



On Totally Projective *QTAG*-modules Characterized by its Submodules

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ABSTRACT: A *QTAG*-module M is called almost totally projective if it has a weak nice system. Here we show that the isotype submodules of a totally projective module which are almost totally projective are precisely those that are separable. From this characterization it follows that every balanced submodule of a totally projective module is almost totally projective. Finally, in some special cases we settle the question of whether a direct summand of an almost totally projective module is again almost totally projective.

Key Words: Totally projective modules, Almost totally projective modules, Isotype submodules, Separable submodules.

Contents

1 Introduction and background material	77
2 * -bases and intersection closure	78
3 A characterization of separable isotype totally projective modules	80
4 The summand problem	82

1. Introduction and background material

Following [8], a unital module M_R is called *QTAG*-module if it satisfies the following condition:

(I) Every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules.

Let all rings discussed here be associative with unity ($1 \neq 0$) and modules are unital *QTAG*-modules. A module in which the lattice of its submodule is totally ordered is called a serial module; in addition, if it has finite composition length it is called a uniserial module. An element $x \in M$ is uniform, if xR is a non-zero uniform (hence uniserial) module and for any R -module M with a unique decomposition series, $d(M)$ denotes its decomposition length. For a uniform element $x \in M$, $e(x) = d(xR)$ and $H_M(x) = \sup \left\{ d \left(\frac{yR}{xR} \right) \mid y \in M, x \in yR \text{ and } y \text{ uniform} \right\}$ are the exponent and height of x in M , respectively. $H_k(M)$ denotes the submodule

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of M generated by the elements of height at least k and $H^k(M)$ is the submodule of M generated by the elements of exponents at most k . M is h -divisible if $M = M^1 = \bigcap_{k=0}^{\infty} H_k(M)$ and it is h -reduced if it does not contain any h -divisible submodule. In other words it is free from the elements of infinite height.

For an ordinal σ , a submodule N of M is said to be σ -pure, if $H_\beta(M) \cap N = H_\beta(N)$ for all $\beta \leq \sigma$ and a submodule N of M is said to be isotype in M , if it is σ -pure for every ordinal σ [7]. A submodule $N \subset M$ is nice [5] in M , if $H_\sigma(M/N) = (H_\sigma(M) + N)/N$ for all ordinals σ , i.e. every coset of M modulo N may be represented by an element of the same height. If N is both isotype and nice, then N is called a balanced submodule of M .

A family \mathcal{N} of nice submodules of M is called a *nice system* [6] in M if

(i) $0 \in \mathcal{N}$;

(ii⁺) if $\{N_i\}_{i \in I}$ is any subset of \mathcal{N} , then $\sum_{i \in I} N_i \in \mathcal{N}$;

(iii⁺) given any $N \in \mathcal{N}$ and any countable subset X of M , there exists $K \in \mathcal{N}$ containing $N \cup X$, such that K/N is countably generated.

An h -reduced $QTAG$ -module M is totally projective if it has a nice system and direct sums and direct summands of totally projective modules are also totally projective [4].

In this paper, we focus on the class of isotype separable submodules of totally projective modules. The object of our study includes, the well known class of balanced submodules of totally projective modules. It is well known that a balanced submodule of a totally projective module need not be totally projective. Nevertheless, we show that the balanced submodules of totally projective modules are almost totally projective. It should be no surprise that this is done in the context of nice system. In addition to proving that every balanced submodule of a totally projective module is almost totally projective, we are able to characterize the isotype submodules of totally projective modules that are almost totally projective. This class will be seen to coincide with the class of isotype separable submodules of totally projective modules. Finally, as an application of this characterization and its method of proof, we obtain results concerning direct summands of almost totally projective modules. Our notations and terminology generally agree with those in [1] and [2].

