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New Types of Soliton Solutions for Space-time Fractional Cubic Nonlinear Schrodinger Equation

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ABSTRACT: New definition for traveling wave transformation and using of new conformable fractional derivative for converting fractional nonlinear evolution equations into the ordinary differential equation are presented in this study. For this aim we consider the time and space fractional derivatives cubic nonlinear Schrodinger equation. Then by using of the efficient and powerful method the exact traveling wave solutions of this equation are obtained. The new definition introduces a promising tool for solving many space-time fractional partial differential equations.

Key Words: Nonlinear Schrodinger equation, Conformable fractional derivative, (G'/G)-expansion method.

Contents

1. Introduction

The examinations of exact solutions of fractional nonlinear evolution equations have a very important place in the enquiry of some physical phenomena. The types of solutions of FNLEEs, that are combined utilizing variety mathematical techniques, are very significant various sciences such as chemistry, technology of space, control engineering problems, physics, applied mathematics and computer engineering.
In this paper, we will use the (G'/G)-expansion method [1-2] to solve nonlinear fractional partial differential equations in the sense of new conformable fractional derivative.

The (G'/G)-expansion method was introduced, by Wang et al. [3], to find the travelling wave solutions of nonlinear evolution equations. This method was further extended [4-5] to find the solutions of fractional order differential equations, the Jacobi elliptic function expansion method [6], the tanh-function method for finding

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solitary wave solutions [7], the homotopy perturbation method [8], the first integral method [9], the solitary wave ansatz [10] and etc[11-13].

The conformable fractional derivative of order α defined by the following expression and theorems.

Definition 1. Let $f^{\alpha}(t)$ stands for$T_{\alpha}(f)(t)$. Hence

$$f^{\alpha}(t) = \lim_{\xi \to 0} \frac{f\left(t + \xi t^{1-\alpha}\right) - f(t)}{\xi}$$

If $f$is α-differentiable in some$(0, a)$, $a > 0$, and $\lim\limits_{t \to 0^+} f^{\alpha}(t)$exists, then by definition

$$f^{\alpha}(0) = \lim_{t \to 0^+} f^{\alpha}(t)$$

We should remark that $T_{\alpha}(t^{\mu}) = \mu t^{\mu-\alpha}$. Further, this definition coincides with the classical definitions of R-L and of Caputo on polynomials (up to a constant multiple).

One can easily show that T_{α} satisfies all the properties in the theorem [14-15].

Theorem 1. Let $\alpha \in [0, 1)$ and f, g beα-differentiable at a point t, Then:

$(i)\, T_{\alpha}(af + bg) = aT_{\alpha}(f) + bT_{\alpha}(g), for \quad all \quad a, b \in \mathrm{R}.$

$(ii)\, T_{\alpha}(t^{\mu}) = \mu t^{\mu-\alpha}, for \quad all \quad \mu \in \mathrm{R}$

$(iii)\, T_{\alpha}(fg) = fT_{\alpha}(g) + gT_{\alpha}(f)$

$(i\nu)\, T_{\alpha}\left(\frac{f}{g}\right) = \frac{fT_{\alpha}(g) - gT_{\alpha}(f)}{g^2}$

If, in addition, f is differentiable, then $T_{\alpha}(f)(t) = t^{1-\alpha}\frac{df}{dt}$.

Theorem 2. Let $f : [0, \infty) \to$ Rbe a function such that f is differentiable and also differentiable. Let g be a function defined in the range of $f$and also differentiable; then, one has the following rule:

$$T_{\alpha}(fog)(t) = t^{1-\alpha} g'(t) f'(g(t)).$$

Definition 2. (Fractional Integral) Let $a \geq 0$ and $t \geq a$. Also, let f be a function defined on $(a, t]$ and $\alpha \in f$. Then the $\alpha-$fractional integral of f is defined by,

$$I_a^{\alpha}(f)(t) = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx$$

if the Riemann improper integral exists. It is interesting to observe that the α -fractional derivative and the $\alpha-$fractional integral are inverse of each other as given in [14-15].

