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On Existence and Stability Results for Nonlinear Fractional Delay Differential Equations

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abstract: We establish existence and uniqueness results for fractional order delay differential equations. It is proved that successive approximation method can also be successfully applied to study Ulam–Hyers stability, generalized Ulam–Hyers stability, Ulam–Hyers–Rassias stability, generalized Ulam–Hyers–Rassias stability, $\mathbb{E}_\alpha$–Ulam–Hyers stability and generalized $\mathbb{E}_\alpha$–Ulam–Hyers stability of fractional order delay differential equations.

Key Words: Fractional delay differential equation, Fixed point theorem, Existence and uniqueness, Successive approximation, Ulam–Hyers stability, $\mathbb{E}_\alpha$–Ulam–Hyers stability.

Contents

1. Introduction

It is well known that many dynamical systems can be described more precisely by using fractional differential equations. Modeling of several physical phenomena appearing in science and engineering can suitably done via fractional differential equations. Hence the study of fractional differential equations have widespread interest.

Several researchers form mathematics community devoted to study existence, uniqueness and other qualitative properties for fractional delay differential equations (FDDEs) by various approaches. At many instances it is very difficult to obtain exact solution of FDDEs and in this case we are intended to obtain the approximate solution for such equations. The answer to the question, " Under

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what condition for every approximate solution to the equation there exists an exact solution near it and what error it gives ? " can be explained via Ulam–Hyers stability theory.

The study of Ulam stability and data dependence for fractional differential equations was initiated by Wang et al. [1]. An overview on the development of theory of Ulam–Hyers–Rassias stability and the Ulam–Hyers stability for fractional differential equations can be found in [1,2] and the references given therein. Subsequently many authors discussed various Ulam–Hyers stability problem for different kinds of fractional integral and fractional differential equations by using different techniques, for instance, Wang and Li [3] established different kinds of $\mathbb{E}_\alpha$–Ulam type stability for fractional order ordinary differential equations, Wang and Zhang [4] proved Ulam–Hyers–Mittag–Leffler stablity for fractional order delay differential equation on compact interval, Wei et al. [5] established Ulam–Hyers stability and Ulam–Hyers–Rassias stability results for fractional Volterra type integral equations with delay by fixed point approach and Brzdek et al. [6] obtained Ulam stability of delayed fractional differential equations through approximate solution.

In recent years, some researchers extended the concept of Ulam type stabilities by using different techniques to various forms of fractional differential and fractional integral equations with different types of fractional derivative operators, for example, Wang and Xu [7] by applying the Laplace transform method have investigated the Hyers–Ulam stability of fractional linear differential equation with Riemann-Liouville fractional derivative, Eghbali and coauthers [8] proved that the fractional order delay integral equation is Mittag–Leffler–Hyers–Ulam stable on a compact interval with respect to the Chebyshev and Bielecki norms, Yu [9] studied β-Ulam–Hyers stability for a class of fractional differential equations with non–instantaneous impulses, Hyers–Ulam stability results for nonlinear fractional systems with coupled nonlocal initial conditions have been investigated in [10], Peng and Wang [11] discussed existence of solutions and Ulam–Hyers stability of Cauchy problem for nonlinear ordinary differential equations involving two Caputo fractional derivatives, Abbas in [12] dealt with existence, uniqueness and Mittag–Leffler–Ulam stablity of fractional integrodifferential equations.

Recently, using successive approximation method, Huang et al. [13] established the Ulam–Hyers stability of integer order delay differential equations, Gachpazan et al. [14] proved the Ulam–Hyers stability for the nonlinear Volterra integral equation of second kind.

In this paper we consider the fractional delay differential equations (FDDEs) of the form:

$$ {}^cD^\alpha x(t) = f(t, x_t),\ t \in [0, b],\ m-1 < \alpha \leq m \in \mathbb{N} \tag{1.1} $$

$$ x(t) = \phi(t),\ t \in [-r, 0], \tag{1.2} $$

where $f : [0, b] \times C([-r, 0], \mathbb{R}^n) \to \mathbb{R}^n$ is a continuous function and ${}^cD^\alpha$ denotes the Caputo fractional derivative of order α with lower terminal at 0.

Taking motivation from [3,13,14], we prove existence and uniqueness of solutions to (1.1)-(1.2) using modified version of contraction principle. Further, by method of successive approximation we obtain Ulam–Hyers stability, Ulam–Hyers–Rassias stability and $\mathbb{E}_\alpha$–Ulam–Hyers stability results for delay differential equation (1.1). Finally, we provide examples to illustrate our obtained results.

Recently, Kucche et al. [15,16] established existence and various qualitative properties of solutions to nonlinear implicit fractional differential equations.

