all i, ℓ and j
$\left| \left\{ (m, n, k) \in I_{i,\ell,j} : (|x_{m+i,n+\ell,k+j} - 0|)^{1/m+n+k} \right\} = 0 \right| \leq$
$\sum_{m\in I_{i,\ell,j}} \sum_{n\in I_{i,\ell,j}} \sum_{k\in I_{i,\ell,j} \; and \, |x_{m+i,n+\ell,k+j}|=0} (|x_{m+i,n+\ell,k+j} - 0|)^{1/m+n+k} \leq$
$\sum_{m\in I_{i,\ell,j}} \sum_{n\in I_{i,\ell,j}} \sum_{k\in I_{i,\ell,j}} (|x_{m+i,n+\ell,k+j} - 0|)^{1/m+n+k}$, for all i, ℓ and j
$P - lim_{i,\ell,j} \frac{1}{h_{i\ell j}} \sum_{m\in I_{i,\ell,j}} \sum_{n\in I_{i,\ell,j}} \sum_{k\in I_{i,\ell,j}} (|x_{m+i,n+\ell,k+j} - 0|)^{1/m+n+k} = 0$
This implies that for all i, ℓ and j

$$P - lim_{i,\ell,j} \frac{1}{h_{i,\ell,j}} \left| \left\{ (m, n, k) \in I_{i,\ell,j} : (|x_{m+i,n+\ell,k+j} - 0|)^{1/m+n+k} = 0 \right\} \right| = 0.$$

(ii)let $x = (x_{mnk})$ be defined as follows:

$$(x_{mnk}) = \begin{pmatrix} 1 & 2 & 3 & ...\frac{\left[\sqrt[4]{h_{i,\ell,j}}\right]^{m+n+k}}{(1)!} & 0 & \dots \\ 1 & 2 & 3 & ...\frac{\left[\sqrt[4]{h_{i,\ell,j}}\right]^{m+n+k}}{(1)!} & 0 & \dots \\ . \\ . \\ . \\ 1 & 2 & 3 & ...\frac{\left[\sqrt[4]{h_{i,\ell,j}}\right]^{m+n+k}}{(1)!} & 0 & \dots \\ . \\ . \\ . \\ 0 & 0 & 0 & ...0 & 0 & \dots \\ . \\ . \\ . \end{pmatrix};$$