Now we consider the following general nonlinear fractional differential equation:

$$G\left(u, D_t^{\alpha}u, D_x^{\beta}u, D_y^{\psi}u, D_t^{\alpha}D_t^{\alpha}u, D_t^{\alpha}D_x^{\beta}u, D_x^{\beta}D_x^{\beta}u, ...\right) = 0, \ 0 < \alpha, \beta, \psi < 1. \quad (1.1)$$

Where u is an unknown function, and G is a polynomial of u. In this equation, the partial fractional derivatives involving the highest order derivatives and the nonlinear terms are included. Next by using the new definition for traveling wave variable

$$u(x,t) = U(\xi)\, e^{i\left(k\frac{x^\beta}{\beta} + c\frac{t^\alpha}{\alpha}\right)}, \qquad \xi = l\frac{x^\beta}{\beta} + \omega\frac{t^\alpha}{\alpha} \tag{1.2}$$

Where k, c, l and ω are non-zero arbitrary constants, we can rewrite Eq. (1.1) as the following nonlinear ODE:

$$Q(U, U', U'', U''', ...) = 0. \tag{1.3}$$

Where the prime denotes the derivation with respect to ξ . If possible, we should integrate Eq. (1.3) term by term one or more times.
Suppose that the solution of ODE (1.3) can be expressed by a polynomial in (G'/G) as follows

$$u(\xi) = \sum_{i=0}^{m} a_i \left(\frac{G'}{G}\right)^i, \qquad a_m \neq 0. \tag{1.4}$$

Where $G = G(\xi)$ satisfies the second order LODE in the form

$$G'' + \lambda G' + \mu G = 0 \tag{1.5}$$

$a_m, \lambda$and μ are constants to be determined later.
The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in Eq. (1.3). By substituting Eq. (1.4) into Eq. (1.3) and using Eq. (1.5), we collect all terms with the same order of (G'/G). By equating each coefficient of the resulting polynomial to zero, we obtain a set of algebraic equations for $a_i\,(i = 0, \pm 1, \pm 2, ...)\,, \lambda, \mu, k, c, l$ and ω. By solving the equation system and substituting the general solutions of Eq. (1.5) into Eq. (1.4), we can obtain a variety of exact solutions of Eq. (1.1).
The rest of this paper is organized as follows. In Sections 2, we use this method to obtain the exact solutions for the time and space fractional derivatives cubic nonlinear Schrodinger equation. Discussion and some conclusions are given in the last section.

2. Outcomes

We consider the following time and space fractional derivatives cubic nonlinear Schrodinger

$$i\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\partial^{2\beta} u}{\partial x^{2\beta}} + 2|u|^2 u = 0, \qquad t > 0, \quad 0 < \alpha, \beta \leq 1, i = \sqrt{-1}. \tag{2.1}$$

Where α is a parameter describing the order of the fractional time derivative. For our purpose, we introduce the following new wave transformations:

$$\begin{gathered} u(x,t) = U(\xi)\, e^{i\left(k\frac{x^\beta}{\beta} + c\frac{t^\alpha}{\alpha}\right)}, \\ \xi = l\frac{x^\beta}{\beta} + \omega\frac{t^\alpha}{\alpha} \end{gathered} \tag{2.2}$$

Where k, c, l and ω are non-zero arbitrary constants. So

$$i\frac{\partial^\alpha u}{\partial t^\alpha} = i\omega U' e^{i\left(k\frac{x^\beta}{\beta}+c\frac{t^\alpha}{\alpha}\right)} - cUe^{i\left(k\frac{x^\beta}{\beta}+c\frac{t^\alpha}{\alpha}\right)} \tag{2.3}$$

$$\frac{\partial^{2\beta} u}{\partial x^{2\beta}} = l^2 U'' e^{i\left(k\frac{x^\beta}{\beta}+c\frac{t^\alpha}{\alpha}\right)} + 2iklU' e^{i\left(k\frac{x^\beta}{\beta}+c\frac{t^\alpha}{\alpha}\right)} - k^2 U e^{i\left(k\frac{x^\beta}{\beta}+c\frac{t^\alpha}{\alpha}\right)} \tag{2.4}$$

By substituting Eqs. (2.2)-(2.4) into Eq. (2.1), Eq. (2.1) is reduced into an ODE

$$l^2 U'' + i\left(2kl + \omega\right) U' - \left(c + k^2\right) U + 2U^3 = 0, \tag{2.5}$$

