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Multiple Solutions For a Quasilinear Schrödinger System of Kirchhoff Type With Critical Exponents

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ABSTRACT: This paper is devoted to the existence of solutions for a class of Kirchhoff type systems involving critical exponents. The proof of the main results is based on concentration compactness principle related to critical elliptic systems due to Kang [12] combined with genus theory.

Key Words: Critical exponents, Quasilinear Schrödinger equations, Kirchhoff type systems.

Contents

1. Introduction and main results

In this article, we are concerned with the multiplicity of nontrivial solutions for the following nonlocal Schrödinger system

$$\begin{cases} -\left(a+b\int_\Omega |\nabla u|^2dx\right)\Delta u - a[\Delta(u^2)]u = & \lambda F_u(x,u,v) \\ & +\eta\frac{\alpha}{\alpha+\beta}|u|^{2(\alpha-1)}uv^{2\beta} \quad \text{in} \quad \Omega \\ -\left(a+b\int_\Omega |\nabla v|^2dx\right)\Delta v - a[\Delta(v^2)]v = & \lambda F_v(x,u,v) \\ & +\eta\frac{\beta}{\alpha+\beta}u^{2\alpha}|v|^{2(\beta-1)}v \quad \text{in} \quad \Omega \\ u=v=0 & \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^N(N\geq 3)$ is a bounded smooth domain, $a,b>0$, α, $\beta>1$ with $\alpha+\beta=2^*:=\frac{2N}{N-2}$, $\nabla F=(F_u,F_v)$ is the gradient of a C^1 function $F:\overline{\Omega}\times\mathbb{R}^2\to\mathbb{R}$ with respect to (u,v).

When $a(\Delta(u^2))=a(\Delta(v^2))=0$, system (1.1) reduces to standard nonlocal problem which is related to the stationary problem of a model presented by Kirchhoff [13]. Recently, Kirchhoff type problems have been studied in many papers, we refer to [5,6,7,18,9,22,23] in which different methods have been used to get the existence and multiplicity of solutions.

On the other hand, problem (1.1) without nonlocal term arises naturally from finding the standing wave solutions for quasilinear Schrödinger equations of the form

$$-i\partial_t z = -\Delta z + V(x)\Delta z - g(|z|^2)z - \kappa\Delta(h(|z|^2))h'(|z|^2)z, \quad x\in\mathbb{R}^N, \tag{1.2}$$

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where κ is a real constant, V is a given potential, g and h are real functions. The study of This type of equations is motivated by its various applications, for example, the case $h(s) = s$ was used to model the time evolution of the condensate wave function in superfluid film, and is called the superfluid film equation in fluid mechanics by Kurihura [14]; in the case $h(s) = (1+s)^{\frac{1}{2}}$, equation (1.2) was used as a model of the self-channeling of ahigh-power ultra short laser in matter, see [2,25] and the references therein. One of the main difficulties of the quasilinear problem with nonhomogeneous term $[\Delta(u^2)]u$ is that there is no suitable space on which the energy functional is well defined. There have been several approaches used in recent years to overcome the difficulties such as minimizations [19,24], the Nehari or Pohozaev manifold [20,26], and change of variables [1,8,21,27]. The critical problems involving nonlocal operators create many difficulties in applying variational methods, these is due to the lack of compactness of the imbedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ and the Palais-Smale condition fails. In a recent paper [12], D. S. Kang establish a variant of concentration compactness principle related to critical elliptic systems, which is based on the ideas by P. L. Lions [16,17]. This result is very useful for the study of the existence of solutions for critical elliptic systems (see e.g., [11]).

Motivated by the above, our purpose is to establish the existence of a sequence of solutions for system (1.1). We will assume that the function F satisfies the following conditions.

(F_0) $F \in C^1(\overline{\Omega} \times \mathbb{R}^2)$, $F(x, 0, 0) = 0$ and $F(x, -s, -t) = F(x, s, t)$ for all $(x, s, t) \in \Omega \times \mathbb{R}^2$;

(F_1) $\lim\limits_{|(s,t)| \to +\infty} \frac{|\nabla F(x,s,t)(s,t)|}{|s|^{2\alpha}|t|^{2\beta}} = 0$ and $\limsup\limits_{|(s,t)| \to +\infty} \frac{F(x,s,t)}{|s|^{2\alpha}|t|^{2\beta}} \leq 0$ uniformly in $x \in \Omega$;

(F_2) $\lim\limits_{|(s,t)| \to 0} \frac{F(x,s,t)}{|(s,t)|^2} = +\infty$ uniformly in $x \in \Omega$.

Let $H_0^1(\Omega)$ be the usual Sobolev space defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm $||u||^2 = \int_\Omega |\nabla u|^2 dx$. Set $X = H_0^1(\Omega) \times H_0^1(\Omega)$. Then X is a Hilbert space with respect to the inner product defined by

$$\langle (u_1, v_1), (u_2, v_2) \rangle = \int_\Omega (\nabla u_1 \nabla u_2 + \nabla v_1 \nabla v_2)\, dx, \text{ for all } (u_1, v_1),\ (u_2, v_2) \in X.$$

and equipped with the norm

$$||(u, v)||_X = \left(\int_\Omega \left(|\nabla u|^2 + |\nabla v|^2 \right) dx \right)^{1/2}.$$

The energy functional $I_{\lambda,\eta} : X \to \mathbb{R}$ corresponding to system (1.1) is given by

$$\begin{aligned} I_{\lambda,\eta}(u,v) =& \frac{a}{2}\int_\Omega \left(|\nabla u|^2 + |\nabla v|^2\right) dx + \frac{b}{4}\left[\left(\int_\Omega |\nabla u|^2 dx\right)^2 + \left(\int_\Omega |\nabla v|^2 dx\right)^2\right] \\ &+ a\int_\Omega (u^2|\nabla u|^2 + v^2|\nabla v|^2) dx \\ &- \frac{\eta}{2(\alpha+\beta)}\int_\Omega |u|^{2\alpha}|v|^{2\beta} dx - \lambda\int_\Omega F(x,u,v)dx \\ =& \frac{a}{2}\int_\Omega \left((1+2u^2)|\nabla u|^2 + (1+2v^2)|\nabla v|^2\right) dx \\ &+ \frac{b}{4}\left[\left(\int_\Omega |\nabla u|^2 dx\right)^2 + \left(\int_\Omega |\nabla v|^2 dx\right)^2\right] \\ &- \frac{\eta}{2(2^*)}\int_\Omega |u|^{2\alpha}|v|^{2\beta} dx - \lambda\int_\Omega F(x,u,v)dx. \end{aligned}$$

Note that a major difficulty associated with (1.1) is that the functional $I_{\lambda,\eta}$ is not well defined in general, for instance, in X. To overcome this difficulty, we use an argument developed by Colin and Jeanjean [8]. We make the changing of variables $(u,v) = (f(w), f(z))$, where f is given by

$$f'(t) = \frac{1}{\sqrt{1+2f^2(t)}} \text{ for } t \in [0,+\infty) \quad \text{and} \quad f(t) = -f(-t) \text{ for } t \in (-\infty, 0].$$

Some properties of the function f are given in the following lemma.