Here x is an trible sequence and for all i, ℓ and j
$P - lim_{i,\ell,j} \frac{1}{h_{k,\ell,j}} \left| \left\{ (m,n,k) \in I_{i,\ell,j} : \left(|x_{m+i,n+\ell,k+j} - 0| \right)^{1/m+n+k} = 0 \right\} \right| =$
$P - lim_{i,\ell,j} \frac{1}{h_{i,\ell,j}} \left(\frac{\left[\sqrt[4]{h_{i,\ell,j}}\right]^{m+n+k}}{(1)!} \right)^{1/m+n+k} = 0.$
Therefore $(x_{mnk}) \xrightarrow{P} \Gamma^3 \left(\widehat{S_{\theta_{i,\ell,j}}} \right)$. Also
$P - lim_{i,\ell,j} \frac{1}{h_{i\ell j}} \sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{i,\ell,j}} \left(|x_{m+i,n+\ell,k+j}| \right)^{1/m+n+k} =$
$P - \frac{1}{2} \left(lim_{i,\ell,j} \frac{1}{h_{i,\ell,j}} \left(\frac{\left[\sqrt[4]{h_{i,\ell,j}}\right]^{m+n+k} \left[\sqrt[4]{h_{i,\ell,j}}\right]^{m+n+k} \left[\sqrt[4]{h_{i,\ell,j}}\right]^{m+n+k}}{(1)!} \right)^{1/m+n+k} + 1 \right) =$
$\frac{1}{2}$.
Therefore $(x_{mnk}) \overset{P}{\not\to} \Gamma^3 \left(AC_{\theta_{i,\ell,j}} \right)$.
(iii) If $x \in \Lambda^3$ and $(x_{mnk}) \xrightarrow{P} \Gamma^3 \left(\widehat{S_{\theta_{i,\ell,j}}} \right)$ then $(x_{mnk}) \xrightarrow{P} \Gamma^3 \left(AC_{\theta_{i,\ell,j}} \right)$.
Suppose $x \in \Lambda^3$ then for all i, ℓ and j, $\left(|x_{m+i,n+\ell,k+j} - 0| \right)^{1/m+n+k} \leq M$ for all m, n, k. Also for given $\epsilon > 0$ and i, ℓ and j large for all i, ℓ and j we obtain the following:
$\frac{1}{h_{i\ell j}} \sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{i,\ell,j}} \left(|x_{m+i,n+\ell,k+j} - 0| \right)^{1/m+n+k} =$
$\frac{1}{h_{i\ell j}} \sum_{m \in I_{k,\ell}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{k,\ell,j} \, and \, |x_{m+i,n+\ell,k+j}| \geq 0} \left(|x_{m+i,n+\ell,k+j} - 0| \right)^{1/m+n+k} +$
$\frac{1}{h_{i\ell j}} \sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{i,\ell,j} \, and \, |x_{m+i,n+\ell,k+j}| \leq 0} \left(|x_{m+i,n+\ell,k+j} - 0| \right)^{1/m+n+k}$
$\leq \frac{M}{h_{i\ell j}} \left| \left\{ (m,n,k) \in I_{i,\ell,j} : \left(|x_{m+i,n+\ell,k+j} - 0| \right)^{1/m+n+k} \right\} = 0 \right| + \epsilon.$
Therefore $x \in \Lambda^3$ and $(x_{mnk}) \xrightarrow{P} \Gamma^3 \left(\widehat{S_{\theta_{i,\ell,j}}} \right)$ then $(x_{mnk}) \xrightarrow{P} \Gamma^3 \left(AC_{\theta_{i,\ell,j}} \right)$.
(iv) $\Gamma^3 \left(\widehat{S_{\theta_{i,\ell,j}}} \right) \bigcap \Lambda^3 = \Gamma^3 \left[AC_{\theta_{i,\ell,j}} \right] \bigcap \Lambda^3$. Follows from (i),(ii) and (iii). □

Theorem 3.6. *If f be any Musielak-Orlicz function then*
$\Gamma^3_f \left[AC_{\theta_{i,\ell,j}}, \| (d(x_1, 0), d(x_2, 0), \cdots, d(x_{n-1}, 0)) \|_p \right] \notin \Gamma^3 \left(\widehat{S_{\theta_{i,\ell,j}}} \right)$

Proof. Let $x \in \Gamma^3_f \left[AC_{\theta_{i,\ell,j}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$, for all i, ℓ and j.
Therefore we have
$\frac{1}{h_{i\ell j}} \sum_{m\in I_{i,\ell,j}} \sum_{n\in I_{i,\ell,j}} \sum_{k\in I_{i,\ell,j}}$
$f\left[|x_{m+i,n+\ell,k+j} - 0|^{1/m+n+k}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$
$\geq \frac{1}{h_{i\ell j}} \sum_{m\in I_{i,\ell,j}} \sum_{n\in I_{i,\ell,j}} \sum_{k\in I_{i,\ell,j}\ and\ |x_{m+r,n+s,k+u}|=0}$
$f\left[|x_{m+i,n+\ell,k+j} - 0|^{1/m+n+k}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] >$
$\frac{1}{h_{i\ell j}} f(0)$
$\left|\left\{(m,n,k) \in I_{i,\ell,j} : |x_{m+i,n+\ell,k+j} - 0|^{1/m+n+k}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right\} = 0\right|$.
Hence $x \notin \Gamma^3\left(\widehat{S_{\theta_{i,\ell,j}}}, \|d(x)\|_p\right)$. □

4. Competing Interests

The authors declare that there is not any conflict of interests regarding the publication of this manuscript.

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