We organize the present work as follows. In Section 2, preliminaries and notations are presented. Section 3 deals with existence and uniqueness of solutions for (1.1)-(1.2). In section 4, we establish Ulam–Hyers, generalized Ulam–Hyers and Ulam–Hyers–Rassias stability results for delay differential equation (1.1). Section 5 deals with $\mathbb{E}_\alpha$–Ulam–Hyers stability of (1.1). In Section 6, we provide few illustrative examples.

2. Preliminaries

In this section we present some basic definitions, notations and preliminaries. Basics of delay differential equations are considered from the monographs by Hale et al. [17] and Naito et al. [18].

Let $\mathbb{R}^n$ is an n- dimensional linear vector space over the reals with the norm

$$\|x\| = \left(\sum_{k=1}^{n} x_k^2\right)^{\frac{1}{2}}, \ x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n.$$

Let $0 \le r < \infty$ be given real number, $C = C([-r,0],\mathbb{R}^n)$ is the Banach space of continuous functions from $[-r,0]$ into $\mathbb{R}^n$ with the norm

$$\|\psi\|_C = \sup_{-r\le\theta\le 0} \|\psi(\theta)\|.$$

Let us denote by $B = C^m([-r,b],\mathbb{R}^n)$, $b > 0$ the Banach space of functions from $[-r,b]$ into $\mathbb{R}^n$ having m^{th} order continuous derivatives, equipped with the supremum norm $\|\cdot\|_B$. For any $x \in B$ and any $t \in [0,b]$, we denote by x_t the element of C defined by $x_t(\theta) = x(t+\theta)$, $\theta \in [-r,0]$.

We use following results in our analysis.

Lemma 2.1 ([13,18])**.** *If $x \in C([-r,b],\mathbb{R}^n)$ then x_t is continuous with respect to $t \in [0,b]$.*

Lemma 2.2 ([13,18])**.** *Let $x : [-r,b) \to \mathbb{R}^n$ be a continuous function with $x_0 = \phi$. If*

$$\|x(t)\| \le \|\phi(0)\| + m(t), \ t \in [0,b)$$

where $m(t)$ is a nondecreasing function, then

$$\|x_t\|_C \le \|\phi\|_C + m(t), \ t \in [0,b)$$

For fundamentals of fractional calculus we refer the research monographs [19, 20,21].

Definition 2.1. *Let $g \in C[0,b]$ and $\alpha \geq 0$ then the Riemann-Liouville fractional integral of order α of a function g is defined as*

$$I^{\alpha}g(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)ds,$$

provided the integral exists. Note that $I^0 g(t) = g(t)$.

Definition 2.2. *Let $m-1 < \alpha \leq m \in \mathbb{N}$ then the Caputo fractional derivative of order α of a function $g \in C^m[0,b]$ is defined as*

$$^{c}D^{\alpha}g(t) = \begin{cases} I^{m-\alpha}D^m g(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{g^{(m)}(s)}{(t-s)^{\alpha-m+1}}ds & \text{if } \ m-1<\alpha<m, \\ g^{(m)}(t) & \text{if } \ \alpha = m. \end{cases}$$

The one parameter Mittag-Leffler function is defined as

$$E_{\gamma}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\gamma k+1)}, \gamma > 0.$$

Following lemmas play important role to obtain our main results.

Lemma 2.3 (See [22]). *Suppose $b \geq 0$, $\beta > 0$ and $a(t)$ is a nonnegative function locally integrable on $0 \leq t < T$, (some $T \leq \infty$), and suppose $u(t)$ is nonnegative and locally integrable on $0 \leq t < T$ with*

$$u(t) \leq a(t) + b\int_0^t (t-s)^{\beta-1}u(s)ds$$

on this interval; then

$$u(t) \leq a(t) + \theta\int_0^t E'_{\beta}(\theta(t-s))a(s)ds, \ 0 \leq t < T,$$

where $\theta = (b\Gamma(\beta))^{\frac{1}{\beta}}, E_{\beta}(z) = \sum_{n=0}^{\infty}\frac{z^{n\beta}}{\Gamma(n\beta+1)}, E'_{\beta}(z) = \dfrac{d}{dz}E_{\beta}(z), E'_{\beta}(z) \equiv \frac{z^{\beta-1}}{\Gamma(\beta)}$ as $z \to 0^+, E'_{\beta}(z) \equiv \frac{1}{\beta}e^z$ as $z \to +\infty$.

Further, if $a(t) \equiv a$, constant, then $u(t) \leq aE_{\beta}(\theta t)$.

Lemma 2.4 (See [23]). *For all $\mu > 0$ and $\nu > -1$,*

$$\int_0^t (t-s)^{\mu-1}s^{\nu}ds = t^{\mu+\nu}\frac{\Gamma(\mu)\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)}$$

3. Existence and Uniqueness Results

To obtain existence and uniqueness of solution to the initial value problem (1.1)-(1.2), we use the following Lemma.