Lemma 1.1. *Concerning the function $f(t)$ and its derivative satisfy the following properties:*

(f_1) *f is uniquely defined, C^∞ and invertible;*

(f_2) *$|f'(t)| \le 1$ for all $t \in \mathbb{R}$;*

(f_3) *$|f(t)| \le |t|$ for all $t \in \mathbb{R}$;*

(f_4) *$\frac{f(t)}{t} \to 1$ as $t \to 0$;*

(f_5) *$\frac{f(t)}{\sqrt{t}} \to 2^{\frac{1}{4}}$ as $t \to +\infty$;*

(f_6) *$\frac{f(t)}{2} \le tf'(t) \le f(t)$ for all $t \ge 0$;*

(f_7) *$\frac{f^2(t)}{2} \le tf'(t)f(t) \le f^2(t)$ for all $t \in \mathbb{R}$;*

(f_8) *$|f(t)| \le 2^{\frac{1}{4}}|t|^{\frac{1}{2}}$ for all $t \in \mathbb{R}$;*

(f_9) *The function f^2 is strictly convex;*

(f_{10}) *There exists a positive constant $C > 0$ such that*

$$|f(t)| \geq \begin{cases} C|t|, & |t| \leq 1, \\ C|t|^{\frac{1}{2}}, & |t| \geq 1; \end{cases}$$

(f_{11}) $|f(t)f'(t)| \leq \frac{1}{\sqrt{2}}$ *for all $t \in \mathbb{R}$.*

So, by the change of variables, from $I_{\lambda,\eta}$, we can define the following functional

$$\begin{aligned}\Phi_{\lambda,\eta}(w,z) :=& \frac{a}{2}\int_\Omega \left(|\nabla w|^2 + |\nabla z|^2\right) dx \\ &+ \frac{b}{4}\left[\left(\int_\Omega |f'(w)|^2|\nabla w|^2 dx\right)^2 + \left(\int_\Omega |f'(z)|^2|\nabla z|^2 dx\right)^2\right] \\ &- \frac{\eta}{2(2^*)}\int_\Omega |f(w)|^{2\alpha}|f(z)|^{2\beta} dx - \lambda\int_\Omega F(x, f(w), f(z)) dx.\end{aligned}$$

Then $\Phi_{\lambda,\eta}$ is well defined. In view of assumptions, it standard to see that $\Phi_{\lambda,\eta} \in C^1(X, \mathbb{R})$ and its derivative at $(\varphi, \psi) \in X$ is given by

$$\begin{aligned}&\langle \Phi'_{\lambda,\eta}(w,z), (\varphi,\psi)\rangle = a\int_\Omega (\nabla w\nabla\varphi + \nabla z\nabla\psi) dx \\ &+ b\left(\int_\Omega \frac{|\nabla w|^2}{1+2f^2(w)}\right)\int_\Omega \frac{(1+2f^2(w))\nabla w\nabla\varphi - 2|\nabla w|^2 f(w)f'(w)\varphi}{(1+2f^2(w))^2} dx \\ &+ b\left(\int_\Omega \frac{|\nabla z|^2}{1+2f^2(z)}\right)\int_\Omega \frac{(1+2f^2(z))\nabla z\nabla\psi - 2|\nabla w|^2 f(z)f'(z)\psi}{(1+2f^2(z))^2} dx \\ &- \frac{\eta\alpha}{2^*}\int_\Omega |f(z)|^{2\beta}|f(w)|^{2(\alpha-1)}f(w)f'(w)\varphi - \frac{\eta\beta}{2^*}\int_\Omega |f(w)|^{2\alpha}|f(z)|^{2(\beta-1)}f(z)f'(z)\psi \\ &- \lambda\int_\Omega [F_u(x, f(w), f(z))f'(w)\varphi + F_v(x, f(w), f(z))f'(z)\psi] dx.\end{aligned}$$

for all $(\varphi, \psi) \in X$. Furthermore, if (w, z) is a critical point of $\Phi_{\lambda,\eta}$, then (w, z) is a weak solution of the following system

$$\begin{cases} -a\Delta w - b\left(\int_\Omega |f'(w)|^2|\nabla w|^2 dx\right)\Lambda[w] &= \lambda F_u(x, f(w), f(z))f'(w) \\ & \quad + \frac{\eta\alpha}{2^*}|f(w)|^{2(\alpha-1)}f(w)f'(w)|f(z)|^{2\beta} \\ -a\Delta z - b\left(\int_\Omega |f'(z)|^2|\nabla z|^2 dx\right)\Lambda[z] &= \lambda F_v(x, f(w), f(z))f'(z) \\ & \quad + \frac{\eta\beta}{2^*}|f(z)|^{2(\beta-1)}f(z)f'(z)|f(w)|^{2\alpha}, \end{cases} \tag{1.3}$$

where

$$\Lambda[w] := |f'(w)|^2\Delta w + 2f'(w)f''(w)|\nabla w|^2 + 2f(w)|f'(w)|^5|\nabla w|^2,$$

and therefore $(u, v) = (f(w), f(z))$ is a solution of problem (1.1).

The main results of this paper are the following theorems.

Theorem 1.2. *Assume that* $(F_0)-(F_2)$ *hold. Then*

(i) *for any* $\eta>0$ *there exists* $\lambda_*>0$ *such that for all* $\lambda\in(0,\lambda_*)$, *system* (1.3) *admits a sequence of nontrivial solutions* $\{(w_n,z_n)\}$ *such that* $(w_n,z_n)\to 0$ *as* $n\to+\infty$;

(ii) *for any* $\lambda>0$ *there exists* $\eta_*>0$ *such that for all* $\eta\in(0,\eta_*)$, *system* (1.3) *admits a sequence of nontrivial solutions* $\{(w_n,z_n)\}$ *such that* $(w_n,z_n)\to 0$ *as* $n\to+\infty$.

Theorem 1.3. *Assume that* $(F_0)-(F_2)$ *hold. Then*

(i) *for any* $\eta>0$ *there exists* $\lambda_*>0$ *such that for all* $\lambda\in(0,\lambda_*)$, *system* (1.1) *admits a sequence of nontrivial solutions* $\{(u_n,v_n)\}$ *such that* $(u_n,v_n)\to 0$ *as* $n\to+\infty$;

(ii) *for any* $\lambda>0$ *there exists* $\eta_*>0$ *such that for all* $\eta\in(0,\eta_*)$, *system* (1.1) *admits a sequence of nontrivial solutions* $\{(u_n,v_n)\}$ *such that* $(u_n,v_n)\to 0$ *as* $n\to+\infty$.