Where$U' = \frac{dU}{d\xi}$. By using the ansatz (2.5), for the linear term of highest order $U''$with the highest order and the nonlinear term U^3, balancing U'' with U^3 in Eq. (2.5)

$$3m = m + 2$$

So $m = 1$. Suppose that the solutions of Eq. (2.5) can be expressed by a polynomial in (G'/G) as follows:

$$U(\xi) = a_1\left(\frac{G'}{G}\right) + a_0, \qquad a_1 \neq 0, \tag{2.6}$$

By using Eq. (1.5), from Eq. (2.6) we have

$$\begin{aligned} U'' &= 2a_1\left(\tfrac{G'}{G}\right)^3 + 3a_1\lambda\left(\tfrac{G'}{G}\right)^2 + \left(a_1\lambda^2 + 2a_1\mu\right)\left(\tfrac{G'}{G}\right) + a_1\lambda\mu \\ U' &= -a_1\left(\tfrac{G'}{G}\right)^2 - a_1\lambda\left(\tfrac{G'}{G}\right) - a_1\mu \end{aligned} \tag{2.7}$$

By substituting Eqs. (2.6)–(2.7) into Eq. (2.5), collecting the coefficients of $(G'/G)^i$ $(i = 0, \ldots, 2)$, and setting them to be zero, and solving this system we have

$$\begin{aligned} a_1 &= \mathrm{i}\sqrt{l}, \\ a_0 &= \tfrac{i(2kl+\omega)-3l^2\lambda}{6\mathrm{i}\sqrt{l}}, \end{aligned} \tag{2.8}$$

$$\omega = -i\left[\left(108l^4\lambda\mu + 6\sqrt{P}\right)^{\frac{1}{3}} - \frac{6cl - 36l^2\mu + 6k^2 l}{108l^4\lambda\mu + 6\sqrt{P}} - 2ikl + 3l^2\lambda\right], \tag{2.9}$$

$$c = \frac{q}{6l} \tag{2.10}$$

Where

$$P = 1296l^6\mu^3 + 648l^5\mu^2k^2 + 648l^5\mu^2c - 108l^4\mu k^4 - 216l^4\mu k^2c - 108l^4\mu c^2 + 6k^6l^3 + 18k^4l^3c + 18k^2l^3c^2 + 6c^3l^3 + 324l^8\lambda^2\mu^2$$

$$q = 6l^3\lambda^2 + 12l^3\mu - 12i\lambda l^2k - 6i\lambda l\omega - 6k^2l + 4k^2l^2 + 4kl\omega + 12ikl^3\lambda + \omega^2 + 6i\omega l^2\lambda - 9l^4\lambda^2$$

By using Eqs. (2.8)- (2.10), and substituting the general solutions of Eq. (1.5) into Eq. (2.6), we have three types of travelling wave solutions of the time and space fractional derivatives cubic nonlinear Schrodinger as follows.

1. When $\lambda^2-4\mu>0$,

$$U(\xi)=\frac{\mathrm{i}\sqrt{l}\sqrt{\lambda^2-4\mu}}{2}\left(\frac{C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}\right)-\frac{\mathrm{i}\sqrt{l}\lambda}{2}+\frac{i(2kl+\omega)-3l^2\lambda}{6\mathrm{i}\sqrt{l}},$$

So

$$u_1(x,t)=\left[\frac{\mathrm{i}\sqrt{l}\sqrt{\lambda^2-4\mu}}{2}\times\left(\frac{C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\left(l\frac{x^\beta}{\beta}+\omega\frac{t^\alpha}{\alpha}\right)+C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\left(l\frac{x^\beta}{\beta}+\omega\frac{t^\alpha}{\alpha}\right)}{C_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\left(l\frac{x^\beta}{\beta}+\omega\frac{t^\alpha}{\alpha}\right)+C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\left(l\frac{x^\beta}{\beta}+\omega\frac{t^\alpha}{\alpha}\right)}\right)-\frac{\mathrm{i}\sqrt{l}\lambda}{2}+\frac{i(2kl+\omega)-3l^2\lambda}{6\mathrm{i}\sqrt{l}}\right]e^{i\left(k\frac{x^\beta}{\beta}+\frac{q}{6l}\frac{t^\alpha}{\alpha}\right)},$$