Lemma 3.1 ([24], Modified version of contraction principle)**.** *Let X be a Banach Space and let D be an operator which maps the element of X into itself for which D^r is a contraction, where r is a positive integer then D has a unique fixed point.*

Definition 3.1. *A function $x \in B$ is said to be a solution of (1.1)-(1.2) if x satisfies the equations ${}^cD^\alpha x(t) = f(t, x_t)$ on $[0, b]$ and $x(t) = \phi(t)$ on $[-r, 0]$.*

The proof of the following Lemma is close to the proof of Lemma 6.2 given in [21].

Lemma 3.2. *If $f : [0, b] \times C \to \mathbb{R}^n$ is continuous then FDDEs (1.1)-(1.2) is equivalent to the following fractional Volterra integral equation*

$$x(t) = \begin{cases} \phi(t), \quad t \in [-r, 0], \\ \sum_{k=o}^{m-1} \dfrac{\phi^{(k)}(0)}{\Gamma(k+1)} t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_s) ds, \quad t \in [0, b]. \end{cases}$$

Next theorem guarantee existence and uniqueness of solution to initial value problem (1.1)-(1.2).

Theorem 3.1. *If $f : [0, b] \times C \to \mathbb{R}^n$ be a continuous function that satisfies Lipschitz condition with respect to second variable*

$$\|f(t, u) - f(t, v)\|| \leq L\|u - v\|_C, \ t \in [0, b]; \ u, v \in C,$$

then FDDEs (1.1)-(1.2) has unique solution $x : [-r, b] \to \mathbb{R}^n$.

Proof: Consider the operator $F : B \to B$ defined by

$$Fx(t) = \begin{cases} \phi(t), \quad t \in [-r, 0], \\ \sum_{k=o}^{m-1} \dfrac{\phi^{(k)}(0)}{\Gamma(k+1)} t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_s) ds, \quad t \in [0, b]. \end{cases}$$

Note that by definition of operator F, for any $x, z \in B$ we have

$$\|F^j x(t) - F^j z(t)\| = 0, \ \text{for all } t \in [-r, 0] \text{ and } \ j \in \mathbb{N}. \tag{3.1}$$

By using mathematical induction, for any $x, z \in B$ and $t \in [0, b]$ we prove that,

$$\|F^j x(t) - F^j z(t)\| \leq \frac{(Lt^\alpha)^j}{\Gamma(j\alpha + 1)} \|x - z\|_B, \forall j \in \mathbb{N}. \tag{3.2}$$

By definition of operator F and using Lipschitz condition, we have for any $x, z \in B$ and $t \in [0, b]$,

$$\begin{aligned} \|Fx(t) - Fz(t)\| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s, x_s) - f(s, z_s)\| ds \\ &\leq \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|x_s - z_s\|_C ds. \end{aligned}$$

For any $t \in [0, b]$ and $\theta \in [-r, 0]$, we have $-r \leq t + \theta \leq b$ and hence

$$\begin{aligned}\|x_t\|_C &= \sup\{x_t(\theta) : \theta \in [-r, 0]\}\\ &= \sup\{x(t+\theta) : \theta \in [-r, 0]\}\\ &\leq \sup\{x(t+\theta) : -r \leq t+\theta \leq b\}\\ &\leq \|x\|_B. \end{aligned} \tag{3.3}$$

Thus

$$\|Fx(t) - Fz(t)\| \leq \frac{L}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{\alpha-1} ds\right)\|x - z\|_B$$

which gives,

$$\|Fx(t) - Fz(t)\| \leq \frac{Lt^\alpha}{\Gamma(\alpha+1)}\|x - z\|_B,\ t \in [0, b].$$

Thus the inequality (3.2) holds for $j = 1$. Let us suppose that the inequality (3.2) holds for $j = r \in \mathbb{N}$, hence

$$\|F^r x(t) - F^r z(t)\| \leq \frac{(Lt^\alpha)^r}{\Gamma(r\alpha+1)}\|x - z\|_B,\ t \in [0, b], \tag{3.4}$$

we prove that (3.2) holds for $j = r + 1$. Let any $x, z \in B$ and denote $x^* = F^r x$, $z^* = F^r z$. Then using definition of operator F and the Lipschitz condition of f, for any $t \in [0, b]$ we get

$$\begin{aligned}\|F^{r+1}x(t) - F^{r+1}z(t)\| &= \|F(F^r x(t)) - F(F^r z(t))\|\\ &= \|F(x^*(s)) - F(z^*(s))\|\\ &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s, x_s^*) - f(s, z_s^*)\| ds\\ &\leq \frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|x_s^* - z_s^*\|_C ds \end{aligned} \tag{3.5}$$

We write from (3.4),

$$\|x^*(t) - z^*(t)\| = \|F^r x(t) - F^r z(t)\| \leq \frac{(Lt^\alpha)^r}{\Gamma(r\alpha+1)}\|x - z\|_B.$$