2. Preliminary lemmas and proof of main results

We start with stating a few known results and giving a preliminary lemmas which we need in our argument. First we recall a variant of concentration compactness principle related to critical elliptic systems of D. S Kang [12].

Lemma 2.1. *Let* $\Omega\subset\mathbb{R}^N$ *be a bounded domain and* $\alpha,\beta>1$ *with* $\alpha+\beta=2^*$. *Let* $(u_n,v_n)\rightharpoonup(u,v)$ *in* X, $|\nabla u_n|^2+|\nabla v_n|^2\rightharpoonup|\nabla u|^2+|\nabla v|^2+\mu$ *and* $|u_n|^\alpha|v_n|^\beta\rightharpoonup|u|^\alpha|v|^\beta+\nu$ *in the sense of measures, where* μ *and* ν *are nonnegative bounded measures on* $\mathbb{R}^N$. *Then there exist an at most countable set* J *and families* $\{x_j\}_{j\in J}\subset\mathbb{R}^N$ *and* $\{\mu_j\}_{j\in J}$, $\{\nu_j\}_{j\in J}\subset[0,+\infty)$ *such that*

$$\mu\geq\sum_{j\in J}\mu_j\delta_{x_j},\quad \nu=\sum_{j\in J}\nu_j\delta_{x_j},\quad \nu_j^{\frac{2}{2^*}}S_{\alpha,\beta}\leq\mu_j,\ \forall j\in J,$$

where δ_{x_j} *is the Dirac mass at* x_j *and* $S_{\alpha,\beta}$ *is given by*

$$S_{\alpha,\beta}=\inf_{(u,v)\in X\setminus\{0\}}\frac{\int_\Omega(|\nabla u|^2+|\nabla v|^2)dx}{\left(\int_\Omega|u|^\alpha|v|^\beta dx\right)^{\frac{2}{\alpha+\beta}}}.$$

From $(F_0)-(F_1)$, for each $\varepsilon>0$, there exists $C(\varepsilon)>0$ such that

$$|\nabla F(x,s,t)(s,t)|\leq\varepsilon|s|^{2\alpha}|t|^{2\beta}+C(\varepsilon),\ \text{for all }(x,s,t)\in\Omega\times\mathbb{R}^2 \tag{2.1}$$

and

$$F(x,s,t)\leq\varepsilon|s|^{2\alpha}|t|^{2\beta}+C(\varepsilon),\ \text{for all }(x,s,t)\in\Omega\times\mathbb{R}^2. \tag{2.2}$$

Then, by (2.1) and (2.2), for any $\varepsilon>0$, we can find $C_0(\varepsilon)>0$ such that

$$F(x,s,t)-\frac{1}{4}[F_u(x,s,t)s+F_v(x,s,t)t]\leq\varepsilon|s|^{2\alpha}|t|^{2\beta}+C_0(\varepsilon),\ \text{for all }(x,s,t)\in\Omega\times\mathbb{R}^2. \tag{2.3}$$

Lemma 2.2. *Assume that* $(F_0)-(F_1)$ *hold. Then for any* $\lambda, \eta > 0$, *the functional* $\Phi_{\lambda,\eta}$ *satisfies the* $(PS)_c$ *condition for all* $c \in \left(-\infty, \frac{\eta}{4N}\left(\frac{aS_{\alpha,\beta}}{2\eta}\right)^{\frac{N}{2}} - \lambda|\Omega|C_0(\varepsilon)\right)$, *where* $\varepsilon = \frac{\eta}{4N\lambda}$.

Proof: Let $\{(w_n, z_n)\} \subset X$ be a sequence such that

$$\Phi_{\lambda,\eta}(w_n, z_n) \to c \quad \text{and} \quad \Phi'_{\lambda,\eta}(w_n, z_n) \to 0 \text{ in } X^*, \text{as } n \to +\infty. \tag{2.4}$$

Let $\widehat{w}_n := \sqrt{1+2f^2(w_n)}f(w_n)$ and $\widehat{z}_n := \sqrt{1+2f^2(z_n)}f(z_n)$. We have $(\widehat{w}_n, \widehat{z}_n) \in X$ and

$$|\nabla \widehat{w}_n| = \left(1 + \frac{2f^2(w_n)}{1+2f^2(w_n)}\right)|\nabla w_n| \le 2|\nabla w_n|,$$

$$|\nabla \widehat{z}_n| = \left(1 + \frac{2f^2(z_n)}{1+2f^2(z_n)}\right)|\nabla z_n| \le 2|\nabla z_n|.$$

Thus

$$||(\widehat{w}_n, \widehat{z}_n)||_X \le C_1||(w_n, z_n)||_X. \tag{2.5}$$

On the other hand, we have

$$\begin{aligned}
&\langle \Phi'_{\lambda,\eta}(w_n, z_n), (\widehat{w}_n, \widehat{z}_n)\rangle \\
&= a\int_\Omega \left[\left(1 + \frac{2f^2(w_n)}{1+2f^2(w_n)}\right)|\nabla w_n|^2 + \left(1 + \frac{2f^2(z_n)}{1+2f^2(z_n)}\right)|\nabla z_n|^2\right]dx \\
&+ b\left[\left(\int_\Omega \frac{|\nabla w_n|^2}{1+2f^2(w_n)}\right)^2 + \left(\int_\Omega \frac{|\nabla z_n|^2}{1+2f^2(z_n)}\right)^2\right] - \eta\int_\Omega |f(w_n)|^{2\beta}|f(z_n)|^{2\alpha} \\
&- \lambda\int_\Omega [F_u(x, f(w_n), f(z_n))f(w_n) + F_v(x, f(w_n), f(z_n))f(z_n)]dx.
\end{aligned}$$

By (2.4)-(2.5), for n large enough

$$\begin{aligned}
&1 + c + ||(w_n, z_n)||_X \ge \Phi_{\lambda,\eta}(w_n, z_n) - \frac{1}{4}\langle \Phi'_{\lambda,\eta}(w_n, z_n), (\widehat{w}_n, \widehat{z}_n)\rangle \\
&= \frac{a}{4}\int_\Omega \left(\frac{|\nabla w_n|^2}{1+2f^2(w_n)} + \frac{|\nabla z_n|^2}{1+2f^2(z_n)}\right)dx \\
&+ \eta\left(\frac{1}{4} - \frac{1}{2(2^*)}\right)\int_\Omega |f(w_n)|^{2\alpha}|f(z_n)|^{2\beta}dx - \lambda\int_\Omega F(x, f(w_n), f(z_n))dx \\
&- \frac{\lambda}{4}\int_\Omega [F_u(x, f(w_n), f(z_n))f(w_n) + F_v(x, f(w_n), f(z_n))f(z_n)]dx \\
&\ge \frac{\eta}{2N}\int_\Omega |f(w_n)|^{2\alpha}|f(z_n)|^{2\beta}dx - \lambda\int_\Omega F(x, f(w_n), f(z_n))dx \\
&- \frac{\lambda}{4}\int_\Omega [F_u(x, f(w_n), f(z_n))f(w_n) + F_v(x, f(w_n), f(z_n))f(z_n)]dx.
\end{aligned}$$

