Bol. Soc. Paran. Mat. (3s.) **v. 38** 1 (2020): 83–100.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v38i1.33534

Oscillation Criteria for Generalized Liénard Type System

Tohid Kasbi, Vahid Roomi and Aliasghar Jodayree Akbarfam

ABSTRACT: In this work we use qualitative theory of differential equations to study the qualitative behavior of the solutions of a generalized Liénard system. Under quite general assumptions we present some sharp conditions under which the solutions of the system are oscillatory. Some examples are presented to illustrate our results.

Key Words: Oscillation, Liénard system, Ordinary differential equation.

Contents

1 Introduction **83**

2 Sufficient conditions for property (X^+) **86**

3 Sufficient conditions for property (Y^+) **94**

4 An Oscillation Theorem **98**

1. Introduction

It is well known that the Liénard system

$$\dot{x} = y - F(x), \quad \dot{y} = -g(x), \tag{1.1}$$

is of grate importance in various applications. Hence, asymptotic and qualitative behavior of this system and some of its extensions have been widely studied by many authors. Various questions on the stability, boundedness, oscillation and periodicity of solutions of (1.1) and its generalizations have received a considerable amount of attention in the last four decades [see 1-19]. We consider the system of two differential equations

$$\dot{x} = h(k(y) - F(x)), \quad \dot{y} = -g(x), \tag{1.2}$$

which is a generalized Liénard system, where F, g , k and h are continuous functions which ensure the existence of a unique solution to the initial value problem. Moreover, the following assumptions hold.

(C$_1$) $F(x)$ and $g(x)$ are continuous on $\mathbb{R}$ with $F(0) = 0$, $xg(x) > 0$ for $x \neq 0$, $h(u)$ is continuously differentiable and strictly increasing with $h(0) = 0$ and $h(\pm\infty) = \pm\infty$ and $k(u)$ is continuously differentiable and strictly increasing with $k(0) = 0$ and $k(\pm\infty) = \pm\infty$.

2010 *Mathematics Subject Classification:* 34A12, 34C10.
Submitted September 18, 2016. Published July 24, 2017

Under these assumptions, the origin is the unique critical point for system (1.2).

In order to study the global asymptotic stability of the zero solution, oscillation problem and existence of periodic solutions of system (1.2) the significant point is to find conditions for deciding whether all trajectories intersect the vertical isocline $k(y) = F(x)$.

In [19], the authors proved a proposition about the existence of a unique solution for initial value problem corresponding to system (1.1). In the following, we prove the same result for system (1.2).

Proposition 1.1. *If (C_1) is satisfied, then for any initial point $p(x_0, y_0)$, system (1.2) has a unique orbit passing through p.*

Proof. By Peano's Theorem (see [13] p. 10]), (1.2) has at least one solution $(x(t), y(t))$ satisfying $x(0) = x_0$ and $y(0) = y_0$. Along such a solution, we have

$$\begin{aligned} &\frac{dy}{dx} = -\frac{g(x)}{h(k(y) - F(x))} \\ &y(x_0) = y_0. \end{aligned} \tag{1.3}$$

In order to prove this proposition, we only have to prove that if $p \neq O = (0, 0)$, then the initial value problem (1.3) has a unique solution.
(i) Suppose $p \notin \Gamma = \{(x, y) \in \mathbb{R}^2 : y = k^{-1}(F(x))\}$, that is, $y_0 \neq k^{-1}(F(x_0))$. Then there exists a rectangle E : $|x - x_0| \leq a \text{ and } |y - y_0| \leq b$ such that E does not intersect Γ. Therefore, (C_1) implies that $\frac{\partial}{\partial y}(\frac{g(x)}{h(k(y)-F(x))})$ is continuous on E. Applying the Picard-Lindelöf Theorem, we know that the initial value problem (1.3) has a unique solution on E.
(ii) Suppose $p \in \Gamma$, that is $y_0 = k^{-1}(F(x_0))$, for example, $x_0 > 0$. If the conclusion is not true in this case, then (1.3) has two solutions $y = y_i(x)$ with $y_i(x_0) = y_0$, for $i = 1, 2$ and $y_1(x) \not\equiv y_2(x)$ for $x_1 \leq x < x_0$. We may assume $y = y_i(x)$ ($[x_1, x_0]$) is under the characteristic curve Γ for $i = 1, 2$. Thus, there is an $x^* \in [x_1, x_0)$ with $y_1(x^*) > y_2(x^*)$. Set

$$\bar{x} = Sup\ \{x : x \in [x^*, x_0) \ \ such \ \ that \ \ y_1(s) > y_2(s) \ \ for \ \ any \ \ s \in [x^*, x]\}.$$

Then, $y_1(x) > y_2(x)$ for $x \in [x^*, \bar{x}]$ and $y_1(\bar{x}) = y_2(\bar{x})$. This shows that (1.3) has two solutions passing through the point $(\bar{x}, y_1(\bar{x}))$.The first step (i) implies that $(\bar{x}, y_1(\bar{x})) \in \Gamma$. Hence, $\bar{x} = x_0$. Using (1.3), we obtain that

$$\frac{d(y_1(x) - y_2(x))}{dx} = \frac{g(x)(h(k(y_1(x) - F(x)) - h(k(y_2(x) - F(x)))}{h(k(y_1(x) - F(x))h(k(y_2(x) - F(x))}. \tag{1.4}$$

It follows from (C_1) that $h(u)$ is strictly increasing with $uh(u) > 0$ for $u \neq 0$. Therefore, from $y_i(x) - F(x) < 0$ for $x \in [x^*, x_0)$ and $y_1(x) > y_2(x)$ and (1.4), we can conclude that

$$\frac{d(y_1(x) - y_2(x))}{dx} > 0 \ \ for \ \ x \in [x^*, x_0).$$

This implies that $y_1(x) - y_2(x)$ is strictly increasing on $[x^*, x_0]$. Thus, $y_1(x) - y_2(x) < y_1(x_0) - y_2(x_0) = 0$, that is, $y_1(x) < y_2(x)$ for $x \in [x^*, x_0)$, a contradiction. This completes the proof.

Definition 1.2. *System* (1.2) *has property* (X^+) *in the right half-plane (resp., in the left half-plane) if for every point* (x_0, y_0) *with* $k(y_0) > F(x_0)$ *and* $x_0 \geq 0$ *(resp.,* $k(y_0) < F(x_0)$ *and* $x_0 \leq 0$*), the positive semi-orbit of* (1.2) *passing through* (x_0, y_0) *crosses the vertical isocline* $k(y) = F(x)$.

Definition 1.3. *System* (1.2) *has property* (Y^+) *in the right half-plane (resp., in the left half-plane) if for every point* $p(x_0, y_0)$ *with* $k(y_0) = F(x_0)$ *and* $x_0 \geq 0$ *(resp.,* $k(y_0) = F(x_0)$ *and* $x_0 \leq 0$*), the positive semi-orbit of* (1.2) *starting at* $p(x_0, y_0)$ *intersects the negative (resp., positive)* y*-axis.*

In this paper we will find conditions for deciding whether system (1.2) has property (X^+) and (Y^+) in the right and left half-planes. Our results extend the results of Villari and Zanolin, and Hara and Sugie for this system with $h(x) = x$ and $k(y) = y$ and improve the results presented by Sugie et al. and Gyllenberg and Ping.