Where

$$\omega=-i\left[\left(108l^4\lambda\mu+6\sqrt{P}\right)^{\frac{1}{3}}-\frac{6cl-36l^2\mu+6k^2l}{108l^4\lambda\mu+6\sqrt{P}}-2ikl+3l^2\lambda\right],$$

1. When $\lambda^2-4\mu<0$,

$$u_2(x,t)=\left[\frac{\mathrm{i}\sqrt{l}\sqrt{4\mu-\lambda^2}}{2}\times\left(\frac{C_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\left(l\frac{x^\beta}{\beta}+\omega\frac{t^\alpha}{\alpha}\right)+C_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\left(l\frac{x^\beta}{\beta}+\omega\frac{t^\alpha}{\alpha}\right)}{C_1\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\left(l\frac{x^\beta}{\beta}+\omega\frac{t^\alpha}{\alpha}\right)+C_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\left(l\frac{x^\beta}{\beta}+\omega\frac{t^\alpha}{\alpha}\right)}\right)-\frac{\mathrm{i}\sqrt{l}\lambda}{2}+\frac{i(2kl+\omega)-3l^2\lambda}{6\mathrm{i}\sqrt{l}}\right]e^{i\left(k\frac{x^\beta}{\beta}+\frac{q}{6l}\frac{t^\alpha}{\alpha}\right)},$$

Where

$$\omega=-i\left[\left(108l^4\lambda\mu+6\sqrt{P}\right)^{\frac{1}{3}}-\frac{6cl-36l^2\mu+6k^2l}{108l^4\lambda\mu+6\sqrt{P}}-2ikl+3l^2\lambda\right],$$

1. When $\lambda^2-4\mu=0$,

$$u_3(x,t)=\left[\frac{\mathrm{i}\sqrt{l}\sqrt{4\mu-\lambda^2}}{2}\left(\frac{C_2\alpha\beta}{C_1\alpha\beta+C_2(l\alpha x^\beta+\omega\beta t^\alpha)}\right)-\frac{\mathrm{i}\sqrt{l}\lambda}{2}+\frac{i(2kl+\omega)-3l^2\lambda}{6\mathrm{i}\sqrt{l}}\right]e^{i\left(k\frac{x^\beta}{\beta}+\frac{q}{6l}\frac{t^\alpha}{\alpha}\right)},$$

In particular, if $C_1\neq 0, C_2=0, \lambda>0$, and$\mu=0$, then u_1 and $u_2$becomes

$$u_1(x,t)=\left[\frac{\mathrm{i}\sqrt{l}\lambda}{2}\times\coth\frac{\lambda}{2}\left(l\frac{x^\beta}{\beta}-i\left[\left(6\sqrt{P}\right)^{\frac{1}{3}}-\frac{6cl+6k^2l}{6\sqrt{P}}-2ikl+3l^2\lambda\right]\frac{t^\alpha}{\alpha}\right)-\frac{\mathrm{i}\sqrt{l}\lambda}{2}+\frac{i(2kl+\omega)-3l^2\lambda}{6\mathrm{i}\sqrt{l}}\right]e^{i\left(k\frac{x^\beta}{\beta}+\frac{q}{6l}\frac{t^\alpha}{\alpha}\right)},$$

And

$$u_2(x,t) = \Big[\frac{\mathrm{i}\sqrt{l}\lambda}{2} \times \cot\frac{\lambda}{2}\left(l\frac{x^\beta}{\beta} - i\left[\left(6\sqrt{P}\right)^{\frac{1}{3}} - \frac{6cl+6k^2l}{6\sqrt{P}} - 2ikl + 3l^2\lambda\right]\frac{t^\alpha}{\alpha}\right) - \frac{\mathrm{i}\sqrt{l}\lambda}{2} + \frac{i(2kl+\omega)-3l^2\lambda}{6\mathrm{i}\sqrt{l}}\Big] e^{i\left(k\frac{x^\beta}{\beta}+\frac{q}{6l}\frac{t^\alpha}{\alpha}\right)},$$

3. Figure Caption

Figure 1-1: The complex variation of u_1 for $l = 1, \lambda = 1, \mu = 0, \alpha = 1, c = 1, k = 0, t = 1, C_1 \neq 0, C_2 = 0$, and different values of $\beta = 0.1$ (red Curve), $\beta = 0.2$ (green Curve), $\beta = 0.3$ (yellow Curve) in region$x = -\pi...\pi$.