2. * -bases and intersection closure

A $QTAG$ -module M is almost totally projective if it has a collection \mathcal{N} of nice submodules such that (i) $0 \in \mathcal{N}$, (ii) \mathcal{N} is closed with respect to unions of ascending chains, and (iii) every countably generated submodule of M is contained in a countably generated submodule from \mathcal{N} . Call a collection \mathcal{N} of nice submodules of a $QTAG$ -module M which satisfies conditions (i), (ii) and (iii) a *weak nice system* for M . It will be convenient to consider those almost totally projective modules that

have the following additional property. We say that an almost totally projective module is *intersection closed* if it has a weak nice system which is closed under the formation of arbitrary intersections. Our first objective in this section is to demonstrate that every totally projective module is intersection closed. This is done somewhat indirectly by exploiting the theory of $*$ -bases as introduced in [3].

Let M be a *QTAG*-module. For each ordinal σ , let B_σ be a set of representatives of the nonzero cosets of $H_\sigma(M)$ mod $H_{\sigma+1}(M)$; in other words, B_σ contains exactly one element from each of the nonzero cosets of $H_{\sigma+1}(M)$ in $H_\sigma(M)$. If each element x in M can be expressed as

$$x = b_1 + b_2 + \cdots + b_n$$

where $b_i \in B_{\sigma(i)}$ with $\sigma(1) < \sigma(2) < \cdots < \sigma(n)$, then $B = \bigcup B_\sigma$ is called a $*$ -basis of M .

A submodule N of a *QTAG*-module M with a $*$ -basis B is called a *secure submodule* if for $0 \neq y \in N$, $y = b_1 + b_2 + \cdots + b_n$ is the unique representation of y with respect to B , then $b_i \in N$ for each i .

The consequences of the results of [3] that concern in this paper are that every secure submodule is nice and that the collection \mathcal{N} of all secure submodules of M constitutes a weak nice system for M . A novelty of our approach in this paper is based on the observation that an arbitrary intersection of secure submodules is again secure. This follows easily from the uniqueness of the representations for the elements of M as described above. Thus, every *QTAG*-module with a $*$ -basis is intersection closed. In [3] it is also shown that every totally projective module has a $*$ -basis. Therefore, we have the following.

Proposition 2.1. *If M is a totally projective module, then M is intersection closed.*

We next establish a property of intersection closed modules that will use in the proof of Theorem 3.1. In order to state this result, we need some further notation and terminology. Suppose that M is a *QTAG*-module. If $x \in M$ we write $H_M(x)$ for the height of x in M . Thus, $H_M(x) = \alpha$ means that $x \in H_\alpha(M) \setminus H_{\alpha+1}(M)$. If $x \in H_\infty(M)$, set $H_M(x) = \infty$ with the understanding that $\alpha < \infty$ for all ordinals α .

Let M be a *QTAG*-module. Two submodules P and Q of M are *compatible*, written $P \parallel Q$, if for each pair $(p, q) \in P \times Q$ there exists $r \in P \cap Q$ such that $H_M(p + q) \leq H_M(p + r)$.

It is easily seen that compatibility is a symmetric relation, and is inductive in the sense that if

$$P_0 \subseteq P_1 \subseteq \cdots \subseteq P_\alpha \subseteq \cdots \quad (\alpha < \beta)$$

is an ascending chain of submodules of M with $P_\alpha \parallel Q$ for all α , then $(\bigcup_{\alpha < \beta} P_\alpha) \parallel Q$.

Lemma 2.1. *Suppose K is a submodule of an intersection closed *QTAG*-module M . Let \mathcal{N} be a weak nice system for M which is closed under arbitrary intersections*

and assume that

$$P_0 \cap K \subseteq P_1 \cap K \subseteq \cdots \subseteq P_\alpha \cap K \dots \ (\alpha < \beta)$$

is an ascending chain of submodules of M with $P_\alpha \in \mathcal{N}$ for all α . Then, there exists a submodule $Q \in \mathcal{N}$ such that the following properties hold.

$$(i) \quad \bigcup_{\alpha < \beta} (P_\alpha \cap K) = Q \cap K.$$

(ii) If $P_\alpha \parallel K$ for all $\alpha < \beta$, then $Q \parallel K$.