Gyllenberg and Yan in [8] proved the following theorem which substantially extended and improved previous results presented by Aghajani and Moradifam [1] which already have included the most of previous sufficient conditions for property (X^+) in the right half-plane for system (1.2) with $h(x) = x$ and $k(y) = y$.

Theorem 1.4. *(see [8]) Suppose that* $G(+\infty) = +\infty$*. Then system* (1.2) *with* $h(x) = x$ *and* $k(y) = y$ *has property* (X^+) *in the right half-plane if*

$$\limsup_{x \to +\infty} \left(\int_b^x \left(\frac{\alpha F(\eta) g(\eta)}{(2G(\eta))^{\frac{2+\alpha}{2}}} + \frac{2\sqrt{\alpha} g(\eta)}{(2G(\eta))^{\frac{1+\alpha}{2}}} \right) d\eta + \frac{F(x)}{(2G(x))^{\frac{\alpha}{2}}} \right) = +\infty \tag{1.5}$$

for some $b > 0$ *and* $\alpha > 0$*, where* $G(x) = \int_0^x g(\eta) d\eta$.

Recently, the authors in [4] proved the following theorem which substantially extend and improve previous results presented in [1,2].

Theorem 1.5. *Assume that* $G(+\infty) = +\infty$*,* $l = h'(0) \neq 0$ *and* $h(x) - h(y) \geq h(x - y)$ *for every* $y < x < 0$*. Then, system* (1.2) *with* $k(y) = y$ *has property* (X^+) *in the right half-plane if there exists* $a(t)$ *with* $a'(t) > 0$ *for* $t \geq \alpha > 0$ *and* $a(\beta) \geq 0$ *for some* $\beta \geq \alpha$*, such that*

$$\begin{aligned} \limsup_{x \to +\infty} \Bigg(\int_b^x \Big(& \frac{h(F(\eta)) a'(\sqrt{2G(\eta)}) g(\eta)}{a^2(\sqrt{2G(\eta)}) \sqrt{2G(\eta)}} + \frac{2\sqrt{l} \sqrt{a'(\sqrt{2G(\eta)}) g(\eta)}}{a^{3/2}(\sqrt{2G(\eta)}) \sqrt[4]{2G(\eta)}} \Big) d\eta \\ & + \frac{h(F(x))}{a(\sqrt{2G(x)})} \Bigg) = +\infty \end{aligned} \tag{1.6}$$

for some $b > 0$*, where* $G(x) = \int_0^x g(\eta) d\eta$.

However, Theorem 1.5 and previous results [1-3] are inapplicable to the system (1.2) with

$$F(x) = -\ln(|x|+1)sgnx, \quad k(y) = \gamma y + \lambda \tanh y, \quad h(x) = 2\sinh x, \quad and \quad g(x) = x, \tag{1.7}$$

for deciding whether all orbits of (1.7) cross the vertical isocline $k(y) = F(x)$ or not (see Example 2.2).

In the next section we will extend and improve Theorems 1.4, 1.5 and the previous results presented in [1-4] and we will derive some new sufficient conditions for property (X^+), which can be applied to system (1.7).

2. Sufficient conditions for property (X^+)

Note: Hereafter we assume that the condition $(\mathbf{C}_1)$ holds.

Let $G(x) = \int_0^x g(\eta)d\eta$. First, we introduce a system which is equivalent to (1.2). Define function $\phi(x)$ by $\phi(x) = \sqrt{2G(x)}sgn(x)$, and map $\Phi : R^2 \to R^2$ by $\Phi(x,y) = (\phi(x), y) \equiv (u,v)$. Changing variables $u = \sqrt{2G(x)}sgn(x)$, $v = y$, $d\tau = \frac{g(x)sgn(x)}{\sqrt{2G(x)}}dt$ and denoting τ by t again, we can transform system (1.2) into the following system

$$\begin{aligned} \dot{u} &= h(k(v) - F^*(u)) \\ \dot{v} &= -u, \end{aligned} \tag{2.1}$$

where F^* is a continuous function defined by $F^*(u) = F(G^{-1}(\frac{1}{2}u^2 sgn(u)))$, and $G^{-1}(w)$ is the inverse function to $G(x)sgn(x)$.

In [10] the authors proved that systems (1.2) and (2.1) with $h(x) = x$ are equivalent. Consequently, we have only to determine whether system (2.1), instead of (1.2), has property (X^+) or not.

Consider the following two conditions on function h.

$(\mathbf{A}_1)$ For every $y < x < 0$, assume that h satisfies the following condition.

$$h(x) - h(y) \geq h(x-y). \tag{2.2}$$

$(\mathbf{A}_2)$ For every $0 < x < y$, assume that h satisfies the following condition.

$$h(x) - h(y) \leq h(x-y). \tag{2.3}$$

Remark 2.1. *Suppose that $h(x)$ is an odd function. Then $(\mathbf{A}_1)$ and $(\mathbf{A}_2)$ are equivalent.*

Theorem 2.2. *Suppose that $(\mathbf{A}_1)$ holds and $h'(x)$ and $k'(x)$ are increasing for $x < 0$. If there exists $a(t)$ with $a'(t) > 0$ for $t \geq \alpha > 0$ and $a(\beta) \geq 0$ for some $\beta \geq \alpha$, such that*

$$\limsup_{u\to+\infty}\left(\int_b^u \frac{h(F^*(s))a'(s)+2\sqrt{a'(s)}\sqrt{sa(s)h'(F^*(s))k'(k^{-1}(F^*(s)))}}{a^2(s)}ds + \frac{h(F^*(u))}{a(u)}\right)=+\infty \tag{2.4}$$

for some $b > 0$. Then, system (2.1) has property (X^+) in the right half-plane.

Proof. We prove the theorem by contradiction. Suppose that there exists a solution $(u(t), v(t))$ of (2.1) whose graph remains in the region $\Lambda = \{(u, v) : u \geq 0 \text{ and } k(v) > F^*(u)\}$ for all future time. Let $(u_0, v_0) = (u(0), v(0)) \in \Lambda$. We claim that

$$\lim_{t\to+\infty} u(t) = +\infty. \tag{2.5}$$

If (2.5) is not true, then $\lim_{t\to+\infty} u(t) = \hat{u} < +\infty$. Let $\hat{P} = (\hat{u}, F^*(\hat{u})) \in \{(u, F^*(u)) : u \geq 0\}$ and $\overline{O\hat{P}}$ be the characteristic curve arc from O to $\hat{P}$. Then the positive semi-orbit of (2.1) starting from $\hat{P}$ is contained in the bounded domain surrounded by $v-$axis, $v = v_0$, $u = \hat{u}$ and arc from O to $\hat{P}$. Thus $\lim_{t\to+\infty}(u(t), v(t))$ must exists and from Poincare-Bendixson Theorem we conclude that it must be an equilibrium of (2.1). But from $\mathbf{(C_1)}$, the origin is the unique equilibrium of (2.1). This implies that $\hat{u} = 0$. Since system (2.1) has no critical points in this region, we have