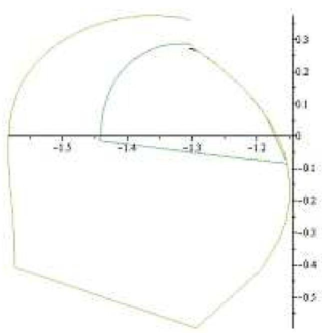

Fig. 1-1

Figure 1-2: The complex variation of u_1 for $l = 1, \lambda = 1, \mu = 0, \alpha = 1, c = 1, k = 0, t = 1, C_1 \neq 0, C_2 = 0$, and different values of $\beta = 0.4$ (red Curve), $\beta = 0.5$ (green Curve), $\beta = 0.6$ (yellow Curve) in region$x = -\pi...\pi$,

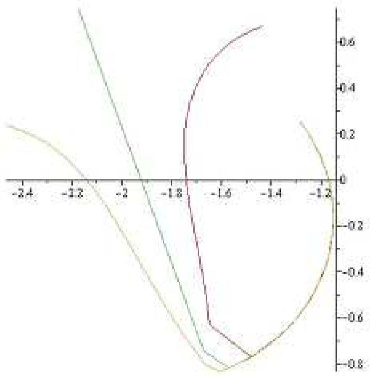

Fig. 1-2

Figure 1-3: The complex variation of u_1 for $l = 1, \lambda = 1, \mu = 0, \alpha = 1, c = 1, k = 0, t = 1, C_1 \neq 0, C_2 = 0$, and different values of $\beta = 0.7$ (red Curve), $\beta = 0.8$ (green Curve), $\beta = 0.9$ (yellow Curve) in region$x = -\pi...\pi$,

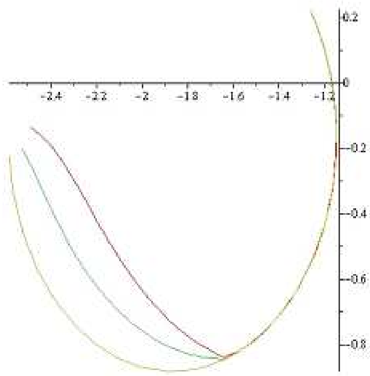

Fig. 1-3

Figure 1-4: The complex variation of u_1 for $l = 1, \lambda = 1, \mu = 0, \alpha = 1, c = 1, k = 0, t = 1, C_1 \neq 0, C_2 = 0$, and different values of $\beta = 0.9$ (red Curve), $\beta = 1$ (yellow Curve), in region$x = -\pi...\pi$,

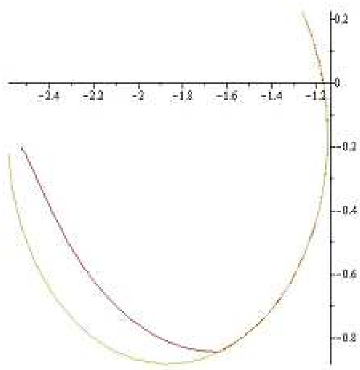

Fig. 1-4

Figure 1-5: The complex variation of u_1 for $l = 1, \lambda = 1, \mu = 0, \alpha = 1, c = 1, k = 0, t = 1, C_1 \neq 0, C_2 = 0$, and $\beta = 1$; complex curve of general form of cubic nonlinear Schrodinger equation in region$x = -\pi...\pi$.

