Bol. Soc. Paran. Mat. (3s.) **v. 37** 4 (2019): 19–23.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v37i4.32631

Composition Operators on Hilbert Spaces of Sequences

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ABSTRACT: In this paper, we will introduce new sequence Hilbertian space and for it we will show boundedness of composition operators.

Key Words: Composition Operators; Weighted Composition Operators; Λ^2-Statistical convergence;Hilbert Sequence Spaces.

Contents

1. Introduction

If (X, Σ, λ) is a finite measure space and Φ is a nonsingular measurable transformation on X into itself, then the composition transformation C_Φ on $L_p(\lambda)$ is defined by

$$C_\Phi f = f \circ \Phi,$$

for every $f \in L_p(\lambda)$. If $C_\Phi \in B(L_p(\lambda))$, the Banach algebra of all bounded linear operators on $L_p(\lambda)$, then it is called a composition operator induced by Φ. In this paper we are interested in the study of composition operators when X is equal to $\mathbb{N}$, the set of all natural numbers and X is the counting measure on $\mathbb{N}$. In this case $L_p(\lambda)$ is equal to ℓ_p.

Let us denote by Ω the vector space of all sequences $x = (x_n)$ of complex numbers. The set of sequence spaces is widely spreading and till now are known many of them. The most important are ℓ_p sequence spaces, which are a Banach spaces of sequences, and defined as follows

$$\ell_p = \left\{ x = (x_n) : \sum_{k=0}^{\infty} |x_k|^p < \infty \right\},$$

for $1 \leq p \leq \infty$. And norm is given by

$$||x||_p = \left(\sum_{n=0}^{\infty} |x_n|^p \right)^{\frac{1}{p}}, 1 \leq p < \infty,$$

2010 *Mathematics Subject Classification:* 47B33; 47B37; 46B45.
Submitted July 08, 2016. Published May 08, 2017

and

$$||x||_\infty = \sup_{0\le n<\infty} |x_n|, \quad \text{for} \quad p=\infty, x\in \ell_p.$$

In case where $p=2$, ℓ_2 is Hilbert space and scalar product is given as follows

$$(x,y)=\sum_{n=0}^{\infty} x_n \overline{y}_n, \forall x,y\in \ell_2.$$

This space is widely studied by several mathematicians in connection with the study of unilateral shift, bilateral shift, multiplication operators, composition operators, cyclic, hyper cyclic operators and weighted composition operators (See [3], [4], [5]). The symbol $k(E)$ denotes the cardinality of the set E.

In [2] is defined the following sequence of real numbers as follows. (λ_k) is a nondecreasing sequence of positive numbers such that $\lambda_k\to\infty$, as $k\to\infty$. The first difference is defined as follows: $\Delta\lambda_k=\lambda_k-\lambda_{k-1}$, where $\lambda_0=\lambda_{-1}=\lambda_{-2}=0$, and the second difference is defined as $\Delta^2(\lambda_k)=\Delta(\Delta(\lambda_k))=\lambda_k-2\lambda_{k-1}+\lambda_{k-2}$. In paper [1], the sequence space $l_p^{\lambda^2}$ was defined as follows:

$$\ell_p^{\lambda^2}=\left\{x=(x_k)\in w: \sum_{n=0}^{\infty}\left|\frac{1}{\lambda_n-\lambda_{n-1}}\sum_{k=0}^{n}(\lambda_k-2\lambda_{k-1}+\lambda_{k-2})x_k\right|^p<\infty\right\}. \tag{1.1}$$

In this paper we will show some properties of the composition operators from sequence space ℓ_p into the subspace $L_p^{\lambda^2}$ of sequence space $\ell_p^{\lambda^2}$ defined as follows

$$L_p^{\lambda^2}=\left\{x=(x_k)\in w: \sum_{k=0}^{\infty}\left|\frac{1}{\lambda_{k^2}-\lambda_{k^2-1}}\sum_{k=0}^{n}(\lambda_{k^2}-2\lambda_{k^2-1}+\lambda_{k^2-2})x_k\right|^p<\infty\right\}, \tag{1.2}$$

and for sequence (λ_n), we will suppose that

$$\sup_k\,(\Delta\lambda_{k^2}-\Delta\lambda_{k^2-1})=K<\infty, \tag{1.3}$$

and

$$\sum_{k=n}^{\infty}\frac{1}{\lambda_{n^2}-\lambda_{n^2-1}}<\infty,$$

for every $n\in\mathbb{N}$.