Proof: For each $\alpha < \beta$, set $Q_\alpha = \bigcap_{\alpha \leq \gamma} P_\gamma$. Then, since \mathcal{N} is closed under intersection,

$$Q_0 \subseteq Q_1 \subseteq \cdots \subseteq Q_\alpha \subseteq \cdots \ (\alpha < \beta)$$

is an ascending chain with $Q_\alpha \in \mathcal{N}$ for each α . Thus, $Q = \bigcup_{\alpha < \beta} Q_\alpha \in \mathcal{N}$. Observe that for each α , $Q_\alpha \cap K = (\bigcap_{\alpha \leq \gamma} P_\gamma) \cap K = \bigcap_{\alpha \leq \gamma} (P_\gamma \cap K) = P_\alpha \cap K$. Now (i) follows easily since $Q \cap K = \bigcup_{\alpha < \beta} (Q_\alpha \cap K) = \bigcup_{\alpha < \beta} (P_\alpha \cap K)$. Observe further that $Q_\alpha \subseteq P_\alpha$ for all $\alpha < \beta$. Thus, if $P_\alpha \parallel K$, $a \in Q_\alpha$ and $b \in K$, then $H_M(a+b) \leq H_M(a+c)$ for some $c \in P_\alpha \cap K = Q_\alpha \cap K$. Consequently, if $P_\alpha \parallel K$ for all α , then $Q_\alpha \parallel K$ for all α . Therefore, (ii) follows since compatibility with K is an inductive property. \square

3. A characterization of separable isotype totally projective modules

We begin with the following elementary lemma.

Lemma 3.1. Suppose K is an isotype submodule of a $QTAG$ -module M . If N is a nice submodule of M satisfying $N \parallel K$, then $N \cap K$ is a nice submodule of K .

Proof: Suppose $x \in K \setminus N \cap K$. Then $H_{K/N \cap K}[x + (N \cap K)] \leq H_{M/N}(x + N) = H_M(x + y)$ for some $y \in N$, since N is nice in M . But $N \parallel K$ and K is isotype in M imply that there exists $z \in N \cap K$ such that $H_M(x + y) \leq H_M(x + z) = H_K(x + z)$. Therefore, $H_{K/N \cap K}[x + (N \cap K)] = H_K(x + z)$ and the result follows. \square

The notion of separability has an important role in the study of $QTAG$ -modules. The authors claim that the same should be true for submodules of the $QTAG$ -modules, and in fact the remaining part of this paper substantially supports this claim. We now define separable submodule in a slightly different way from that in [2].

A submodule N of a $QTAG$ -module M is a *separable submodule* if for each $x \in M$ there is a corresponding countably generated submodule K of N such that

$$\sup\{H_M(x + y) : y \in N\} = \sup\{H_M(x + z) : z \in K\}.$$

There are two crucial properties of separability.

(A) An almost totally projective module is a separable submodule of any QTAG-module in which it appears as an isotype submodule.

(B) Suppose N is a separable submodule of a QTAG-module M . If S is a countably generated submodule of M , there exists a countably generated submodule T of M such that $S \subseteq T$ and $T \parallel N$.

Our next result characterizes those isotype submodules of a totally projective module that are almost totally projective.

Theorem 3.1. *Suppose K is an isotype submodule of a totally projective module M . Then K is almost totally projective if and only if K is separable in M .*

Proof: If K is almost totally projective, then K is separable in M by property (A). Conversely, assume that K is separable in M . By Proposition 2.1, M is intersection closed and so has a weak nice system \mathcal{N} which is closed under intersection. Set

$$\mathcal{N}_K = \{T \cap K : T \in \mathcal{N} \text{ and } T \parallel K\}$$

We claim that \mathcal{N}_K is a weak nice system for K .