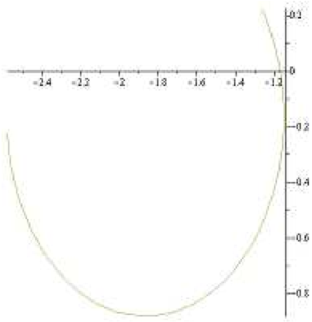

Fig. 1-5

Figure 2-1: The complex variation of u_2 for $l = 1, \lambda = 1, \mu = 0, \alpha = 1, c = 1, k = 0, t = 1, C_1 \neq 0, C_2 = 0$, and different values of $\beta = 0.1$ (red Curve), $\beta = 0.2$ (green Curve), $\beta = 0.3$ (yellow Curve) in region$x = -\pi ... \pi$,

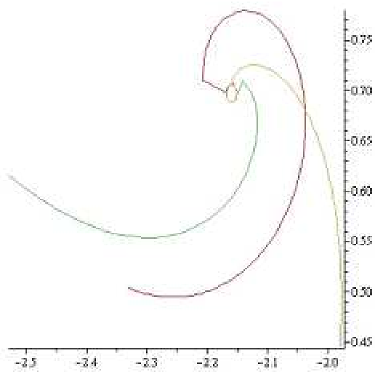

Fig. 2-1

Figure 2-2: The complex variation of u_2 for $l = 1, \lambda = 1, \mu = 0, \alpha = 1, c = 1, k = 0, t = 1, C_1 \neq 0, C_2 = 0$, and different values of $\beta = 0.4$ (red Curve), $\beta = 0.5$ (green Curve), $\beta = 0.6$ (yellow Curve) in region$x = -\pi ... \pi$,

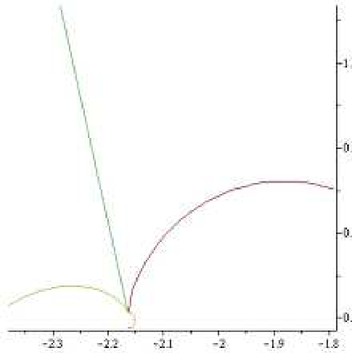

Fig. 2-2

Figure 2-3: The complex variation of u_2 for $l = 1, \lambda = 1, \mu = 0, \alpha = 1, c = 1, k = 0, t = 1, C_1 \neq 0, C_2 = 0$, and different values of $\beta = 0.7$ (red Curve), $\beta = 0.8$ (green Curve), $\beta = 0.9$ (yellow Curve) in region$x = -\pi...\pi$,

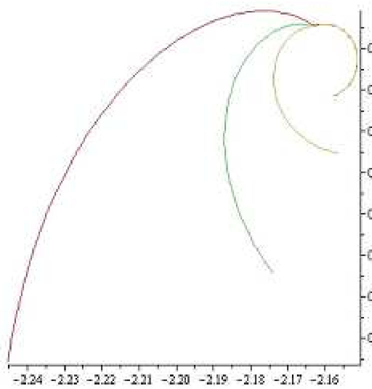

Fig. 2-3

Figure 2-4: The complex variation of u_2 for $l = 1, \lambda = 1, \mu = 0, \alpha = 1, c = 1, k = 0, t = 1, C_1 \neq 0, C_2 = 0$, and different values of $\beta = 0.9$ (red Curve), $\beta = 1$ (yellow Curve), in region$x = -\pi...\pi$,

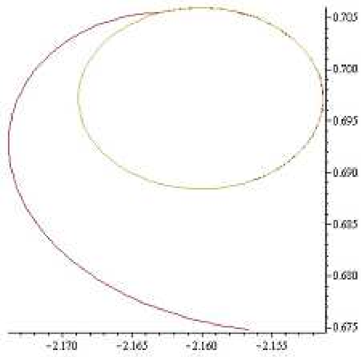

Fig. 2-4

4. Discussion and conclusion

The graphs of the solutions related to u_1 and u_2 show that with changing β (if β tends to one) the graphs of the solutions of fractional cubic nonlinear Schrodinger equation is near to graph of solution of cubic nonlinear Schrodinger equation in general form and finally for $\beta = 1$ it coincide with the graph of the general form of cubic nonlinear Schrodinger equation.
Summary, in this paper we successfully introduce new definition for wave transformation and by using this definition the time and space fractional derivatives cubic nonlinear Schrodinger equation converted to the ordinary differential equation. Also, we successfully use the (G'/G)-expansion method to solve time and space fractional derivatives cubic nonlinear Schrodinger with using conformable fractional derivative. Finally, to our knowledge, the solutions obtained in this paper have not been reported in the literature so far.

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