Remark 1.1. *The proposed sequence (λ_n) exists, let $\Delta\lambda_n\sim n^{\frac{\alpha}{2}}$, for $1<\alpha<2$. Then it satisfies conditions given above for the sequence (λ_n).*

The sequence space $\ell_p^{\lambda^2}$ is a BK space(see [1]) related to the norm

$$||x||_{\ell_p^{\lambda^2}}=\sum_{n=0}^{\infty}\left|\frac{1}{\lambda_n-\lambda_{n-1}}\sum_{k=0}^{n}(\lambda_k-2\lambda_{k-1}+\lambda_{k-2})x_k\right|^p.$$

For $p = 2$, $L_p^{\lambda^2}$ is a Hilbert space under inner product

$$(x, y) = (\Lambda^2 x, \Lambda^2 y),$$

where

$$(\Lambda^2 x)(n) = \frac{1}{\lambda_{n^2} - \lambda_{n^2-1}} \sum_{k=0}^{n} (\lambda_{k^2} - 2\lambda_{k^2-1} + \lambda_{k^2-2}) x_k, \forall n \in \mathbb{N}.$$

2. Bounded composition operators on Hilbert Spaces

Theorem 2.1. *Let Φ be a mapping on $\mathbb{N}$ into itself. Then $C_\Phi : \ell_p \to L_p^{\Lambda^2}$, for $1 \le p < \infty$, if and only if there exists an integer $M > 0$ such that $k(\Phi^{-1}(\{n\})) \le M$ for every $n \in \mathbb{N}$. And*

$$||C_\Phi||_{\Lambda^2} = \inf\{M : k(\Phi^{-1}(\{n\}) \le M \quad \text{for all} \quad n \in \mathbb{N}\}.$$

Proof. First we will prove the boundedness of the operator then we will evaluate norm. Let $x \in \ell_p$, then

$$\begin{aligned} ||C_\Phi x||_{\Lambda^2} &= \sum_{n=0}^{\infty} \left| \sum_{k=0}^{n} \frac{\lambda_{k^2} - 2\lambda_{k^2-1} + \lambda_{k^2-2}}{\lambda_{n^2} - \lambda_{n^2-1}} x_{\Phi(k)} \right|^p \\ &\le \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \frac{\lambda_{k^2} - 2\lambda_{k^2-1} + \lambda_{k^2-2}}{\lambda_{n^2} - \lambda_{n^2-1}} |x_{\Phi(k)}|^p \left(\sum_{k=0}^{n} \frac{\lambda_{k^2} - 2\lambda_{k^2-1} + \lambda_{k^2-2}}{\lambda_{n^2} - \lambda_{n^2-1}} \right)^{\frac{p}{q}} \right] \\ &\le \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{\lambda_{k^2} - 2\lambda_{k^2-1} + \lambda_{k^2-2}}{\lambda_{n^2} - \lambda_{n^2-1}} |x_{\Phi(k)}|^p \\ &\le \sum_{k=0}^{\infty} (\lambda_{k^2} - 2\lambda_{k^2-1} + \lambda_{k^2-2}) |x_{\Phi(k)}|^p \sum_{n=k}^{\infty} \frac{1}{\lambda_{n^2} - \lambda_{n^2-1}} \\ &\le K \sum_{k=0}^{\infty} (\lambda_{n^2} - 2\lambda_{n^2-1} + \lambda_{n^2-2}) |x_{\Phi(k)}|^p, \quad \text{where} \quad K = \sum_{n=k}^{\infty} \frac{1}{\lambda_{n^2} - \lambda_{n^2-1}}. \end{aligned}$$

Respectively

$$||C_\Phi x||_{\Lambda^2} \le K_1 \sum_{k=0}^{\infty} |x_{\Phi(k)}|^p = K_1 \sum_{k=0}^{\infty} \sum_{n \in T^{-1}(k)} |x_{\Phi(n)}|^p = K_1 \sum_{k=0}^{\infty} \sum_{n \in T^{-1}(k)} |x_n|^p \le$$

$$K_1 \cdot M \sum_{k=0}^{\infty} |x_k|^p = K_1 \cdot M ||x||_p^p,$$

where $K_1 = K \cdot \sup_k (\lambda_{k^2} - 2\lambda_{k^2-1} + \lambda_{k^2-2})$. Hence C_Φ is a bounded operator. Now we will determinate it's norm.

$$k(\Phi^{-1}(\{n\})) = ||C_\Phi(e_n)||_{\Lambda^2}^p \le ||C_\Phi||_{\Lambda^2}^p \cdot ||e_n||^p = ||C_\Phi||_{\Lambda^2}^p,$$

where (e_n) is a bases for the sequence space l_p. Since this is valid for any $n \in \mathbb{N}$, then we get

$$\inf\{M : k(\Phi^{-1}(\{n\})) \leq M, \quad \text{for all} \quad n \in \mathbb{N}\} \leq ||C_\Phi||^p_{\Lambda^2}.$$

On other side, suppose that $k(\Phi^{-1}(\{n\})) \leq M$, for all $n \in \mathbb{N}$. Then we have

$$||C_\Phi f||^p_{\Lambda^2} \leq M||f||^p, \quad \text{for all} \quad f \in \ell_p.$$

Hence

$$||C_\Phi||^p_{\Lambda^2} \leq \inf\{M : k(\Phi^{-1}(\{n\})) \leq M, \quad \text{for all} \quad n \in \mathbb{N}\},$$

and theorem is proved. □

Proof of the following two corollaries, are immediately implication of the above theorem, so we omit the proof.