By Lemma 3.1, \mathcal{N}_K consists of nice submodules of K . Also, it is clear that $0 \in \mathcal{N}_K$. Moreover, that \mathcal{N}_K is closed with respect to ascending unions follows immediately from Lemma 2.1. Therefore, to establish the claim, and thereby complete the proof, it suffices to show that every countably generated submodule S of K is contained in a countably generated submodule from \mathcal{N}_K . Since S is a countably generated submodule of M , there exists a countably generated submodule $T_0 \in \mathcal{N}$ such that $S \subseteq T_0$. Next, K is separable in M and T_0 is countably generated, so it follows from property (B) that there exists a countably generated submodule S_0 of M such that $T_0 \subseteq S_0$ and $S_0 \parallel K$. In a similar fashion, we can select countably generated submodule T_1 and S_1 such that $T_0 \subseteq S_0 \subseteq T_1 \subseteq S_1$, $T_1 \in \mathcal{N}$ and $S_1 \parallel K$. Continuing in this way, we obtain an ascending chain

$$S \subseteq T_0 \subseteq S_0 \subseteq T_1 \subseteq S_1 \subseteq \dots T_n \subseteq S_n \subseteq \dots \quad (n < \omega_0),$$

of countably generated submodules such that $T_n \in \mathcal{N}$ and $T_n \parallel K$ for all n . If we take T to be the union of the chain, then $T = \bigcup_{n < \omega_0} T_n$ and so $T \in \mathcal{N}$. Also,

$T = \bigcup_{n < \omega_0} S_n$ implies that $T \parallel K$, since compatibility with K is an inductive property. Therefore, $T \cap K$ is a countably generated submodule from \mathcal{N}_K which contains the countable generated submodule S of K . \square

Remark 3.1. *The careful reader will observe that Theorem 3.1 remains valid, with the same proof, if M is replaced by any QTAG-module with a $*$ -basis. It is unknown at this time whether a QTAG-module with a $*$ -basis is necessarily totally projective. However, it is true that if M has a $*$ -basis and $H(M) \leq \aleph_1$, then M is totally projective (see [3]).*

Remark 3.2. *There exists almost totally projective modules which do not appear as isotype submodules of totally projective modules. To see this, take K to be a h -reduced almost totally projective module of length ω_0 which is not a direct sum of uniserial modules. Suppose to the contrary that K embeds as an isotype submodule in a totally projective module M . Then, $H_{\omega_0}(M) \cap K = H_{\omega_0}(K) = 0$ implies that K embeds in $M/H_{\omega_0}(M)$, a h -reduced totally projective module of length ω_0 and hence a direct sum of uniserial modules. But this contradicts the fact that every submodule of a direct sum of uniserial modules is also a direct sum of uniserial modules.*

Note that if K is a nice submodule of a $QTAG$ -module M , then K is a separable submodule of M . This is because, for a nice submodule K , $\sup\{H_M(x+y) : y \in K\}$ is actually attained by $H_M(x+z)$ for some $z \in K$. Therefore, we have the following as an immediate corollary of Theorem 3.1.

Corollary 3.1. *If B is a balanced submodule of the totally projective module M , then B is almost totally projective.*

4. The summand problem

In this section, we address the question of whether a direct summand of an almost totally projective module is almost totally projective. Even though a complete solution has so far resisted our efforts, we present two special cases in which the question can be answered affirmatively. In our first result, we consider the case when almost totally projective module appears as an isotype submodule of a totally projective module. Recall, as demonstrated in Section 3, that there exists almost totally projective modules which do not appear in this manner.