An application of Lemma 2.2 gives,

$$\|x_t^* - z_t^*\|_C \leq \frac{(Lt^\alpha)^r}{\Gamma(r\alpha+1)}\|x - z\|_B.$$

By using above inequality in (3.5) and then applying Lemma 2.4, we get

$$\begin{aligned}\|F^{r+1}x(t) - F^{r+1}z(t)\| &\leq \frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\frac{(Ls^\alpha)^r}{\Gamma(r\alpha+1)}\|x - z\|_B ds\\ &= \frac{L^{r+1}}{\Gamma(\alpha)\Gamma(r\alpha+1)}\left(\int_0^t (t-s)^{\alpha-1}s^{r\alpha} ds\right)\|x - z\|_B\\ &= \frac{L^{r+1}}{\Gamma(\alpha)\Gamma(r\alpha+1)}t^{(r+1)\alpha}\frac{\Gamma(\alpha)\Gamma(r\alpha+1)}{\Gamma((r+1)\alpha+1)}\|x - z\|_B. \end{aligned}$$

Thus,

$$\|F^{r+1}x(t) - F^{r+1}z(t)\| \leq \frac{(Lt^{\alpha})^{r+1}}{\Gamma((r+1)\alpha+1)}\|x-z\|_B,\ t \in [0,b].$$

We have proved that the inequality (3.2) holds for $j = r+1$. By the principle of mathematical induction the proof of inequality (3.2) is completed. Combining (3.1) and (3.2), we obtain

$$\|F^{j}x(t) - F^{j}z(t)\| \leq \frac{(Lt^{\alpha})^{j}}{\Gamma(j\alpha+1)}\|x-z\|_B,\ t \in [-r,b],\ j \in \mathbb{N}. \tag{3.6}$$

This gives,

$$\|F^{j}x - F^{j}z\|_B = \sup_{t\in[-r,b]} \|F^{j}x(t) - F^{j}z(t)\| \leq \frac{(Lb^{\alpha})^{j}}{\Gamma(j\alpha+1)}\|x-z\|_B.$$

By definition of one parameter Mittag-Leffler function, we have

$$E_{\alpha}(Lb^{\alpha}) = \sum_{j=0}^{\infty} \frac{(Lb^{\alpha})^{j}}{\Gamma(j\alpha+1)}.$$

Note that $\frac{(Lb^{\alpha})^{j}}{\Gamma(j\alpha+1)}$ is the j^{th} term of the convergent series of nonnegative real numbers, hence we must have

$$\lim_{j\to\infty} \frac{(Lb^{\alpha})^{j}}{\Gamma(j\alpha+1)} = 0.$$

Thus we can choose $j \in \mathbb{N}$ such that $\frac{(Lb^{\alpha})^{j}}{\Gamma(j\alpha+1)} < 1$ so that F^j is a contraction. Therefore by modified version of contraction principle, F has a unique fixed point $x : [-r,b] \to \mathbb{R}^n$ in B, which is the unique solution of the FDDEs (1.1)-(1.2). □

4. Ulam-Hyers Stability of FDDE

We adopt the definitions of Ulam–Hyers stability, generalized Ulam–Hyers stability and Ulam–Hyers–Rassias stability given in [2].

Definition 4.1. *We say that the equation (1.1) has Ulam–Hyers stability if there exists a real number $K_f > 0$ such that for each $\epsilon > 0$, if $y : [-r,b] \to \mathbb{R}^n$ in B satisfies*

$$||^{c}D^{\alpha}y(t) - f(t, y_t)|| \leq \epsilon,\ t \in [0,b],$$

then there exists a solution $x : [-r,b] \to \mathbb{R}^n$ of equation (1.1) in B with

$$||y(t) - x(t)|| \leq K_f\,\epsilon,\ t \in [-r,b].$$

Moreover if $x^{(k)}(0) = y^{(k)}(0), k = 0, 1, 2, ..., m-1$, equation (1.1) is Ulam–Hyers stable with initial conditions.

Definition 4.2. *We say that the equation (1.1) has generalized Ulam–Hyers stability if there exists* $\psi_f \in C([0,b],\mathbb{R}_+), \psi_f(0) = 0$ *such that for each* $\epsilon > 0$*, if* $y : [-r,b] \to \mathbb{R}^n$ *in* B *satisfies*

$$||^c D^\alpha y(t) - f(t, y_t)|| \leq \epsilon,\ t \in [0,b],$$

then there exists a solution $x : [-r,b] \to \mathbb{R}^n$ *of equation (1.1) in* B *with*

$$||y(t) - x(t)|| \leq \psi_f(\epsilon),\ t \in [-r,b].$$

Definition 4.3. *We say that the equation (1.1) has Ulam–Hyers–Rassias stability with respect to* $\psi \in C([0,b],\mathbb{R}_+)$ *if there exists* $K_{f,\psi} > 0$ *such that for each* $\epsilon > 0$*, if* $y : [-r,b] \to \mathbb{R}^n$ *in* B *satisfies*