$$u(t) \to +\infty \quad as\ t \to +\infty.$$

So, we may assume that u_0 is sufficiently large. We also assume $v_0 < 0$. Because

$$v(t) \leq v_0 - u_0 t \to -\infty \quad as \quad t \to +\infty.$$

Thus, it turns out that $u(t) \geq u_0 > 0$ and $v(t) \leq v_0 < 0$. for $t \geq 0$. Now let

$$w(t) = \int_\gamma^{a(u(t))} \frac{h(F^*(a^{-1}(s)))}{s^2} ds + \frac{h(k(v(t)))}{a(u(t))},$$

for some $\gamma \geq \beta$. Then

$$\begin{aligned}
\dot{w}(t) &= \frac{\dot{u}(t)a'(u(t))h(F^*(u(t)))}{a^2(u(t))} \\
&\quad + \frac{\dot{v}(t)k'(v(t))h'(k(v(t))a(u(t)) - \dot{u}(t)a'(u(t))h(k(v(t))}{a^2(u(t))} \\
&= \frac{-u(t)k'(v(t)h'(k(v(t))a(u(t)) - \dot{u}(t)a'(u(t))(h(k(v(t)) - h(F^*(u(t))))}{a^2(u(t))}.
\end{aligned}$$

Since $F^*(u(t)) < k(v(t)) < 0$ for $t > 0$ and by the assumptions $\mathbf{(A_1)}$ and $h'(x)$ and $k'(x)$ are increasing for $x < 0$, we get $h(k(v(t)) - h(F^*(u(t))) \geq h(k(v(t)) - F^*(u(t)))$

and $h'(k(v(t)) \geq h'(F^*(u))$ and $k'(v(t) \geq k'(k^{-1}(F^*(u(t))))$. Then

$$\begin{aligned}
&\dot{w}(t)\\
&\leq \frac{-u(t)k'(k^{-1}(F^*(u(t))h'(F^*(u(t)))a(u(t)) - \dot{u}(t)a'(u(t))h(k(v(t)) - F^*(u(t)))}{a^2(u(t))}\\
&= \frac{-u(t)k'(k^{-1}(F^*(u(t))h'(F^*(u(t)))a(u(t)) - \dot{u}^2(t)a'(u(t))}{a^2(u(t))}\\
&\leq \frac{-2\dot{u}(t)\sqrt{a'(u(t))}\sqrt{u(t)k'(k^{-1}(F^*(u(t)))h'(F^*(u(t)))}}{a^{\frac{3}{2}}(u(t))}.
\end{aligned}$$

Thus

$$\frac{d}{dt}\left(w(t) + \int_\gamma^{u(t)} \frac{2\sqrt{a'(s)}\sqrt{sk'(k^{-1}(F^*(s)))h'(F^*(s))}}{a^{\frac{3}{2}}(s)}ds\right) \leq 0.$$

Therefore,

$$\begin{aligned}
&\int_\gamma^{a(u(t))} \frac{h(F^*(a^{-1}(s)))}{s^2}ds + \frac{h(k(v(t)))}{a(u(t))}\\
&\quad + \int_\gamma^{u(t)} \frac{2\sqrt{a'(s)}\sqrt{sk'(k^{-1}(F^*(s)))h'(F^*(s))}}{a^{\frac{3}{2}}(s)}ds\\
&\leq \int_\gamma^{a(u_0)} \frac{h(F^*(a^{-1}(s)))}{s^2}ds + \frac{h(k(v_0))}{a(u_0)}\\
&\quad + \int_\gamma^{u_0} \frac{2\sqrt{a'(s)}\sqrt{sk'(k^{-1}(F^*(s)))h'(F^*(s))}}{a^{\frac{3}{2}}(s)}ds < +\infty.
\end{aligned}$$

for $t \geq 0$.
Since, $k(v(t)) > F^*(u(t))$ and $u(t) \to +\infty$ as $t \to +\infty$,

$$\begin{aligned}
&\int_\gamma^{a(u(t))} \frac{h(F^*(a^{-1}(s)))}{s^2}ds + \frac{h(F^*(u))}{a(u(t))}\\
&\quad + \int_\gamma^{u(t)} \frac{2\sqrt{k'(s)}\sqrt{sk'(k^{-1}(F^*(s)))h'(F^*(s))}}{a^{\frac{3}{2}}(s)}ds < +\infty.
\end{aligned}$$

Now by taking $z = a^{-1}(s)$ and $b = \max\{\gamma, a^{-1}(\gamma)\}$ we have

$$\begin{aligned}
\limsup_{u\to+\infty} \Bigg(&\int_b^u \frac{h(F^*(s))a'(s) + 2\sqrt{a'(s)}\sqrt{sa(s)h'(F^*(s))k'(k^{-1}(F^*(s)))}}{a^2(s)}ds\\
&+ \frac{h(F^*(u))}{a(u)}\Bigg) < +\infty.
\end{aligned}$$

This contradiction completes the proof.

Notice that if h satisfies $(\mathbf{A}_1)$, then by dividing two sides of (2.2) by $x - y$ and then $y \to x$ we conclude that $h'(x) \geq h'(0) \geq 0$ for $x < 0$. Now by the same way as in the proof of the theorem above we can prove the following theorem which does not need the condition $h'(x)$ be increasing.

Theorem 2.3. *Suppose that* $(\mathbf{A}_1)$ *holds and* $l = h'(0) \neq 0$ *and* $k'(x)$ *is increasing for* $x < 0$*. If there exists* $a(t)$ *with* $a'(t) > 0$ *for* $t \geq \alpha > 0$ *and* $a(\beta) \geq 0$ *for some* $\beta \geq \alpha$*, such that*

$$\limsup_{u\to+\infty}\left(\int_b^u \frac{h(F^*(s))a'(s) + 2\sqrt{a'(s)}\sqrt{sa(s)k'(k^{-1}(F^*(s)))l}}{a^2(s)}ds + \frac{h(F^*(u))}{a(u)}\right) = +\infty \tag{2.6}$$

for some $b > 0$*, then system* (2.1) *has property* (X^+) *in the right half-plane.*

The following two corollaries are obtained by straightforward application of the theorem above.