It follows from (2.3) that

$$\frac{\eta}{2N}\int_\Omega |f(w_n)|^{2\alpha}|f(z_n)|^{2\beta}dx \leq \lambda\varepsilon\int_\Omega |f(w_n)|^{2\alpha}||f(z_n)|^{2\beta}dx + \lambda|\Omega|C_0(\varepsilon) + 1 + c + ||(w_n, z_n)||_X.$$

By choosing $\varepsilon = \frac{\eta}{4N\lambda}$, we obtain

$$\frac{\eta}{4N}\int_\Omega |f(w_n)|^{2\alpha}|f(z_n)|^{2\beta}dx \leq \lambda|\Omega|C_0\left(\frac{\eta}{4N\lambda}\right) + 1 + c + ||(w_n, z_n)||_X.$$

Combine this with (2.2), for n large enough

$$\begin{aligned}
1 + c \geq& \Phi_{\lambda,\eta}(w_n, z_n)\\
=&\frac{a}{2}\int_\Omega \left[\left(1 + \frac{2f^2(w_n)}{1+2f^2(w_n)}\right)|\nabla w_n|^2 + \left(1 + \frac{2f^2(z_n)}{1+2f^2(z_n)}\right)|\nabla z_n|^2\right]dx\\
&- \frac{\eta}{2(2^*)}\int_\Omega |f(w_n)|^{2\alpha}|f(z_n)|^{2\beta}dx - \lambda\int_\Omega F(x, f(w_n), f(z_n))dx\\
\geq&\frac{a}{2}||(w_n, z_n)||_X^2 - \left(\frac{\eta}{2(2^*)} + \varepsilon\lambda\right)\int_\Omega |f(w_n)|^{2\alpha}|f(z_n)|^{2\beta}dx - \lambda|\Omega|C(\varepsilon)\\
\geq&\frac{a}{2}||(w_n, z_n)||_X^2 - (N-1)\left(\lambda|\Omega|C_0\left(\frac{\eta}{4N\lambda}\right) + 1 + c + ||(w_n, z_n||_X\right)\\
&- \lambda|\Omega|C\left(\frac{\eta}{4N\lambda}\right).
\end{aligned} \tag{2.6}$$

This last inequality shows that $\{(w_n, z_n)\}$ is bounded in X. Therefore $\{w_n\}$ and $\{z_n\}$ are bounded in $H_0^1(\Omega)$ and hence $\{(f^2(w_n), f^2(z_n))\}$ is bounded in X. Then passing to a subsequence if necessary, we may assume that

$$\begin{cases} w_n \rightharpoonup w \text{ in } H_0^1(\Omega)\\ w_n \to w \text{ a.e. in } \Omega\\ z_n \rightharpoonup z \text{ in } H_0^1(\Omega)\\ z_n \to z \text{ a.e. in } \Omega. \end{cases} \tag{2.7}$$

By using the fact that f is continuous, it follows that $(f^2(w_n), f^2(z_n)) \to (f^2(w), f^2(z))$ a.e. in Ω. Since $\{(f^2(w_n), f^2(z_n))\}$ is bounded in X, we deduce that $(f^2(w_n), f^2(z_n)) \rightharpoonup (f^2(w), f^2(z))$ in X and

$$\begin{cases} |\nabla f^2(w_n)|^2 + |\nabla f^2(z_n)|^2 \rightharpoonup |\nabla f^2(w)|^2 + |\nabla f^2(z)|^2 + \mu\\ |f(w_n)|^{2\alpha}|f(z_n)|^{2\beta} \rightharpoonup |f(w)|^{2\alpha}|f(z)|^{2\beta} + \nu \end{cases} \tag{2.8}$$

in the sense of measures, where μ and ν are nonnegative bounded measures on $\mathbb{R}^N$. According to Lemma 2.1, there exist an at most countable set J and families $\{x_j\}_{j\in J} \subset \mathbb{R}^N$ and $\{\mu_j\}_{j\in J}$, $\{\nu_j\}_{j\in J} \subset [0, +\infty)$ such that

$$\mu \geq \sum_{j\in J}\mu_j\delta_{x_j}, \quad \nu = \sum_{j\in J}\nu_j\delta_{x_j}, \quad \nu_j^{\frac{2}{2^*}}S_{\alpha,\beta} \leq \mu_j,\ \forall j \in J. \tag{2.9}$$

Let $\phi \in C_0^\infty\left(\mathbb{R}^N\right)$ such that

$$0 \leq \phi \leq 1, \quad \phi \equiv 1 \text{ in } B(0,1), \quad \phi = 0 \text{ in } \mathbb{R}^N \backslash B(0,2),\ |\nabla \phi|_\infty \leq 2.$$

For $\varepsilon > 0$ and $j \in J$ denote

$$\phi_\varepsilon^j(x) := \phi\left(\frac{x - x_j}{\varepsilon}\right), \text{ for all}\, x \in \mathbb{R}^N.$$

By (2.5), $\left(\widehat{w}_n \phi_\varepsilon^j, \widehat{z}_n \phi_\varepsilon^j\right)$ is bounded in X and therefore

$$\left\langle \Phi'_{\lambda,\eta}(w_n, z_n), \left(\widehat{w}_n \phi_\varepsilon^j, \widehat{z}_n \phi_\varepsilon^j\right)\right\rangle \underset{n\to+\infty}{\longrightarrow} 0.$$