$$||^c D^\alpha y(t) - f(t, y_t)|| \leq \epsilon\, \psi(t),\ t \in [0,b],$$

then there exists a solution $x : [-r,b] \to \mathbb{R}^n$ *of equation (1.1) in* B *with*

$$||y(t) - x(t)|| \leq K_{f,\psi}\, \epsilon\, \psi(t),\ t \in [-r,b].$$

Definition 4.4. *We say that the equation (1.1) has generalized Ulam–Hyers–Rassias stability with respect to* $\psi \in C([0,b],\mathbb{R}_+)$ *if there exists* $K_{f,\psi} > 0$ *such that, if* $y : [-r,b] \to \mathbb{R}^n$ *in* B *satisfies*

$$||^c D^\alpha y(t) - f(t, y_t)|| \leq \psi(t),\ t \in [0,b],$$

then there exists a solution $x : [-r,b] \to \mathbb{R}^n$ *of equation (1.1) in* B *with*

$$||y(t) - x(t)|| \leq K_{f,\psi}\, \psi(t),\ t \in [-r,b].$$

Remark 4.1. *Note that, Definition 4.1* $\Rightarrow$ *Definition 4.2, Definition 4.3* $\Rightarrow$ *Definition 4.4 and Definition 4.3* $\Rightarrow$ *Definition 4.1.*

In the following theorem by method of successive approximation we prove that FDDE (1.1) is Ulam–Hyers stable.

Theorem 4.1. *Let* $f : [0,b] \times C \to \mathbb{R}^n$ *be a continuous function that satisfies Lipschitz condition*

$$\|f(t,u) - f(t,v)\|| \leq L\|u - v\|_C,\ u, v \in C,\ t \in [0,b].$$

For every $\epsilon > 0$*, if* $y : [-r,b] \to \mathbb{R}^n$ *in* B *satisfies,*

$$||^c D^\alpha y(t) - f(t, y_t)|| \leq \epsilon,\ t \in [0,b],$$

then there exists unique solution $x : [-r,b] \to \mathbb{R}^n$ *of equation (1.1) in* B *with* $x^{(k)}(0) = y^{(k)}(0), k = 0,1,2,...,m-1,$ *such that*

$$\|y(t) - x(t)\| \leq \left(\frac{E_\alpha(Lb^\alpha) - 1}{L}\right)\epsilon,\ t \in [-r,b].$$

Proof: For every $\epsilon > 0$, let $y : [-r, b] \to \mathbb{R}^n$ in B satisfies,

$$\|{}^cD^\alpha y(t) - f(t, y_t)\| \leq \epsilon, \ t \in [0, b]. \tag{4.1}$$

Then there exists a function $\sigma_y \in B$ (depending on y) such that,

$$\|\sigma_y(t)\| \leq \epsilon, \ t \in [0, b],$$

and

$${}^cD^\alpha y(t) = f(t, y_t) + \sigma_y(t), \ t \in [0, b]. \tag{4.2}$$

If $y(t)$ satisfies (4.2) then in view of Lemma 3.2 it satisfies equivalent fractional integral equation

$$y(t) = \sum_{k=0}^{m-1} \frac{y^{(k)}(0)}{\Gamma(k+1)} t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_s) ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sigma_y(s) ds, \ t \in [0, b]. \tag{4.3}$$

Define ,

$$x^0(t) = y(t), \ t \in [-r, b],$$

and consider the sequence $\{x^j\} \subseteq B$ defined by,

$$x^j(t) = \begin{cases} y(t), & t \in [-r, 0], \\ \sum_{k=o}^{m-1} \frac{y^{(k)}(0)}{\Gamma(k+1)} t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_s^{j-1}) ds, & t \in [0, b]. \end{cases}$$

Using mathematical induction firstly we prove that,

$$\|x^j(t) - x^{j-1}(t)\| \leq \frac{\epsilon}{L} \frac{(Lt^\alpha)^j}{\Gamma(j\alpha+1)}, \ t \in [0, b], \ j \in \mathbb{N}. \tag{4.4}$$

By definition of successive approximations given above and (4.3) we have,

$$\begin{aligned} \|x^1(t) - x^0(t)\| &= \left\| \sum_{k=0}^{m-1} \frac{y^{(k)}(0)}{\Gamma(k+1)} t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_s^0) ds - y(t) \right\| \\ &= \left\| \sum_{k=0}^{m-1} \frac{y^{(k)}(0)}{\Gamma(k+1)} t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_s) ds - y(t) \right\| \\ &= \left\| \int_0^t (t-s)^{\alpha-1} \sigma_y(s) ds \right\| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|\sigma_y(s)\| ds \\ &\leq \frac{\epsilon t^\alpha}{\Gamma(\alpha+1)} \end{aligned}$$

Therefore,

$$\|x^1(t) - x^0(t)\| \leq \frac{\epsilon}{L} \frac{Lt^\alpha}{\Gamma(\alpha+1)}, \ t \in [0, b],$$

which proves the inequality (4.4) for $j = 1$. Let us suppose that the inequality (4.4) hold for $j = r \in \mathbb{N}$, we prove it also hold for $j = r + 1 \in \mathbb{N}$.