Corollary 2.4. *Assume* $(\mathbf{A}_1)$ *holds and* $l = h'(0) \neq 0$ *and* $k'(x)$ *is increasing for* $x < 0$*. If there exists* $a(t)$ *with* $a'(t) > 0$ *for* $t \geq \alpha > 0$ *and* $a(\beta) \geq 0$ *for some* $\beta \geq \alpha$*, such that*

$$\liminf_{u\to+\infty} \frac{h(F^*(u))}{a(u)} > -\infty \text{ and}$$
$$\limsup_{u\to+\infty} \int_b^u \frac{h(F^*(s))a'(s) + 2\sqrt{a'(s)}\sqrt{sa(s)k'(k^{-1}(F^*(s)))l}}{a^2(s)}ds = +\infty,$$

for some $b > 0$*. Then, system* (2.1) *has property* (X^+) *in the right half-plane.*

Corollary 2.5. *Suppose that* $(\mathbf{A}_1)$ *holds and* $l = h'(0) \neq 0$ *and* $k'(x)$ *is increasing for* $x < 0$*. If there exists* $a(t)$ *with* $a'(t) > 0$ *for* $t \geq \alpha > 0$*,* $a(\beta) \geq 0$ *for some* $\beta \geq \alpha$ *and* $\int_b^\infty \frac{1}{a(s)}ds = +\infty$ *for some* $b > 0$*, such that*

$$\liminf_{u\to+\infty} \frac{h(F^*(u))}{a(u)} > -\infty \text{ and}$$
$$\liminf_{u\to+\infty} \frac{h(F^*(u))\sqrt{a'(u)}}{\sqrt{a(u)}\sqrt{uk'(k^{-1}F^*(u))}} > -2\sqrt{l},$$

then, system (2.1) *has property* (X^+) *in the right half-plane.*

Recall the definition of function $F^*(u)$ under the assumption $G(+\infty) = +\infty$ as follows:

$$F^*(u) = F\left(G^{-1}\left(\frac{1}{2}u^2\right)\right) \quad for \quad u \geq 0.$$

Put $x = G^{-1}(\frac{1}{2}u^2)$. Then we have the following result for the system (1.2) which is equivalent to Theorem 2.3.

Theorem 2.6. *Assume that* $G(+\infty) = +\infty$*,* $l = h'(0) \neq 0$ *and* $(\mathbf{A}_1)$ *holds and* $k'(x)$ *is increasing for* $x < 0$*. Then, system* (1.2) *has property* (X^+) *in the right half-plane if there exists* $a(t)$ *with* $a'(t) > 0$ *for* $t \geq \alpha > 0$ *and* $a(\beta) \geq 0$ *for some*

$\beta \geq \alpha$, such that

$$\limsup_{x\to+\infty}\Bigg(\int_b^x \Big(\frac{h(F(\eta))a'(\sqrt{2G(\eta)})g(\eta)}{a^2(\sqrt{2G(\eta)})\sqrt{2G(\eta)}}+ \frac{2\sqrt{l}\sqrt{a'(\sqrt{2G(\eta)})}\sqrt{k'(k^{-1}(F(\eta)))}g(\eta)}{a^{3/2}(\sqrt{2G(\eta)})\sqrt[4]{2G(\eta)}}\Big)d\eta + \frac{h(F(x))}{a(\sqrt{2G(x)})}\Bigg) = +\infty, \tag{2.7}$$

for some $b > 0$.

Corollaries 2.4 and 2.5 can be formulated for system (1.2) in the same manner. The following theorem is useful in applications.

Theorem 2.7. *Suppose that $h_1(x) \leq h_2(x)$ for $x > 0$. If system (1.2) with $h_2(x)$ as $h(x)$ has property (X^+) in the right half-plane, then it has property (X^+) in the right half-plane with $h_1(x)$ as $h(x)$ too.*

Proof. We prove the theorem by contradiction. Suppose that system (1.2) with $h_1(x)$ fails to have property (X^+) in the right half-plane. Then there exists a positive semi-orbit $O_1^+(p)$ of (1.2) starting at a point $p(x_0, y_0)$ with $h_1(y_0) > F(x_0)$, which does not meet the characteristic curve $h_1(y) = F(x)$. Suppose that $O_2^+(p)$ is a positive semi-orbit of (1.2) with $h_2(x)$ as $h(x)$. From $h_1(x) \leq h_2(x)$ we have:

$$(\frac{\dot{y}}{\dot{x}})_{h_1} = \frac{-g(x)}{h_1(k(y)-F(x))} \leq \frac{-g(x)}{h_2(k(y)-F(x))} = (\frac{\dot{y}}{\dot{x}})_{h_2} \leq 0.$$

The last relation shows that the slope of $O_1^+(p)$ is less than the slope of $O_2^+(p)$. Therefore, system (1.2) corresponding to $h_2(x)$ fails to have property (X^+) in the right half-plane. This contradiction completes the proof.

The same results can be proven for functions k, F and g.

Now, we show that how our results are related to those listed in the introduction. Also, we give some examples which the previous results are inapplicable.

Example 2.1. *Consider system (1.2) with*

$$F(x) = 6x - 5x\sin^2 x, \quad k(y) = y, \quad h(x) = g(x) = x,$$

Note that for every $b > 0$,

$$\begin{aligned}\liminf_{x\to+\infty}\frac{1}{F(x)}\int_b^x \frac{g(\eta)}{F(\eta)}d\eta &\leq \lim_{n\to+\infty}\frac{\int_b^{2n\pi}\frac{1}{6-5\sin^2\eta}d\eta}{12n\pi}\\ &\leq \lim_{n\to+\infty}\frac{1}{12n\pi}\int_b^{2n\pi}d\eta = \frac{1}{6} < \frac{1}{4}.\end{aligned}$$

Thus, all of the results presented by Filippov [5], Hara and Yoneyama [[10], Lemma 4.2], Villari and Zanolin [[17], Theorem 2.6] and Hara and Sugie [[11], *Theorem 5.1] are inapplicable to this system. From Corollary 2.4 with $a(t) = t$ we have*

$$\liminf_{u\to+\infty} \frac{h(F^*(u))}{a(u)} = \liminf_{u\to+\infty}(6 - 5\sin^2 u) = 1 > -\infty, \text{ and}$$

$$\limsup_{u\to+\infty} \int_b^u \frac{h(F^*(s))a'(s) + 2\sqrt{a'(s)}\sqrt{sa(s)k'(k^{-1}(F^*(s))l}}{a^2(s)} ds$$

$$= \limsup_{u\to+\infty} \int_b^u \frac{8 - 5\sin^2 s}{s} ds = +\infty \text{ for some } b > 0,$$

Hence system (1.2) has property (X^+) in the right half-plane.

Example 2.2. *Consider system (1.2) with*

$$F(x) = -\ln(|x| + 1)sgn(x), \quad k(y) = \lambda \tanh y + \gamma y \text{ with } \gamma > \frac{1}{8} \text{ and } \lambda > 0,$$
$$h(x) = e^x - e^{-x}, \quad \text{and} \quad g(x) = x,$$

Obviously, $h'(0) = 2 > 0$ and if $y < x < 0$ we have

$$e^y \le 1 \Rightarrow -e^{y-x} \ge -e^{-x} \Rightarrow \frac{e^{y-x} - 1}{e^{x-y} - 1} \ge \frac{e^{-x} - 1}{e^x - 1}.$$

Since $(e^x - 1).(e^{x-y} - 1) < 0$ we have

$$(e^{y-x} - 1)(e^x - 1) \le (e^{-x} - 1)(e^{x-y} - 1) \Rightarrow$$
$$e^y - e^x - e^{-y} + e^{-x} \le e^{y-x} - e^{x-y} \Rightarrow$$
$$h(x) - h(y) \ge h(x - y).$$

Thus, h satisfies $(\mathbf{A}_1)$. Now, by choosing $a(t) = t$, we have

$$\limsup_{u\to+\infty} \left(\int_b^u \frac{h(F^*(s)) + 2s\sqrt{2k'(k^{-1}(F^*(s))}}{s^2} ds + \frac{h(F^*(u))}{u} \right)$$

$$\ge \limsup_{u\to+\infty} \left(\int_b^u \frac{(\frac{1}{s+1} - (s+1) + 2\sqrt{2\gamma}s}{s^2} ds + \frac{\frac{1}{u+1} - (u+1)}{u} \right)$$

$$= \limsup_{u\to+\infty} \left(\int_b^u \left(\frac{2\sqrt{2\gamma} - 1}{s+1} + \frac{2\sqrt{2\gamma} - 2}{s(s+1)} \right) ds \right) - 1 = +\infty$$

Therefore, by Theorem 2.3 this system has property (X^+) in the right half-plane.