Thus

$$\begin{aligned}
o_n(1) &- a\int_\Omega \left(\sqrt{1+2f^2(w_n)}f(w_n)\nabla w_n + \sqrt{1+2f^2(z_n)}f(z_n)\nabla z_n\right)\nabla\phi_\varepsilon^j dx \\
&- b\left(\int_\Omega \frac{|\nabla w_n|^2}{1+2f^2(w_n)}dx\right)\int_\Omega \frac{f(w_n)\nabla w_n \nabla \phi_\varepsilon^j}{\sqrt{1+2f^2(w_n)}}dx \\
&- b\left(\int_\Omega \frac{|\nabla z_n|^2}{1+2f^2(z_n)}dx\right)\int_\Omega \frac{f(z_n)\nabla z_n \nabla \phi_\varepsilon^j}{\sqrt{1+2f^2(z_n)}}dx \\
&+ \lambda\int_\Omega [F_u(x, f(w_n), f(z_n))f(w_n) + F_v(x, f(w_n), f(z_n))f(z_n)]\phi_\varepsilon^j dx \\
=a&\int_\Omega \left[\left(1+\frac{2f^2(w_n)}{1+2f^2(w_n)}\right)|\nabla w_n|^2 + \left(1+\frac{2f^2(z_n)}{1+2f^2(z_n)}\right)|\nabla z_n|^2\right]\phi_\varepsilon^j dx \\
&+ b\left(\int_\Omega \frac{|\nabla w_n|^2}{1+2f^2(w_n)}dx\right)\int_\Omega \frac{|\nabla w_n|^2\phi_\varepsilon^j}{1+2f^2(w_n)}dx \\
&+ b\left(\int_\Omega \frac{|\nabla z_n|^2}{1+2f^2(z_n)}dx\right)\int_\Omega \frac{|\nabla z_n|^2\phi_\varepsilon^j}{1+2f^2(z_n)}dx - \eta\int_\Omega |f(w_n)|^{2\alpha}|f(z_n)|^{2\beta}\phi_\varepsilon^j dx.
\end{aligned} \tag{2.10}$$

In view of Lemma 1.1 (f_5), Hölder's inequality and Lebesgue's dominated convergence theorem,

$$\begin{aligned}
&\limsup_{n\to+\infty}\left|\int_\Omega \sqrt{1+2f^2(w_n)}f(w_n)\nabla w_n \nabla\phi_\varepsilon^j dx\right| \\
&\leq C_2 \limsup_{n\to+\infty}\int_\Omega |w_n \nabla w_n \nabla \phi_\varepsilon^j| dx \\
&\leq C_2 \limsup_{n\to+\infty}\left(\int_\Omega |\nabla w_n|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega |w_n\nabla\phi_\varepsilon^j|^2 dx\right)^{\frac{1}{2}} \\
&\leq C_3\left(\int_{B(x_j,2\varepsilon)} |w\nabla\phi_\varepsilon^j|^2 dx\right)^{\frac{1}{2}}
\end{aligned}$$

$$\leq C_4 \left(\int_{B(x_j, 2\varepsilon)} |w|^{2^*} dx \right)^{\frac{1}{2^*}} \underset{\varepsilon \to 0}{\longrightarrow} 0.$$

Hence, up to subsequence

$$\lim_{\varepsilon \to 0} \lim_{n \to +\infty} \int_\Omega \sqrt{1 + 2f^2(w_n)} f(w_n) \nabla w_n \nabla \phi_\varepsilon^j dx = 0, \tag{2.11}$$

and we also have

$$\lim_{\varepsilon \to 0} \lim_{n \to +\infty} \int_\Omega \sqrt{1 + 2f^2(z_n)} f(z_n) \nabla z_n \nabla \phi_\varepsilon^j dx = 0. \tag{2.12}$$

In the similar way, we get

$$\lim_{\varepsilon \to 0} \lim_{n \to +\infty} \left(\int_\Omega \frac{|\nabla w_n|^2}{1 + 2f^2(w_n)} dx \right) \int_\Omega \frac{f(w_n) \nabla w_n \nabla \phi_\varepsilon^j}{\sqrt{1 + 2f^2(w_n)}} dx = 0, \tag{2.13}$$

$$\lim_{\varepsilon \to 0} \lim_{n \to +\infty} \left(\int_\Omega \frac{|\nabla z_n|^2}{1 + 2f^2(z_n)} dx \right) \int_\Omega \frac{f(z_n) \nabla z_n \nabla \phi_\varepsilon^j}{\sqrt{1 + 2f^2(z_n)}} dx = 0. \tag{2.14}$$

Since $f^2(w_n)$ and $f^2(z_n)$ are bounded in $H_0^1(\Omega)$, Hölder's inequality yields

$$\int_\Omega |f(w_n)|^{2\alpha} |f(z_n)|^{2\beta} dx \leq \left(\int_\Omega |f(w_n)|^{2(2^*)} dx \right)^{\frac{\alpha}{2^*}} \left(\int_\Omega |f(w_n)|^{2(2^*)} dx \right)^{\frac{\beta}{2^*}} \leq C_5. \tag{2.15}$$

Furthermore, by the continuity of F_u and F_v, we have

$$\lim_{n \to +\infty} F_u(x, f(w_n), f(z_n)) f(w_n) + F_v(x, f(w_n), f(z_n)) f(z_n) =$$
$$F_u(x, f(w), f(z)) f(w) + F_v(x, f(w), f(z)) f(z) \ a.e. \text{ in } \Omega.$$

Therefore, by (2.1), (2.15) and Egorov's theorem, we obtain

$$\lim_{n \to +\infty} \int_\Omega [F_u(x, f(w_n), f(z_n)) f(w_n) + F_v(x, f(w_n), f(z_n)) f(z_n)] dx$$
$$= \int_\Omega F_u(x, f(w), f(z)) f(w) + F_v(x, f(w), f(z)) f(z) dx, \tag{2.16}$$

and hence

$$\lim_{\varepsilon \to 0} \lim_{n \to +\infty} \int_\Omega [F_u(x, f(w_n), f(z_n)) f(w_n) + F_v(x, f(w_n), f(z_n)) f(z_n)] \phi_\varepsilon^j dx = 0. \tag{2.17}$$

Tacking account that

$$\frac{1}{2} |\nabla f^2(w_n)|^2 \leq \left(1 + \frac{2f^2(w_n)}{1 + 2f^2(w_n)} \right) |\nabla w_n|^2,$$

$$\frac{1}{2}|\nabla f^2(z_n)|^2 \leq \left(1+\frac{2f^2(z_n)}{1+2f^2(z_n)}\right)|\nabla z_n|^2,$$

it follows from (2.8)-(2.14) and (2.17) that

$$\begin{aligned}
0 = &\lim_{\varepsilon\to 0}\lim_{n\to+\infty}\Big[a\int_\Omega\Big[\left(1+\frac{2f^2(w_n)}{1+2f^2(w_n)}\right)|\nabla w_n|^2 \\
&+\left(1+\frac{2f^2(z_n)}{1+2f^2(z_n)}\right)|\nabla z_n|^2\Big]\phi_\varepsilon^j dx \\
&+b\left(\int_\Omega \frac{|\nabla w_n|^2}{1+2f^2(w_n)}dx\right)\int_\Omega\frac{|\nabla w_n|^2\phi_\varepsilon^j}{1+2f^2(w_n)}dx \\
&+b\left(\int_\Omega \frac{|\nabla z_n|^2}{1+2f^2(z_n)}dx\right)\int_\Omega\frac{|\nabla z_n|^2\phi_\varepsilon^j}{1+2f^2(z_n)}dx-\eta\int_\Omega |f(w_n)|^{2\alpha}|f(z_n)|^{2\beta}\phi_\varepsilon^j dx\Big] \\
\geq &\lim_{\varepsilon\to 0}\lim_{n\to+\infty}\Big[\frac{a}{2}\int_\Omega\left(|\nabla f^2(w_n)|^2+|\nabla f^2(z_n)|^2\right)\phi_\varepsilon^j dx \\
&-\eta\int_\Omega |f(w_n)|^{2\alpha}|f(z_n)|^{2\beta}\phi_\varepsilon^j dx\Big] \\
\geq &\frac{a}{2}\mu_j-\eta\nu_j.
\end{aligned}$$