Proposition 4.1. *Suppose K is an isotype submodule of a totally projective module M . If K is almost totally projective, then every direct summand of K is almost totally projective.*

Proof: Write $K = S \oplus T$ and observe that S is isotype in M . Therefore, by Theorem 3.1, it is enough to show that S is separable in M . In order to see that S is separable in M , suppose $x \in M$ and observe that K is separable in M by Property (A). Thus, there is a countably generated submodule $L = \{y_n : n < \omega_0\}$ of K such that for every $y \in K$, there exists an $n < \omega_0$ such that $H_M(x+y) \leq H_M(x+y_n)$. For each n , write $y_n = s_n + t_n$, where $s_n \in S$ and $t_n \in T$. We claim that if $s \in S$, there exists $n < \omega_0$ such that $H_M(x+s) \leq H_M(x+s_n)$. Indeed, it is well known that there exists an n such that $H_M(x+s) \leq H_M(x+s_n+t_n)$. So, to establish the claim and thereby complete the proof, we may assume that $H_M(x+s_n) < H_M(x+s_n+t_n)$. In this case, observe that $H_M(t_n) = H_M(x+s_n)$,

Therefore,

$$\begin{aligned}
H_M(x + s) &\leq H_M[(x + s) - (x + s_n + t_n)] \\
&= H_M[(s - s_n) - t_n] \\
&= H_K[(s - s_n) - t_n] \\
&\leq H_K(t_n) \\
&= H_M(t_n) \\
&= H_M(x + s_n).
\end{aligned}$$

□

The following lemma is one of the main ingredient in the proof of second summand result.

Lemma 4.1. *Suppose $M = S \oplus T$ is almost totally projective. Then, M has a weak nice system \mathcal{N} satisfying for every $P \in \mathcal{N}$, $P = (P \cap S) \oplus (P \cap T)$. Moreover, if $H(T) \leq \aleph_1$, then T is totally projective.*

Proof: Let \mathcal{N}_M be a weak nice system for M and let

$$\mathcal{N} = \{P \in \mathcal{N}_M : P = (P \cap S) \oplus (P \cap T)\}.$$

Clearly $0 \in \mathcal{N}$ and \mathcal{N} is closed under ascending unions. Therefore, to establish that \mathcal{N} is a weak nice system for M , it suffices to show that every countably generated submodule K of M is contained in a countably generated member of \mathcal{N} .

For every $x \in M$, let $x = s_x + t_x$ ($s_x \in S, t_x \in T$) be the unique representation of x with respect to the decomposition $M = S \oplus T$. Define an ascending chain

$$P_0 \subseteq P_1 \subseteq \dots \subseteq P_n \subseteq \dots \quad (n < \omega_0)$$

of countably generated submodules from \mathcal{N}_M as follows. Take P_0 to be a countably generated submodule from \mathcal{N}_M such that $P_0 \supseteq K \cup \{s_x : x \in K\} \cup \{t_x : x \in K\}$. Now, if $1 \leq n \leq \omega_0$ and P_{n-1} has been defined, take P_n to be a countably generated submodule from \mathcal{N}_M such that $P_n \supseteq P_{n-1} \cup \{s_x : x \in P_{n-1}\} \cup \{t_x : x \in P_{n-1}\}$. Then $P = \bigcup_{n < \omega_0} P_n \in \mathcal{N}_M$ contains K and is countably generated. Moreover, it is clear from the construction that $P = (P \cap S) \oplus (P \cap T)$ so that P is actually in \mathcal{N} .

Next suppose that $H(T) \leq \aleph_1$ and take \mathcal{N} to be the weak nice system constructed above. Since every countably generated module is totally projective, we may assume that $H(T) = \aleph_1$. Select a smooth chain

$$0 = T_0 \subseteq T_1 \subseteq \dots \subseteq T_\alpha \subseteq \dots \quad (\alpha < \omega_1)$$

of countably generated submodules of $T = \bigcup_{\alpha < \omega_1} T_\alpha$. We now use this chain to construct a smooth chain

$$0 = L_0 \subseteq L_1 \subseteq \dots L_\alpha \subseteq \dots \quad (\alpha < \omega_1)$$

of countably generated submodules from \mathcal{N} as follows. Starting with $L_0 = 0$, suppose $1 \leq \alpha \leq \omega_1$ and that L_β has been constructed for all $\beta < \alpha$. If $\alpha < \omega_1$ is a limit, we set $L_\alpha = \bigcup_{\beta < \alpha} L_\beta$ as we must, and if $\alpha - 1$ exists, we take L_α to be a countably generated submodule from \mathcal{N} containing $L_{\alpha-1} + T_\alpha$. In either case, $L_\alpha \in \mathcal{N}$ and $H(L_\alpha) \leq \aleph_0$. Thus,