Example 2.3. *Consider system (1.2) with*

$$F(x) = -x\sin^2(x)\ln(|\, x\,| + 1), \quad k(y) = y \text{ and } h(x) = g(x) = x.$$

Notice that for any $\alpha > 0$ and $b > 0$ we have

$$\limsup_{x\to+\infty} \left(\int_b^x \left(\frac{\alpha F(\eta)g(\eta)}{(2G(\eta))^{\frac{2+\alpha}{2}}} + \frac{2\sqrt{\alpha}g(\eta)}{(2G(\eta))^{\frac{1+\alpha}{2}}} \right) d\eta + \frac{F(x)}{(2G(x))^{\frac{\alpha}{2}}} \right)$$

$$= \limsup_{x\to+\infty} \left(\int_b^x \left(\frac{-\alpha\sin^2(\eta)\ln(\eta+1) + 2\sqrt{\alpha}}{\eta^\alpha} \right) d\eta + \frac{-\sin^2(x)\ln(x+1)}{x^{\alpha-1}} \right)$$

$$\ne +\infty.$$

Therefore, (1.5) *does not hold. So, Theorem 1.4 and all previous results are inapplicable to this system. However, let* $a(t) = \frac{t}{t+1}$*, then*

$$\begin{aligned}
&\limsup_{u\to+\infty}\left(\int_b^u \frac{h(F^*(s))a'(s)+2\sqrt{a'(s)}\sqrt{sa(s)k'(k^{-1}(F^*(s)))h'(F^*(s))}}{a^2(s)}ds\right.\\
&\left.+\frac{h(F^*(u))}{a(u)}\right)\\
&=\limsup_{u\to+\infty}\left(\int_b^u \frac{2\sqrt{s+1}-\sin^2(s)\ln(s+1)}{s}ds-(1+u)\sin^2(u)\ln(u+1)\right)\\
&\geq \limsup_{u=k\pi\to+\infty}\left(\int_b^u \frac{2\sqrt{s+1}-\sin^2(s)\ln(s+1)}{s}ds\right)=+\infty \quad for \quad some \quad b>0.
\end{aligned}$$

Therefore, by Theorem 2.2 system (1.2) *has property* (X^+) *in the right half-plane.*

The following analogous results are obtained with respect to property (X^+) in the left half-plane.

Theorem 2.8. *Suppose that* $(\mathbf{A}_2)$ *holds and* $h'(x)$ *is increasing and* $k'(x)$ *is decreasing for* $x > 0$*. If there exists* $a(t)$ *with* $a'(t) > 0$ *for* $t \leq \alpha < 0$ *and* $a(\beta) \leq 0$ *for some* $\beta \leq \alpha$*, such that*

$$\begin{aligned}
&\liminf_{u\to-\infty}\left(\int_u^b \frac{h(F^*(s))a'(s)-2\sqrt{a'(s)}\sqrt{sa(s)h'(F^*(s))k'(k^{-1}(F^*(s))}}{a^2(s)}ds\right.\\
&\qquad\left.-\frac{h(F^*(u))}{a(u)}\right)=-\infty,
\end{aligned} \tag{2.8}$$

for some $b < 0$*, then system* (2.1) *has property* (X^+) *in the left half-plane.*

Notice that if h satisfies $(\mathbf{A}_2)$, then by dividing two sides of (2.3) by $x - y$ and then $y \to x$, we conclude that $h'(x) \geq h'(0) \geq 0$ for $x > 0$. Similar to the Theorem 2.3 we have the following theorem which does not need the condition $h'(x)$ be increasing.

Theorem 2.9. *Suppose that* $(\mathbf{A}_2)$ *holds and* $l = h'(0) \neq 0$ *and* $k'(x)$ *is decreasing for* $x > 0$*. If there exists* $a(t)$ *with* $a'(t) > 0$ *for* $t \leq \alpha < 0$ *and* $a(\beta) \leq 0$ *for some* $\beta \leq \alpha$*, such that*

$$\liminf_{u\to-\infty}\left(\int_u^b \frac{h(F^*(s))a'(s)-2\sqrt{a'(s)}\sqrt{sa(s)k'(k^{-1}(F^*(s))l}}{a^2(s)}ds-\frac{h(F^*(u))}{a(u)}\right)=-\infty \tag{2.9}$$

for some $b < 0$*, then system* (2.1) *has property* (X^+) *in the left half-plane.*

Corollary 2.10. *Assume* $(\mathbf{A}_2)$ *holds and* $l = h'(0) \neq 0$ *and* $k'(x)$ *is decreasing for* $x > 0$*. If there exists* $a(t)$ *with* $a'(t) > 0$ *for* $t \leq \alpha < 0$ *and* $a(\beta) \leq 0$ *for some*

$\beta \le \alpha$, such that

$$\liminf_{u\to-\infty} \frac{h(F^*(u))}{a(u)} > -\infty \text{ and}$$
$$\liminf_{u\to-\infty} \int_u^b \frac{h(F^*(s))a'(s) - 2\sqrt{a'(s)}\sqrt{sa(s)k'(k^{-1}(F^*(s)))l}}{a^2(s)} ds = -\infty,$$

for some $b < 0$. Then system (2.1) has property (X^+) in the left half-plane.

Corollary 2.11. *Suppose that $(\mathbf{A}_2)$ holds and $l = h'(0) \neq 0$ and $k'(x)$ is decreasing for $x > 0$. If there exists $a(t)$ with $a'(t) > 0$ for $t \le \alpha < 0$, $a(\beta) \le 0$ for some $\beta \le \alpha$ and $\int_{-\infty}^b \frac{1}{a(s)} ds = -\infty$ for some $b < 0$, such that*

$$\liminf_{u\to-\infty} \frac{h(F^*(u))}{a(u)} > -\infty \text{ and}$$
$$\limsup_{u\to-\infty} \frac{h(F^*(u))\sqrt{a'(u)}}{\sqrt{a(u)}\sqrt{uk'(k^{-1}(F^*(u)))}} < 2\sqrt{l},$$

then system (2.1) has property (X^+) in the left half-plane.

Now recall defining the function $F^*(u)$ under the assumption $G(-\infty) = +\infty$ as follows:

$$F^*(u) = F\left(G^{-1}\left(-\frac{1}{2}u^2\right)\right) \quad for \quad u < 0$$

and put $x = G^{-1}(-\frac{1}{2}u^2)$. Then we have the following result for the system (1.2) which is equivalent to Theorem 2.9.