By (2.9), we conclude that

$$\nu_j \geq \left(\frac{aS_{\alpha,\beta}}{2\eta}\right)^{\frac{N}{2}} \quad \text{or} \quad \nu_j = 0. \tag{2.18}$$

Suppose by contradiction that $\nu_j \geq \left(\frac{aS_{\alpha,\beta}}{2\eta}\right)^{\frac{N}{2}}$ for some $j\in J$. Then, by (2.3) with $\varepsilon = \frac{\eta}{4N\lambda}$, (2.8)-(2.9) and using the fact that $0\leq \phi_\varepsilon^j\leq 1$,

$$\begin{aligned}
c = &\lim_{n\to+\infty}\left(\Phi_{\lambda,\eta}(w_n,z_n)-\frac{1}{4}\langle \Phi'_{\lambda,\eta}(w_n,z_n),(\widehat{w}_n,\widehat{z}_n)\rangle\right) \\
\geq &\left(\frac{\eta}{2N}-\lambda\varepsilon\right)\lim_{n\to+\infty}\int_\Omega |f(w_n)|^{2\alpha}|f(z_n)|^{2\beta}dx-\lambda|\Omega|C_0(\varepsilon) \\
\geq &\frac{\eta}{4N}\lim_{n\to+\infty}\int_\Omega |f(w_n)|^{2\alpha}|f(z_n)|^{2\beta}\phi_\varepsilon^j dx-\lambda|\Omega|C_0\left(\frac{\eta}{4N\lambda}\right) \\
\geq &\frac{\eta}{4N}\int_\Omega |f(w)|^{2\alpha}|f(z)|^{2\beta}\phi_\varepsilon^j dx+\frac{\eta}{4N}\nu_j-\lambda|\Omega|C_0\left(\frac{\eta}{4N\lambda}\right) \\
\geq &\frac{\eta}{4N}\left(\frac{aS_{\alpha,\beta}}{2\eta}\right)^{\frac{N}{2}}-\lambda|\Omega|C_0\left(\frac{\eta}{4N\lambda}\right),
\end{aligned}$$

which is impossible. Therefore $\nu_j = 0$ and hence

$$\lim_{n\to+\infty}\int_\Omega |f(w_n)|^{2\alpha}|f(z_n)|^{2\beta}dx = \int_\Omega |f(w)|^{2\alpha}|f(z)|^{2\beta}dx. \tag{2.19}$$

By the weak lower semicontinuity of the norm and $f \in C^{\infty}$, we entail that

$$\liminf_{n\to+\infty} \int_\Omega \frac{|\nabla w_n|^2}{1+2f^2(w_n)}dx \geq \int_\Omega \frac{|\nabla w|^2}{1+2f^2(w)}dx,$$
$$\liminf_{n\to+\infty} \int_\Omega \frac{f^2(w_n)|\nabla w_n|^2}{1+2f^2(w_n)}dx \geq \int_\Omega \frac{f^2(w)|\nabla w|^2}{1+2f^2(w)}dx,$$
$$\liminf_{n\to+\infty} \int_\Omega \frac{|\nabla z_n|^2}{1+2f^2(z_n)}dx \geq \int_\Omega \frac{|\nabla z|^2}{1+2f^2(z)}dx,$$
$$\liminf_{n\to+\infty} \int_\Omega \frac{f^2(z_n)|\nabla z_n|^2}{1+2f^2(z_n)}dx \geq \int_\Omega \frac{f^2(z)|\nabla z|^2}{1+2f^2(z)}dx. \tag{2.20}$$

It follows from (2.16) and (2.19)-(2.20) that

$$\begin{aligned} 0 =& \lim_{n\to+\infty} \langle \Phi'_{\lambda,\eta}(w_n, z_n), (\widehat{w}_n, \widehat{z}_n)\rangle \\ \geq& a \liminf_{n\to+\infty} ||(w_n, z_n)||_X^2 + a \int_\Omega \left[\frac{2f^2(w)}{1+2f^2(w)}|\nabla w|^2 + \frac{2f^2(z)}{1+2f^2(z)}|\nabla z|^2\right] dx \\ &+ b\left(\int_\Omega \frac{|\nabla w|^2}{1+2f^2(w)}dx\right)^2 + b\left(\int_\Omega \frac{|\nabla z|^2}{1+2f^2(z)}dx\right)^2 - \eta \int_\Omega |f(w)|^{2\alpha}|f(z)|^{2\beta}dx \\ &- \lambda \int_\Omega (F_u(x, f(w), f(z))f(w) - F_v(x, f(w), f(z))f(z))dx. \end{aligned} \tag{2.21}$$

On the other hand, up to subsequence, Brezis-Lieb's Lemma [3] leads to

$$\liminf_{n\to+\infty} ||(w_n, z_n)||_X^2 = \lim_{n\to+\infty} ||(w_n, z_n)||_X^2 = \lim_{n\to+\infty} ||(w_n-w, z_n-z)||_X^2 + ||(w, z)||_X^2.$$

Combining this with (2.21), we obtain

$$\begin{aligned} 0 \geq& a \lim_{n\to+\infty} ||(w_n - w, z_n - z)||_X^2 + a||(w, z)||_X^2 \\ &+ a \int_\Omega \left[\frac{2f^2(w)}{1+2f^2(w)}|\nabla w|^2 + \frac{2f^2(z)}{1+2f^2(z)}|\nabla z|^2\right] dx \\ &+ b\left(\int_\Omega \frac{|\nabla w|^2}{1+2f^2(w)}dx\right)^2 + b\left(\int_\Omega \frac{|\nabla z|^2}{1+2f^2(z)}dx\right)^2 - \eta \int_\Omega |f(w)|^{2\alpha}|f(z)|^{2\beta}dx \\ &- \lambda \int_\Omega (F_u(x, f(w), f(z))f(w) - F_v(x, f(w), f(z))f(z))dx \\ =& \lim_{n\to+\infty} ||(w_n - w, z_n - z)||_X^2 + \langle \Phi_{\lambda,\eta}(w, z), (\widehat{w}, \widehat{z})\rangle, \end{aligned} \tag{2.22}$$

where $\widehat{w} := \sqrt{1+2f^2(w)}f(w)$ and $\widehat{z} := \sqrt{1+2f^2(z)}f(z)$. Using the same arguments as above, we can prove that

$$0 = \lim_{n\to+\infty} \langle \Phi'_{\lambda,\eta}(w_n, z_n), (\varphi, \psi)\rangle = \langle \Phi'_{\lambda,\eta}(w, z), (\varphi, \psi)\rangle \ \forall(\varphi, \psi) \in X.$$