Corollary 2.2. *Let Φ be a mapping on $\mathbb{N}$ into itself. Then $C_\Phi : \ell_p \to L_p^{\Lambda^2}$ is one-to-one if and only if C_Φ is a bounded operator and $||C_\Phi|| = 1$.*

Corollary 2.3. *If $C_\Phi : \ell_p \to L_p^{\Lambda^2}$ is a bounded operator, then the range of Φ has infinitely many elements.*

Theorem 2.4. *Let C_Φ be a bounded operator defined as $C_\Phi : \ell_p \to L_p^{\Lambda^2}$. Then C_Φ is invertible if and only if Φ is invertible.*

Proof. We omit it, because it is similar to the proof of Theorem 2.2 given in [4]. □

Corollary 2.5. *Under conditions given in the above theorem, if $\Phi : \mathbb{N} \to \mathbb{N}$ is a constant function, then C_Φ is a not bounded operator in $L_p^{\Lambda^2}$.*

Proof. Let us suppose that $\Phi : \mathbb{N} \to \mathbb{N}$ is a constant function given by $\Phi(n) = n_0$, for all $n \in \mathbb{N}$. Take $x \in L_p^{\Lambda^2}$, such that $x_{n_0} \neq 0$. Then we have

$$||C_\Phi x||_{\Lambda^2} = \sum_{n=0}^{\infty} \left| \frac{1}{\lambda_{n^2} - \lambda_{n^2-1}} \sum_{k=0}^{n} (\lambda_{k^2} - 2\lambda_{k^2-1} + \lambda_{k^2-2}) x_{\Phi(k)} \right|^p =$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{\lambda_{n^2} - \lambda_{n^2-1}} \right)^p \left| \sum_{k=0}^{n} (\lambda_{k^2} - 2\lambda_{k^2-1} + \lambda_{k^2-2}) x_{k_0} \right|^p =$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{\lambda_{n^2} - \lambda_{n^2-1}} \right)^p |(\lambda_{n^2} - \lambda_{n^2-1}) x_{n_0}|^p$$

$$= \sum_{n=0}^{\infty} |x_{k_0}|^p = \infty.$$

□

Corollary 2.6. *Under conditions given in the above theorem, if the transformation Φ is unilateral shift operator, then composition $C_\Phi : \ell_2 \to L_2^{\Lambda^2}$, is bounded operator for every $x \in \ell_2$.*

Proof. Let $x \in \ell_2$, then

$$||C_\Phi x||^2_{\Lambda^2} = \sum_{n=0}^{\infty} \left| \frac{1}{\lambda_{n^2} - \lambda_{n^2-1}} \sum_{k=0}^{n} (\lambda_{k^2} - 2\lambda_{k^2-1} + \lambda_{k^2-2}) x_{\Phi(k)} \right|^2 =$$

$$\sum_{n=0}^{\infty} \left| \sum_{k=0}^{n} \left(\frac{\lambda_{k^2} - 2\lambda_{k^2-1} + \lambda_{k^2-2}}{\lambda_{n^2} - \lambda_{n^2-1}} \right)^{\frac{1}{2}} \cdot \left(\frac{\lambda_{k^2} - 2\lambda_{k^2-1} + \lambda_{k^2-2}}{\lambda_{n^2} - \lambda_{n^2-1}} \right)^{\frac{1}{2}} x_{k+1} \right|^2 \leq$$

(from Holder's inequality)

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{\lambda_{k^2} - 2\lambda_{k^2-1} + \lambda_{k^2-2}}{\lambda_{n^2} - \lambda_{n^2-1}} \cdot \sum_{k=0}^{n} \frac{\lambda_{k^2} - 2\lambda_{k^2-1} + \lambda_{k^2-2}}{\lambda_{n^2} - \lambda_{n^2-1}} |x_{k+1}|^2 =$$

$$\sum_{k=0}^{\infty} (\lambda_{k^2} - 2\lambda_{k^2-1} + \lambda_{k^2-2}) |x_{k+1}|^2 \sum_{n=k}^{\infty} \frac{1}{\lambda_{n^2} - \lambda_{n^2-1}} \leq K \sum_{k=0}^{\infty} |x_{k+1}|^2$$

$$= K ||x||^2_{l_2} < \infty,$$

where

$$K = \sup_k (\lambda_{k^2} - 2\lambda_{k^2-1} + \lambda_{k^2-2}) \sum_{n=k}^{\infty} \frac{1}{\lambda_{n^2} - \lambda_{n^2-1}} < \infty.$$

Hence $C_\Phi : \ell_2 \to L_2^{\Lambda^2}$, is bounded operator for every $x \in \ell_2$. □

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