Theorem 2.12. *Assume that $G(-\infty) = +\infty$, $l = h'(0) \neq 0$ and $(\mathbf{A}_2)$ holds and $k'(x)$ is decreasing for $x > 0$. Then, system (1.2) has property (X^+) in the left half-plane if there exists $a(t)$ with $a'(t) > 0$ for $t \le \alpha < 0$ and $a(\beta) \le 0$ for some $\beta \le \alpha$, such that*

$$\liminf_{x\to-\infty}\Bigg(\int_x^b \Bigg(-\frac{h(F(\eta))a'(-\sqrt{2G(\eta)})g(\eta)}{a^2(-\sqrt{2G(\eta)})\sqrt{2G(\eta)}} + \frac{2\sqrt{l}\sqrt{a'(-\sqrt{2G(\eta)})}\sqrt{k'(k^{-1}(F(\eta))}g(\eta)}{(-a(-\sqrt{2G(\eta)}))^{3/2}\sqrt[4]{2G(\eta)}}\Bigg)d\eta - \frac{h(F(x))}{a(-\sqrt{2G(x)})}\Bigg) = -\infty, \tag{2.10}$$

for some $b < 0$.

Corollaries 2.10 and 2.11 can be formulated for system (1.2) in the same maner.

Example 2.4. *Consider system (1.2) with functions given in Example 2.2. In Example 2.2 it is proved that, function h satisfies $(\mathbf{A}_1)$. Since h is an odd function,* *by Remark 2.1, h satisfies* $(\mathbf{A}_2)$. *Now, by choosing* $a(t)=t$, *we have*

$$\liminf_{u\to-\infty}\frac{h(F(u))}{a(u)}=\frac{(1-u)-\frac{1}{1-u}}{u}=-1>-\infty \quad and$$
$$\liminf_{u\to-\infty}\int_u^b\frac{h(F(s))a'(s)-2\sqrt{a'(s)}\sqrt{sa(s)k'(k^{-1}(F(s))l}}{a^2(s)}ds$$
$$\le\liminf_{u\to-\infty}\int_u^b\frac{(1-s)-\frac{1}{1-s}-2\mid s\mid\sqrt{2\gamma}}{s^2}ds$$
$$\le\liminf_{u\to-\infty}\left(\int_u^b\left(\frac{1-2\sqrt{2\gamma}}{1-s}+\frac{2\sqrt{2\gamma}-2}{s(1-s)}\right)ds\right)=-\infty \quad for \quad some \quad b<0,$$

Therefore, by Corollary 2.10 this system has property (X^+) *in the left half-plane.*

3. Sufficient conditions for property (Y^+)

In this section we present some sufficient conditions for system (1.2) to have property (Y^+) in the right and left half-planes. Many authors have also investigated property (Y^+) and several interesting sufficient conditions have been given. Hara and Yoneyama in [10] proved that if there exists an $a>0$ such that $F(x)>0$ for $0<x\le a$ and some $\alpha>\frac{1}{4}$ such that

$$\frac{1}{F(x)}\int_0^x\frac{g(\eta)}{F(\eta)}d(\eta)\ge\alpha,$$

then system (1.2) with $k(y)=y$ and $h(u)=u$ has property (Y^+) (see also [5,9,16]).

In this section we will extend the above result and will obtain some sufficient conditions which can be applied when none of the sufficient conditions presented in the pervious literatures are applicable. First, we have a lemma about asymptotic behavior of solution of (1.2) and then we will give some results about the property of (Y^+) in the right half-plane for system (1.2).

Lemma 3.1. *For each point* $P(p,k^{-1}(F(p))$ *with* $p>0$, *the positive semi-orbit of* (1.2) *starting at* P *crosses the negative* y*-axis if the following condition hold.*
$(\mathbf{A}_3)$ *There exists a* $\delta>0$ *such that* $F(x)<0$ *for* $0<x<\delta$ *or* $F(x)$ *has an infinite number of positive zeroes clustering at* $x=0$.

Proof. We prove the theorem by contradiction. Suppose that there exists a point $P(p,k^{-1}(F(p))$ with $p>0$, such that the positive semi-orbit of (1.2) starting at P does not cross the negative y-axis. Let $(x(t),y(t))$ be a solution of (1.2) defined on interval $[t_0,+\infty)$ with $(x(t_0),y(t_0))=P$. Then we have

$$0<x(t)<y(t_0) \quad for \quad t\ge t_0.$$

and

$$y(t)\to+\infty \quad as \quad t\to-\infty.$$

Hence, it follows from the first equation of (1.2) that

$$\dot{x}(t) \to -\infty \quad as \quad t \to -\infty.$$

And therefore, there exists $t_1 > t_0$ such that

$$\dot{x}(t) \leq -1 \quad for \quad t \geq t_1.$$

Integrating leads to

$$-x(t_1) < x(t) - x(t_1) \leq t_1 - t \to -\infty \quad as \quad t \to -\infty.$$

This is a contradiction and the proof is complete.

Hereafter we assume that there exists a $\delta > 0$ such that $F(x) > 0$ for $0 < x < \delta$.

Theorem 3.2. *Suppose that* $(\mathbf{A}_2)$ *holds and* $h'(x)$ *is increasing for* $x > 0$ *and* $k'(x)$ *is decreasing for* $x > 0$. *If there exists* $a(t)$ *with* $a'(t) > 0$ *for* $t \geq \alpha > 0$ *and* $a(\beta) > 0$ *for* $\beta \geq \alpha$ *such that*

$$\liminf_{x \to 0^+} \left(\int_x^b \left(\frac{h(F(s))g(s)a'(s)}{(2G(a(s))^{3/2}} - \frac{\sqrt{a'(s)}g(s)\sqrt{h'(F(s))k'(k^{-1}(F(s))}}{G(a(s))} \right) ds \right) = -\infty \tag{3.1}$$

for some $b > 0$, *then system* (1.2) *has property* (Y^+) *in the right half-plane.*

Proof. We prove the theorem by contradiction. Suppose that there exists a point $P = (p, k^{-1}(F(p))$ with $p > 0$ such that the semi-orbit of (1.2) passing through $P = (p, k^{-1}(F(p))$ does not intersect the positive y-axis. Now suppose that there exists a solution $(x(t), y(t))$ of (1.2) which starts at point (x_0, y_0) with $k(y_0) = F(x_0)$ and whose graph remains in the region $\{(x, y) : x > 0 \ and \ k(y) < F(x)\}$ for all future time. Taking the vector field of (1.2) into account, we see that if the positive semi-orbit of this system crosses the x-axis, then it also meets the negative y-axis. Since $\dot{x} < 0$ in this region, we have

$$\lim_{t \to \infty} x(t) = \lim_{t \to \infty} y(t) = 0.$$

So, we may assume that u_0 is sufficiently small and

$$0 < x(t) \leq u_0 \quad for \quad t > 0.$$

Now let

$$\chi(t) = \int_\gamma^{a(x(t))} \frac{h(F(a^{-1}(s)))g(a^{-1}(s))}{(2G(s)^{3/2})} ds + \frac{h(k(y(t)))}{\sqrt{2G(a(x(t)))}},$$