From this and (2.22), we deduce that $(w_n, z_n) \to (w, z)$ strongly in X. This completes the proof of Lemma 2.2. □

Now we use minimax procedure to prove Theorem 1.2. For a Banach space X, let

$$\Sigma = \{E \subset X\backslash\{0\} : E \text{ is closed in } X \text{ and symmetric with respect to the origin}\}.$$

For each $E \in \Sigma$, define

$$\gamma(E) = \inf\{k \in \mathbb{N} : \exists \varphi \in C(E, \mathbb{R}^k\backslash\{0\}),\ \varphi(x) = -\varphi(-x)\}.$$

If there is nomapping φ as above for any $k \in \mathbb{N}$, then $\gamma(E) = +\infty$. Set

$$\Sigma_k = \{E \in \Sigma : \gamma(E) \geq k\}.$$

This next proposition is a version of the symmetric mountain-pass lemma [10].

Proposition 2.3. *Let X be an infinite dimensional space and $\Phi \in C^1(X, \mathbb{R})$ and assume the following assertions holds.*

- (i) *Φ is even, $\Phi(0) = 0$, bounded from below and satisfies the (PS_c) condition for $c < \widetilde{c}$, for some $\widetilde{c} > 0$;*
- (ii) *For each $k \in \mathbb{N}$ there exists $E_k \in \Sigma_k$ such that $\sup_{u \in E_k} \Phi(u) < 0$.*

Then, either (R_1) or (R_2) below holds.

(R_1) *There exists a sequence $\{u_k\}$ such that $\Phi'(u_k) = 0$, $\Phi(u_k) < 0$ and $u_k \to 0$;*
(R_2) *There exist two sequences $\{u_k\}$ and $\{v_k\}$ such that $\Phi'(u_k) = 0$, $\Phi(u_k) = 0$, $u_k \neq 0$, $u_k \to 0, \Phi'(v_k) = 0$, $\Phi(v_k) < 0$ and $\{v_k\}$ converges to a non-zero limit.*

Now, choosing $\varepsilon = \frac{\eta}{2(2^*)\lambda}$, by (2.2) and Young's inequality, we have

$$\begin{aligned}
&\Phi_{\lambda,\eta}(w,z) \\
&\geq \frac{a}{2}\int_\Omega \left(|\nabla w|^2 + |\nabla z|^2\right) dx - \left(\frac{\eta}{2(2^*)} + \lambda\varepsilon\right)\int_\Omega |f(w)|^{2\alpha}|f(z)|^{2\beta} dx - \lambda C(\varepsilon)|\Omega| \\
&= \frac{a}{2}\int_\Omega \left(|\nabla w|^2 + |\nabla z|^2\right) dx - \frac{\eta}{2^*}\int_\Omega |f(w)|^{2\alpha}|f(z)|^{2\beta} dx - \lambda C(\varepsilon)|\Omega| \\
&\geq \frac{a}{2}||(w,z)||_X^2 - \frac{\eta}{2^*}\left(\frac{\alpha}{2^*}\int_\Omega |f(w)|^{2(2^*)} dx + \frac{\beta}{2^*}\int_\Omega |f(z)|^{2(2^*)} dx\right) - \lambda C(\varepsilon)|\Omega|
\end{aligned}$$

In view of Lemma 1.1 and the Sobolev embedding theorem, we get

$$\Phi_{\lambda,\eta}(w,z) \geq A_0||(w,z)||_X^2 - \eta A_1||(w,z)||_X^{2^*} - \lambda A_2, \text{ for some } A_0, A_1, A_2 > 0. \quad (2.23)$$

Set

$$h(t) = A_0 t^2 - \eta A_1 t^{2^*} - \lambda A_2, \text{ for all } t \geq 0.$$

Then for any $\eta > 0$, there exists $\lambda_* := \frac{2A_0}{NA_2}\left(\frac{2A_0}{2^*\eta A_1}\right)^{(N-2)/2}$ such that for all $\lambda \in (0, \lambda_*)$, there exists $t_* := \left(\frac{2A_0}{2^*\eta A_1}\right)^{(N-2)/4}$ such that

$$s_* := h(t_*) = \max_{t\geq 0} h(t) > 0.$$

Analogously, for any $\lambda > 0$, there exists $\eta_* := \frac{2A_0}{2^* A_1}\left(\frac{2A_0}{N\lambda A_2}\right)^{2/(N-2)}$ such that for all $\eta \in (0, \eta_*)$,

$$h(t_*) = \max_{t\geq 0} h(t) > 0.$$

Therefore, for $s_0 \in (0, s_*)$, we can find $t_0 < t_*$ such that $h(t_0) = s_0$. Let us now define

$$Q(t) = \begin{cases} 1, & 0 \leq t \leq t_0 \\ \frac{A_0 t^2 - s_* - \lambda A_2}{\eta A_1 t^{2^*}} & t \geq t_* \\ l(t) \in [0,1], & t_0 \leq t \leq t_*, \text{ where } l \in C^\infty. \end{cases}$$

Clearly, $0 \leq Q \leq 1$ and $Q \in C^\infty$. Consider the functional

$$\begin{aligned} \widetilde{\Phi}_{\lambda,\eta}(w,z) :=& \frac{a}{2}\int_\Omega \left(|\nabla w|^2 + |\nabla z|^2\right) dx \\ &+ \frac{b}{4}\left[\left(\int_\Omega |f'(w)|^2|\nabla w|^2 dx\right)^2 + \left(\int_\Omega |f'(z)|^2|\nabla z|^2 dx\right)^2\right] \\ &- \frac{\eta}{2(2^*)} Q(||(w,z)||_X)\int_\Omega |f(w)|^{2\alpha}|f(z)|^{2\beta} dx \\ &- \lambda Q(||(w,z)||_X)\int_\Omega F(x, f(w), f(z)) dx. \end{aligned} \tag{2.24}$$

Thus, (2.23) implies

$$\begin{aligned} \widetilde{\Phi}_{\lambda,\eta}(w,z) &\geq A_0||(w,z)||_X^2 - \eta A_1 Q(||(w,z)||_X)||(w,z)||_X^{2^*} - \lambda A_2 \\ &= \widetilde{h}(||(w,z)||_X), \end{aligned}$$

where $\widetilde{h}(t) = A_0 t^2 - \eta A_1 Q(t) t^{2^*} - \lambda A_2$. Observe that

$$\widetilde{h}(t) = \begin{cases} h(t), & 0 \leq t \leq t_0 \\ s_* & t \geq t_*. \end{cases}$$

We then have the following lemma.