By using definition of successive approximations and Lipschitz condition of f, for any $t \in [0, b]$ we obtain,

$$\begin{aligned} \|x^{r+1}(t) - x^r(t)\| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\| f(s, x_s^r) - f(s, x_s^{r-1}) \right\| ds \\ &\leq \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\| x_s^r - x_s^{r-1} \right\|_C ds. \end{aligned} \tag{4.5}$$

Since (4.4) hold for $j = r$, we have

$$\|x^r(t) - x^{r-1}(t)\| \leq \frac{\epsilon}{L} \frac{(Lt^\alpha)^r}{\Gamma(r\alpha+1)}, \ t \in [0, b].$$

Therefore by using Lemma 2.2 we get,

$$\|x_t^r - x_t^{r-1}\|_C \leq \frac{\epsilon}{L} \frac{(Lt^\alpha)^r}{\Gamma(r\alpha+1)}, \ t \in [0, b]$$

Thus the inequality (4.5) reduces to

$$\begin{aligned} \|x^{r+1}(t) - x^r(t)\| &\leq \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{\epsilon}{L} \frac{(Ls^\alpha)^r}{\Gamma(r\alpha+1)} ds \\ &= \frac{\epsilon L^r}{\Gamma(\alpha)\Gamma(r\alpha+1)} \int_0^t (t-s)^{\alpha-1} s^{r\alpha} ds \end{aligned}$$

Using Lemma 2.4, in the above inequality, we get,

$$\|x^{r+1}(t) - x^r(t)\| \leq \frac{\epsilon}{\Gamma(\alpha)} \frac{L^r}{\Gamma(r\alpha+1)} t^{(r+1)\alpha} \frac{\Gamma(\alpha)\Gamma(r\alpha+1)}{\Gamma((r+1)\alpha+1)}.$$

Therefore,

$$\|x^{r+1}(t) - x^r(t)\| \leq \frac{\epsilon}{L} \frac{(Lt^\alpha)^{r+1}}{\Gamma((r+1)\alpha+1)}, \ t \in [0, b].$$

which is the inequality (4.4) for $j = r+1$. Using principle of mathematical induction the proof of the inequality (4.4) is completed.

Now using the estimation (4.4) for any $t \in [0, b]$,

$$\sum_{j=1}^{\infty} \|x^j(t) - x^{j-1}(t)\| \leq \frac{\epsilon}{L} \sum_{j=1}^{\infty} \frac{(Lt^\alpha)^j}{\Gamma(j\alpha+1)} = \frac{\epsilon}{L} (E_\alpha(Lt^\alpha) - 1). \tag{4.6}$$

Hence the series,

$$x^0(t) + \sum_{j=1}^{\infty}[x^j(t) - x^{j-1}(t)]$$

converges absolutely and uniformly on $[0, b]$ with respect to the norm $\|\cdot\|$. Let us suppose

$$\tilde{x}(t) = x^0(t) + \sum_{j=1}^{\infty}[x^j(t) - x^{j-1}(t)],\ t \in [0, b]. \tag{4.7}$$

Then,

$$x^r(t) = x^0(t) + \sum_{j=1}^{r}[x^j(t) - x^{j-1}(t)] \tag{4.8}$$

is the r^{th} partial sum of the series (4.7), therefore we can write,

$$\lim_{r\to\infty} \|x^r(t) - \tilde{x}(t)\| = 0, \quad t \in [0, b].$$

Further by definition of successive approximations we have,

$$x^r(t) = y(t),\ t \in [-r, 0]$$

Therefore,

$$\lim_{r\to\infty} x^r(t) = y(t), \quad t \in [-r, 0].$$

Define,

$$x(t) = \begin{cases} y(t), & t \in [-r, 0], \\ \tilde{x}(t), & t \in [0, b]. \end{cases}$$

Clearly $x \in B$. We prove that this limit function is the solution of fractional integral equation

$$x(t) = \begin{cases} y(t), & t \in [-r, 0], \\ \sum_{k=o}^{m-1} \dfrac{y^{(k)}(0)}{\Gamma(k+1)} t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_s)ds, & \text{if} \quad t \in [0, b]. \end{cases} \tag{4.9}$$

Using definition of successive approximations for any $t \in [0,b]$, we have

$$\begin{aligned}
&\left\| x(t) - \sum_{k=0}^{m-1} \frac{y^{(k)}(0)}{\Gamma(k+1)} t^k - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_s) ds \right\| \\
&= \left\| \tilde{x}(t) - \left(x^r(t) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_s^{r-1}) ds \right) \right. \\
&\left. -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_s) ds \right\| \\
&\le \|\tilde{x}(t) - x^r(t)\| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s, x_s^{r-1}) - f(s, x_s)\| ds \\
&\le \|\tilde{x}(t) - x^r(t)\| + \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|x_s^{r-1} - x_s\|_C ds \qquad (4.10)
\end{aligned}$$