$$0 = L_0 \cap T \subseteq L_1 \cap T \subseteq \dots L_\alpha \cap T \subseteq \dots \quad (\alpha < \omega_1)$$

is a smooth chain of countably generated submodules of $T = \bigcup_{\alpha < \omega_1} (L_\alpha \cap T)$.

Since $L_\alpha = (L_\alpha \cap S) \oplus (L_\alpha \cap T)$ is nice in M for each α , each $L_\alpha \cap T$ is nice in T . Therefore, $\{T\} \cup \{L_\alpha \cap T\}_{\alpha < \omega_1}$ is a nice system for T , and T is totally projective. \square

We are now in a position to state and prove the second summand result.

Proposition 4.2. *Suppose $M = S \oplus T$ and T has cardinality not exceeding \aleph_1 . If M is almost totally projective, then S is almost totally projective and T is totally projective.*

Proof: We may assume that $H(T) = \aleph_1$. Since T is totally projective by Lemma 4.1, there exists a smooth chain

$$0 = L_0 \subseteq L_1 \subseteq \dots L_\alpha \subseteq \dots \quad (\alpha < \omega_1)$$

of countably generated nice submodules of T such that $\bigcup_{\alpha < \omega_1} L_\alpha = T$. Let us use Lemma 4.1 to select a weak nice system \mathcal{N} for M with the property that $P = (P \cap S) \oplus (P \cap T)$ for every $P \in \mathcal{N}$.

We now set $L_{\omega_1} = T$ and define

$$\mathcal{N}_M = \{P \in \mathcal{N} : P = (P \cap S) \oplus L_\alpha \text{ for some } \alpha \leq \omega_1\}.$$

Observe that every countably generated submodule K of M is contained in a countably generated member of \mathcal{N}_M . To see this, select a countably generated submodule $P_0 \in \mathcal{N}$ such that $K \subseteq P_0$. Next select $\alpha(0) < \omega_1$ so that $L_{\alpha(0)} \supseteq P_0 \cap T$. Moreover, since $P_0 + L_{\alpha(0)} = (P_0 \cap S) \oplus L_{\alpha(0)}$ is countably generated, there exists a countably generated submodule $P_1 \in \mathcal{N}$ and $\alpha(1) < \omega_1$ such that $P_1 \supseteq P_0 + L_{\alpha(0)}$ and $L_{\alpha(1)} \supseteq P_1 \cap T \supseteq L_{\alpha(0)}$. Continuing in this way, we obtain an ascending chain of countably generated submodules

$$P_0 \subseteq P_0 + L_{\alpha(0)} \subseteq P_1 \subseteq P_1 + L_{\alpha(1)} \subseteq \dots \subseteq P_n \subseteq P_n + L_{\alpha(n)} \subseteq \dots \quad (n < \omega_0),$$

such that for each $n < \omega_0$, $P_n \in \mathcal{N}$, $L_{\alpha(n)} \subseteq L_{\alpha(n+1)}$ and $P_n \cap T \subseteq L_{\alpha(n)}$ so that $P_n + L_{\alpha(n)} = (P_n \cap S) \oplus L_{\alpha(n)}$. If we set $P = \bigcup_{n < \omega_0} P_n$, then P is a countably generated member of \mathcal{N} with $K \subseteq P$. Moreover,

$$\begin{aligned} P &= \bigcup_{n < \omega_0} [(P_n \cap S) \oplus L_{\alpha(n)}] \\ &= [(\bigcup_{n < \omega_0} P_n) \cap S] \oplus (\bigcup_{n < \omega_0} L_{\alpha(n)}) \\ &= (P \cap S) \oplus L_\alpha \end{aligned}$$

for some $\alpha < \omega_1$. Therefore, $P \in \mathcal{N}_M$ as well.