Lemma 2.4. *Assume that $(F_0) - (F_1)$ hold. Then the functional $\widetilde{\Phi}_{\lambda,\eta}$ given by (2.24) satisfies the following proprieties:*

i) $\widetilde{\Phi}_{\lambda,\eta} \in C^1(X, \mathbb{R})$ and $\widetilde{\Phi}_{\lambda,\eta}$ is even and bounded from below;

ii) If $\widetilde{\Phi}_{\lambda,\eta}(w,z) < s_0$, then $||(w,z)||_X < t_0$ and $\widetilde{\Phi}_{\lambda,\eta}(w,z) = \Phi_{\lambda,\eta}(w,z)$;

iii) For all $\lambda \in (0, \lambda_*)$, $\widetilde{\Phi}_{\lambda,\eta}$ *satisfies* $(PS)_c$ *condition for* $c < s_0$ *with*

$$s_0 \in \left(0, \min\left\{s_*, \frac{\eta}{4N}\left(\frac{aS_{\alpha,\beta}}{2\eta}\right)^{N/2} - \lambda C_0(\varepsilon)|\Omega|\right\}\right), \text{ where } \varepsilon = \frac{\eta}{4N\lambda}.$$

Lemma 2.5. *Assume that* $(F_0) - (F_2)$ *hold. Then for any* $k \in \mathbb{N}$, *there is* $\delta_k > 0$ *such that*

$$\gamma\left(\left\{(w,z) \in X : \widetilde{\Phi}_{\lambda,\eta}(w,z) \le -\delta_k\right\} \setminus \{0\}\right) \ge k. \tag{2.25}$$

Proof: For each $k \in \mathbb{N}$, we can choose X_k k-dimensional subspace of X such that $X_k \subset L^\infty(\Omega) \times L^\infty(\Omega)$. Then for some $\varrho_k, \varsigma_k > 0$,

$$|(w,z)|_{L^\infty(\Omega)\times L^\infty(\Omega)} \le \varrho_k ||(w,z)||_X, \text{ for all } (w,z) \in X_k, \tag{2.26}$$
$$||(w,z)||_X \le \varsigma_k |(w,z)|_{L^2(\Omega)\times L^2(\Omega)}, \text{ for all } (w,z) \in X_k. \tag{2.27}$$

By (F_2), for any $\xi > 0$, there exists $0 < \vartheta < 1$ such that

$$F(x,s,t) \ge \xi^{-1}|s,t|^2, \text{ for all } |s,t| < \vartheta \text{ and all } x \in \Omega.$$

According to Lemma 2.1 (f_3), (f_{10}), for some $C > 0$ we have

$$C|(s,t)| \le |(f(t), f(s))| \le |(s,t)|, \text{ for all } |s,t| \le 1.$$

Therefore

$$F(x, f(s), f(t)) \ge C^2\xi^{-1}|s,t|^2, \text{ for all } |s,t| < \vartheta < 1 \text{ and all } x \in \Omega. \tag{2.28}$$

Let $(w,z) \in X_k$ such that $||(w,z)||_X = \tau < \min\left\{\frac{\vartheta}{\varrho_k}, 1, t_0\right\}$ with t_0 is given in Lemma 2.4. Then, by Lemma 2.1 (f_2) and (2.26)-(2.28), for $\xi > 0$ small enough,

$$\begin{aligned}
\widetilde{\Phi}_{\lambda,\eta}(w,z) =& \frac{a}{2}\int_\Omega \left(|\nabla w|^2 + |\nabla z|^2\right) dx \\
&+ \frac{b}{4}\left[\left(\int_\Omega |f'(w)|^2|\nabla w|^2 dx\right)^2 + \left(\int_\Omega |f'(z)|^2|\nabla z|^2 dx\right)^2\right] \\
&- \frac{\eta}{2(2^*)} Q(||(w,z)||_X) \int_\Omega |f(w)|^{2\alpha}|f(z)|^{2\beta} dx \\
&- \lambda Q(||(w,z)||_X) \int_\Omega F(x, f(w), f(z)) dx \\
\le& \frac{a}{2}\tau^2 + \frac{b}{4}\tau^4 - \lambda Q(\tau) C^2\xi^{-1}\int_\Omega |(w,z)|^2 dx \\
\le& \frac{a}{2}\tau^2 + \frac{b}{4}\tau^4 - \frac{\lambda Q(\tau)C^2}{\varsigma_k^2}\xi^{-1}||(w,z)||_X^2 \\
\le& \left(\frac{a}{2} + \frac{b}{4} - \frac{\lambda C^2}{\varsigma_k^2}\xi^{-1}\right)\tau^2 \\
=&: -\delta_k < 0,
\end{aligned} \tag{2.29}$$

here we use the fact that $Q(\tau) = 1$, and $\xi^{-1} \to +\infty$ as $\xi \to 0^+$. This last equality shows that

$$S^k(0,\tau) := \{(w,z) \in X_k : ||(w,z)||_X = \tau\} \subset \left\{(w,z) \in X : \widetilde{\Phi}_{\lambda,\eta}(w,z) \leq -\delta_k\right\} \setminus \{0\}$$

and hence

$$\gamma\left(\left\{(w,z) \in X : \widetilde{\Phi}_{\lambda,\eta}(w,z) \leq -\delta_k\right\} \setminus \{0\}\right) \geq k.$$

□

Proof of Theorem 1.2. Setting

$$\Sigma_k := \{E \subset X \setminus \{0\} : E \text{ is closed and } E = -E,\ \gamma(E) \geq k\}$$

and

$$c_k := \inf_{E \in \Sigma_k} \sup_{(w,z) \in E} \widetilde{\Phi}_{\lambda,\eta}(w,z).$$

By Lemma 2.5 and Lemma 2.4 (i), $-\infty < c_k < 0$. Therefore the functional $\widetilde{\Phi}_{\lambda,\eta}$ satisfies all assumptions of Proposition 2.3, and consequently, $\widetilde{\Phi}_{\lambda,\eta}$ has a sequence of critical points $\{(w_n, z_n)\} \subset X \setminus \{0\}$ such that $(w_n, z_n) \to 0$. Thanks to Lemma 2.4 (ii), $\{(w_n, z_n)\}$ is a solution of problem (1.3).
Proof of Theorem 1.3. This is a consequence of Theorem 1.2.

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