Now for any $t \in [0,b]$, we write from equations (4.7) and (4.8),

$$\|\tilde{x}(t) - x^r(t)\| = \left\| \sum_{j=r+1}^{\infty} [x^j(t) - x^{j-1}(t)] \right\| \le \sum_{j=r+1}^{\infty} \left\| x^j(t) - x^{j-1}(t) \right\|$$

Using inequality (4.4), we obtain

$$\|x(t) - x^r(t)\| = \|\tilde{x}(t) - x^r(t)\| \le \sum_{j=r+1}^{\infty} \frac{\epsilon}{L} \frac{(Lt^\alpha)^j}{\Gamma(j\alpha+1)}, \ t \in [0,b], \qquad (4.11)$$

Applying Lemma 2.2, we get

$$\|x_t - x_t^r\| \le \sum_{j=r+1}^{\infty} \frac{\epsilon}{L} \frac{(Lt^\alpha)^j}{\Gamma(j\alpha+1)} \qquad (4.12)$$

Using (4.11) and (4.12) in (4.10), we obtain

$$\begin{aligned}
&\left\| x(t) - \sum_{k=0}^{m-1} \frac{y^{(k)}(0)}{\Gamma(k+1)} t^k - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_s) ds \right\| \\
&\le \sum_{j=r+1}^{\infty} \frac{\epsilon}{L} \frac{(Lt^\alpha)^j}{\Gamma(j\alpha+1)} + \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sum_{j=r+1}^{\infty} \frac{\epsilon}{L} \frac{(Ls^\alpha)^j}{\Gamma(j\alpha+1)} ds \\
&= \sum_{j=r+1}^{\infty} \frac{\epsilon}{L} \frac{(Lt^\alpha)^j}{\Gamma(j\alpha+1)} + \frac{L}{\Gamma(\alpha)} \frac{\epsilon}{L} \sum_{j=r+1}^{\infty} \frac{L^j}{\Gamma(j\alpha+1)} \int_0^t (t-s)^{\alpha-1} s^{\alpha j} ds \\
&= \sum_{j=r+1}^{\infty} \frac{\epsilon}{L} \frac{(Lt^\alpha)^j}{\Gamma(j\alpha+1)} + \frac{\epsilon}{\Gamma(\alpha)} \sum_{j=r+1}^{\infty} \frac{L^j}{\Gamma(j\alpha+1)} t^{(\alpha+1)j} \frac{\Gamma(\alpha)\Gamma(j\alpha+1)}{\Gamma(j(\alpha+1)+1)} \\
&= \frac{\epsilon}{L} \sum_{j=r+1}^{\infty} \frac{(Lt^\alpha)^j}{\Gamma(j\alpha+1)} + \frac{\epsilon}{L} \sum_{j=r+1}^{\infty} \frac{(Lt^{\alpha+1})^j}{\Gamma(j(\alpha+1)+1)}.
\end{aligned}$$

Thus

$$\left\|x(t) - \sum_{k=0}^{m-1} \frac{y^{(k)}(0)}{\Gamma(k+1)} t^k - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_s) ds\right\|$$
$$\leq \frac{\epsilon}{L} \sum_{j=r+1}^{\infty} \frac{(Lt^\alpha)^j}{\Gamma(j\alpha+1)} + \frac{\epsilon}{L} \sum_{j=r+1}^{\infty} \frac{(Lt^{\alpha+1})^j}{\Gamma(j(\alpha+1)+1)}, \ t \in [0, b]. \tag{4.13}$$

Since both the series on the right hand side of above inequality are convergent, by taking limit as $j \to \infty$, we obtain

$$\left\|x(t) - \sum_{k=0}^{m-1} \frac{y^{(k)}(0)}{\Gamma(k+1)} t^k - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_s) ds\right\| \leq 0, \ t \in [0, b].$$

This implies

$$x(t) = \sum_{k=0}^{m-1} \frac{y^{(k)}(0)}{\Gamma(k+1)} t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_s) ds, \ t \in [0, b]. \tag{4.14}$$

Therefore $x(t)$ is solution of (1.1) with initial condition $x^{(k)}(0) = y^{(k)}(0), \ k = 0, 1, ..., m-1$. Further, from equations (4.6), (4.7) and (4.9), we have

$$\|y(t) - x(t)\| \leq \left(\frac{E_\alpha(Lb^\alpha) - 1}{L}\right)\epsilon, \ t \in [-r, b].$$

This proves that the equation (1.1) is Ulam–Hyers stable. Moreover as $x^{(k)}(0) = y^{(k)}(0), \ k = 0, 1, ..., m-1$, the equation (1.1) has Ulam–Hyers stability with the initial conditions.