then

$$\begin{aligned}
\dot{\chi}(t) &= \dot{x}(t)a'(x(t))\frac{h(F(x(t)))g(x(t))}{(2G(a(x(t))^{3/2})} \\
&+ \frac{\dot{y}(t)k'(y(t))h'(k(y(t)))\sqrt{G(a(x(t))}}{2G(a(x(t)))} \\
&- \frac{\dot{x}(t)a'(x(t))g(x(t))\frac{1}{\sqrt{2G(a(x(t))}}h(k(y(t)))}{2G(a(x(t)))} \\
&= -\frac{2g(x(t))k'(y(t))h'(k(y(t))G(a(x(t))}{(2G(a(x(t))^{3/2})} \\
&+ \frac{a'(x(t))g(x(t))\big(h(k(y(t)) - h(F(x(t))\big)\dot{x}(t)}{(2G(a(x(t))^{3/2})} \\
&\leq \frac{\sqrt{a'(x(t)}g(x(t))\sqrt{k'(k^{-1}(F(x(t)))h'(F(x(t)))}}{\sqrt{2G(a(x(t)))}}\dot{x}(t).
\end{aligned}$$

Thus,

$$\frac{d}{dt}\left(\chi(t) - \int_{\gamma}^{x(t)} \frac{\sqrt{a'(s)}g(s)\sqrt{k'(k^{-1}(F(s)))h'(F(s))}}{\sqrt{2G(a(s))}}ds\right) \leq 0.$$

Therefore,

$$\begin{aligned}
&\int_{\gamma}^{a(x(t))} \frac{h(F(a^{-1}(s)))g(a^{-1}(s))}{(2G(s)^{3/2})}ds + \frac{h(k(y(t)))}{\sqrt{2G(a(x(t)))}} \\
&\quad - \int_{\gamma}^{x(t)} \frac{\sqrt{a'(s)}g(s)\sqrt{k'(k^{-1}(F(s)))h'(F(s))}}{\sqrt{2G(a(s))}}ds \\
&\leq \int_{\gamma}^{a(u_0)} \frac{h(F(a^{-1}(s)))g(a^{-1}(s))}{(2G(s)^{3/2})}ds + \frac{h(k(v_0))}{\sqrt{2G(a(u_0))}} \\
&\quad - \int_{\gamma}^{u_0} \frac{\sqrt{a'(s)}g(s)\sqrt{k'(k^{-1}(F(s)))h'(F(s))}}{\sqrt{2G(a(s))}}ds < +\infty
\end{aligned}$$

for $t > 0$. Now by taking $z = a^{-1}(s)$ and $b = max\{\gamma, a^{-1}(\gamma)\}$ we have

$$\begin{aligned}
\limsup_{x\to 0^+}\Bigg(\int_b^{x(t)}\Bigg(\frac{h(F(s))g(s)a'(s)}{(2G(a(s))^{3/2}} - \frac{\sqrt{a'(s)}g(s)\sqrt{h'(F(s))k'(k^{-1}(F(s))}}{G(a(s))}\Bigg)ds \\
+ \frac{h(k(y(t))}{\sqrt{(G(a(x(t))}}\Bigg) < +\infty.
\end{aligned} \tag{3.2}$$

Since, $\frac{h(k(y(t)))}{\sqrt{(G(a(x(t)))}} > 0$ and $x(t) \to 0$ as $t \to +\infty$, (3.2) contradicts (3.1). This contradiction completes the proof.

Now by the same way as in the proof of Theorem 3.2 we can prove the following theorem which does not need the condition $h'(x)$ be increasing.

Theorem 3.3. *Suppose that* $(\mathbf{A}_2)$ *holds and* $l = h'(0) \neq 0$ *and* $k'(x)$ *is decreasing for* $x > 0$*. If there exists* $a(t)$ *with* $a'(t) > 0$ *for* $t \geq \alpha > 0$ *and* $a(\beta) > 0$ *for* $\beta \geq \alpha$ *such that*

$$\liminf_{x\to 0^+}\left(\int_x^b \left(\frac{h(F(s))g(s)a'(s)}{(2G(a(s))^{3/2}} - \frac{\sqrt{a'(s)}g(s)\sqrt{lk'(k^{-1}(F(s)))}}{G(a(s))}\right)ds\right) = -\infty \qquad (3.3)$$

for some $b > 0$*, then system* (1.2) *has property* (Y^+) *in the right half-plane.*

By choosing $a(t) = t$ we have following corollary.

Corollary 3.4. *Suppose that* $(\mathbf{A}_2)$ *holds and* $h'(x)$ *is increasing for* $x > 0$ *sufficiently small and* $k'(x)$ *is decreasing for* $x > 0$ *sufficiently small. Then system* (1.2) *has property* (Y^+) *in the right half-plane if there exists an* $\alpha < 0$ *such that*

$$\frac{h(F(x))}{2\sqrt{2G(x)}} - \sqrt{h'(F(x))k'(k^{-1}(F(x))} < \alpha,$$

for $x > 0$ *sufficiently small.*

Similarly, turning our attention to the left half-plane, we have the following results about property of (Y^+) in the left half-plane.

Lemma 3.5. *For each point* $P(-p, k^{-1}(F(-p)))$ *with* $p > 0$*, the positive semi-orbit of* (1.2) *starting at* P *crosses the positive* y*-axis if the following condition holds.*
$(\mathbf{A}_4)$ *There exists a* $\delta > 0$ *such that* $F(x) > 0$ *for* $-\delta < x < 0$ *or* $F(x)$ *has an infinite number of negative zeroes clustering at* $x = 0$*.*

Example 3.1. *Consider system* (1.2) *with functions given in Example 2.2. Since* $F(x) < 0$ *for* $x > 0$ *and* $F(x) > 0$ *for* $x < 0$*, by Lemmas 3.1 and 3.5 this system has property* (Y^+) *in the both right and left half-planes.*

Example 3.2. *Consider system* (1.2) *with*

$$F(x) = x, \quad k(y) = \alpha \arctan y + \beta y, \quad h(x) = x^3 + x, \quad \textit{and} \quad g(x) = x,$$
$$\textit{with} \quad \alpha > 0,\ \beta > 0 \quad \textit{and} \quad \beta > \frac{1}{4}.$$

By choosing $a(t) = t$ *and using Theorem 3.3 we have:*

$$\begin{aligned}
&\liminf_{x\to 0^+}\left(\int_x^b \left(\frac{h(F(s))g(s)a'(s)}{(2G(a(s)))^{\frac{3}{2}}} - \frac{\sqrt{a'(s)}g(s)\sqrt{lk'(k^{-1}(F(s)))}}{G(a(s))}\right)ds\right)\\
&\leq \liminf_{x\to 0^+}\left(\int_x^b \left(\frac{(s^3+s)s}{s^3} - \frac{2s\sqrt{\beta}}{s^2}\right)ds\right)\\
&= \liminf_{x\to 0^+}\left(\int_x^b \left(s + (1-2\sqrt{\beta})\frac{1}{s}\right)ds\right) = -\infty,
\end{aligned}$$

for some $b > 0$*. Therefore, this system has property* (Y^+) *in the right half-plane.*

Hereafter we assume that there exists a $\delta > 0$ such that $F(x) < 0$ for $-\delta < x < 0$.