Finally, we construct a weak nice system for S . To do this, set

$$\mathcal{N}_S = \{P \cap S : P \in \mathcal{N} \text{ and } P = (P \cap S) \oplus L_\alpha \text{ for some } \alpha \leq \omega_1\}.$$

Since each $P \in \mathcal{N}$ is nice in M , it follows that \mathcal{N}_S consists of nice submodules of S . Moreover, $0 \in \mathcal{N}_S$ and each countably generated submodule of S is contained in a countably generated member of \mathcal{N}_S , since \mathcal{N}_M has the corresponding properties. Therefore, to complete the proof, it suffices to show that \mathcal{N}_S is closed under ascending unions. Towards this end, suppose that

$$Q_0 \subseteq Q_1 \subseteq \dots Q_\beta \subseteq \dots \quad (\beta < \mu)$$

is an ascending chain in \mathcal{N}_S such that, for each $\beta < \mu$, $Q_\beta = P_\beta \cap S$, with $P_\beta \in \mathcal{N}$, and $P_\beta = (P_\beta \cap S) \oplus L_{\alpha(\beta)} = Q_\beta \oplus L_{\alpha(\beta)}$ for some $\alpha(\beta) \leq \omega_1$. Thus, associated with the chain there is a function $\alpha : \mu \rightarrow [0, \omega_1]$ with the property that if $\beta < \mu$, then $P_\beta = Q_\beta \oplus L_{\alpha(\beta)}$. Now define

$$\Gamma = \{\gamma \leq \omega_1 : \alpha(\beta) \geq \gamma \text{ for arbitrarily large } \beta < \mu\}$$

and observe that $0 \in \Gamma$ so that Γ is not empty. We now consider three cases.

Case 1: Γ has a least upper bound $\gamma_1 \in \Gamma$. Select $\beta_1 < \mu$ such that $\gamma_1 \leq \alpha(\beta_1)$. If there is a cofinal subset C of μ such that $\alpha(\beta) = \alpha(\beta_1)$ for all $\beta \in C$ (which in particular would hold if $\gamma_1 = \omega_1$), we readily obtain the conclusion that $\bigcup_{\beta < \mu} Q_\beta \in \mathcal{N}_S$.

Otherwise we may assume without loss that $\gamma_1 < \alpha(\beta_1)$ so that $\alpha(\beta_1) \notin \Gamma$. In this case, there exists a cofinal subset C of μ such that $\beta > \beta_1$ for every $\beta \in C$, and whenever $\beta, \eta \in C$, with $\beta < \eta$ then $\alpha(\beta) > \alpha(\eta) > \gamma_1$. But then we have the contradiction $\inf\{\alpha(\beta) : \beta \in C\} = \gamma_1 \notin \Gamma$.

Case 2: Γ has a least upper bound $\gamma_1 \notin \Gamma$. By passing to cofinal subchain if necessary, we may assume that $\alpha(\beta) < \gamma_1$ for all $\beta < \mu$. In this case, for every $\beta < \mu$, there exists $\eta > \beta$ such that $\alpha(\eta) \geq \gamma \geq \alpha(\beta)$ for some $\gamma \in \Gamma$. Thus we can pass to a further cofinal subchain where the P_β 's and the $L_{\alpha(\beta)}$'s ascend. It now follows that $\bigcup_{\beta < \mu} Q_\beta \in \mathcal{N}_S$.

Case 3: $\omega_1 \notin \Gamma$ and Γ is unbounded. In this case, for each $\beta < \mu$, there exists $\eta < \mu$

such that $\beta < \eta$ and $\alpha(\beta) < \alpha(\eta)$. Thus, by passing to a cofinal subchain, we may assume that the P_β 's and the $L_{\alpha(\beta)}$'s ascend. It again follows that $\bigcup_{\beta < \mu} Q_\beta \in \mathcal{N}_S$. \square

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