Theorem 3.6. *Suppose that* $(\mathbf{A}_1)$ *holds and* $h'(x)$ *and* $k'(x)$ *are increasing for* $x < 0$. *If there exists* $a(t)$ *with* $a'(t) > 0$ *for* $t \leq \alpha < 0$ *and* $a(\beta) \leq 0$ *for some* $\beta \leq \alpha$ *such that*

$$\liminf_{x\to 0^-}\left(\int_b^x \left(\frac{h(F(s))g(s)a'(s)}{(2G(a(s)))^{3/2}} + \frac{\sqrt{a'(s)}g(s)\sqrt{h'(F(s))k'(k^{-1}(F(s))}}{G(a(s))}\right)ds\right) = -\infty \tag{3.4}$$

for some $b < 0$, *then system* (1.2) *has property* (Y^+) *in the left half-plane.*

Theorem 3.7. *Suppose that* $(\mathbf{A}_1)$ *holds and* $l = h'(0) \neq 0$ *and* $k'(x)$ *is increasing for* $x < 0$. *If there exists* $a(t)$ *with* $a'(t) > 0$ *for* $t \leq \alpha < 0$ *and* $a(\beta) < 0$ *for* $\beta \leq \alpha$ *such that*

$$\liminf_{x\to 0^-}\left(\int_b^x \left(\frac{h(F(s))g(s)a'(s)}{(2G(a(s)))^{3/2}} + \frac{\sqrt{a'(s)}g(s)\sqrt{lk'(k^{-1}(F(s)))}}{G(a(s))}\right)ds\right) = -\infty \tag{3.5}$$

for some $b < 0$, *then system* (1.2) *has property* (Y^+) *in the left half-plane.*

By choosing $a(t) = t$ we have following corollary.

Corollary 3.8. *Suppose that* $(\mathbf{A}_1)$ *holds and* $h'(x)$ *and* $k'(x)$ *are increasing for* $x > 0$ *sufficiently small. Then system* (1.2) *has property* (Y^+) *in the left half-plane if there exists an* $\alpha > 0$ *such that*

$$\frac{h(F(x))}{2\sqrt{2G(x)}} + \sqrt{h'(F(x))k'(k^{-1}(F(x)))} < \alpha,$$

for $x < 0$ *sufficiently small.*

4. An Oscillation Theorem

In this section we will present our main result and will give examples to illustrate our results.

Theorem 4.1. *Assume that* $G(\pm\infty) = +\infty$, $l = h'(0) \neq 0$ *and* $(\mathbf{A}_1)$, $(\mathbf{A}_2)$ *hold. Then, all nontrivial solutions of system* (1.2) *are oscillatory if* (2.7),(2.10),(3.3) *and* (3.5) *hold for some functions satisfying in the conditions of Theorem 2.6, Theorem 2.12, Theorem 3.3 and Theorem 3.7 respectively.*

Notice that the other obtained results in section 2 and section 3 can be formulated for the existence of the oscillatory solutions for systems (1.2) and (2.1) in the same manner.

Example 4.1. *Consider system* (1.2) *with functions given in Example 2.2. By Examples 2.2, 2.4 and 3.1 this system has property* (X^+) *and* (Y^+) *in the both half-planes. Hence, all nontrivial solutions of this system are oscillatory.*

References

1. A. Aghajani, A. Moradifam, *Some sufficient conditions for the intersection with the vertical isocline in the Lienard plane,* Appl. Math. Letters 19, 491-497, (2006).

2. A. Aghajani, A. Moradifam, *Intersection with the vertical isocline in the the Liénard plane,* Nonlinear Analysis: Theory, Methods and Applications, 68, 3475-3484, (2008).

3. A. Aghajani, V. Roomi, *On the property of (X^+) for generalized Liénard system,* International Journal of Dynamical Systems and Differential Equations,Vol.1, No.4, 276-282, (2008).

4. A. Aghajani, V. Roomi, *Oscillation theorems for generalized Lienard system,* Mathematical Sciences, Vol.2, No.3, 243-250, (2008).

5. A.F. Filippov, *Sufficient conditions for the existence of stable limit cycles of second order equations*, Math. Sb. 30, (72), 171-180, (1952).

6. J.R. Graef, *On the generalized Liénard equation with negative damping*, J. Differential Equations. 12, 34-62, (1972).

7. M. Gyllenberg, P. Yan, J. Jiang, *The qualitative behavior of the second-order system with zero diagonal coefficient*, J. Math. Anal. Appl. 291, 322-340, (2004).

8. M. Gyllenberg, P. Yan, *New conditions for the intersection of orbits with the vertical isocline of the Liénard system*, Mathematical and Computer Modelling, 49, 906-911, (2009).

9. M. W. Hirsch, S. Smale, . Differential Equations, Dynamical System, and linear algebra. Academic Press, New York (1964).

10. T. Hara, T. Yoneyama, *On the global center of generalized Liénard equation and its application to stability problems*, Funkcial. Ekvac. 28, 171-192, (1985).

11. T. Hara, J. Sugie, *When all trajectories in the Liénard plane cross the vertical isocline?* , Nonlinear Differential Equations Appl. 2(4), 527-551, (1995).

12. T. Hara, T. Yoneyama, J. Sugie, *A necessary and sufficient condition for osillation of the generalized Liénard equation*, Ann. Math. Pura Appl. 154, 223-230, (1989).

13. P. Hartman . Ordinary Differential Equations. Wiley, New York (1964).

14. P.J. Ponzo, N. Wax, *On peridic solution of the system* $\dot{x} = y - F(x),\ \dot{y} = -g(x)$, J. Differential Equations. 10 (1971) 262-269.

15. J. Sugie, et al., *On global asymptotic stability of systems of Lienard type*, J.Math. Anal. Appl. 219, 140-164, (1998).

16. J. Sugie, T. Hara, *Existence and non-existence of homoclinic trajectories of the Lienard system*, J. Discrete Contin. Dyn. Syst., 2, 237-254, (1996).

17. G. Villari, F. Zanolin, *On a dynamical system in the Liénard plane. Necessary and sufficient conditions for the intersection with the vertical isocline and applications*, Funkcial. Ekvac. 33 (1990) 19-38.

18. G. Villari, *On the qualitative behavior of solutions of Liénard equations*, J. Differential Equations. 67 (1987) 269-277.

19. P. Yan, J. Jiang, *On global asymptotic stability of second order nonlinear differential systems*, Appl. Anal. 81 (2002) 681-703.

T. Kasbi,
Department of Mathematics,
University of Tabriz,
Iran.
E-mail address: Tohid.Kasbi@gmail.co

and

Vahid Roomi,
Department of Mathematics,
Azarbaijan Shahid Madani University,
Iran.
E-mail address: roomi@azaruniv.ac.ir

and

A. J. Akbarfam,
Department of Mathematics,
University of Tabriz,
Iran.
E-mail address: Akbarfam@yahoo.com