It remains to prove the uniqueness of $x(t)$. Assume $\bar{x}(t)$ is another solution of (1.1) with the initial conditions $\bar{x}^{(k)}(0) = y^{(k)}(0), \ k = 0, 1, \cdots, m-1$. Then

$$\bar{x}(t) = \begin{cases} y(t) & \text{if} \quad t \in [-r, 0], \\ \sum_{k=o}^{m-1} \frac{y^{(k)}(0)}{\Gamma(k+1)} t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \bar{x}_s) ds & \text{if} \quad t \in [0, b]. \end{cases}$$

Note that

$$\|x(t) - \bar{x}(t)\| = 0, \ t \in [-r, 0].$$

By using Lipschitz condition we find that

$$\|x(t) - \bar{x}(t)\| \leq \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|x_s - \bar{x}_s\|_C \, ds, \ \text{for all } t \in [0, b].$$

Using Lemma 2.2,

$$\|x_t - \bar{x}_t\|_C \leq \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|x_s - \bar{x}_s\|_C \, ds, \ \text{for all } t \in [0, b].$$

An application of Lemma 2.3 to above inequality with $u(t) = \|x_t - \bar{x}_t\|_C$ and $a(t) = 0$, we obtain

$$\|x_t - \bar{x}_t\|_C = 0 \text{ for all } t \in [0, b].$$

Hence $\|x(t) - \bar{x}(t)\| = 0$ for all $t \in [-r, b]$. This completes the proof. □

Remark 4.2. *If we set* $\psi_f(\epsilon) = \left(\dfrac{E_\alpha(Lb^\alpha) - 1}{L}\right)\epsilon$ *then* $\psi_f(0) = 0$. *Hence FDDE (1.1) is generalized Ulam–Hyers stable with initial conditions.*

Note 1. *An example is given in last section to illustrate without initial condition* $x(t)$ *is neither unique nor necessarily the best approximate solution to FDDE (1.1).*

Next we obtain Ulam–Hyers–Rassias stability result for the equation (1.1) by method of successive approximation.

Theorem 4.2. *Let* $f : [0, b] \times C \to \mathbb{R}^n$ *be a continuous function that satisfies Lipschitz condition*

$$\|f(t, u) - f(t, v)\|| \leq L\|u - v\|_C,\ u, v \in C,\ t \in [0, b].$$

For every $\epsilon > 0$, *if* $y : [-r, b] \to \mathbb{R}^n$ *in* B *satisfies,*

$$||{}^cD^\alpha y(t) - f(t, y_t)|| \leq \epsilon\psi(t),\ t \in [0, b],$$

where $\psi \in C([0, b], \mathbb{R}_+)$ *is a nondecreasing function such that,*

$$\left|\frac{1}{\Gamma(\alpha)}\int_0^t (t - s)^{\alpha-1}\psi(s)ds\right| \leq \lambda\psi(t),\ t \in [0, b],$$

and $\lambda > 0$ *is constant satisfying* $0 < \lambda L < 1$, *then there exists unique solution* $x : [-r, b] \to \mathbb{R}^n$ *of equation (1.1) in* B *with* $x^{(k)}(0) = y^{(k)}(0), k = 0, 1, 2, ..., m - 1$, *that satisfies,*

$$\|y(t) - x(t)\| \leq \frac{\lambda}{(1 - \lambda L)}\,\epsilon\,\psi(t),\ t \in [-r, b].$$

Proof: For every $\epsilon > 0$, let $y : [-r, b] \to \mathbb{R}^n$ in B satisfies

$$\|{}^cD^\alpha y(t) - f(t, y_t)\| \leq \psi(t),\ t \in [0, b], \tag{4.15}$$

then there exists a function $\sigma_y \in B$ (depending on y) such that ,

$$\|\sigma_y(t)\| \leq \psi(t),\ t \in [0, b]$$

and

$${}^cD^\alpha y(t) = f(t, y_t) + \sigma_y(t),\ t \in [0, b].$$

Then y satisfies the fractional integral equation

$$y(t) = \sum_{k=0}^{m-1} \frac{y^{(k)}(0)}{\Gamma(k+1)} t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_s) ds \\ + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sigma_y(s) ds, \; t \in [0, b]. \tag{4.16}$$

Define the sequence of approximation $\{x^j\} \subseteq B$ as in proof of Theorem 4.1 starting with zeroth order approximation $x^0(t) = y(t), t \in [-r, 0]$. By using mathematical induction we prove that,

$$\|x^j(t) - x^{j-1}(t)\| \le \frac{\epsilon}{L} (\lambda L)^j \psi(t), \; t \in [0, b], \; j \in \mathbb{N}. \tag{4.17}$$

Using definition of successive approximations and (4.16) we have,

$$\begin{aligned} \|x^1(t) - x^0(t)\| &= \left\| \sum_{k=0}^{m-1} \frac{y^{(k)}(0)}{\Gamma(k+1)} t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x^0(s)) ds - y(t) \right\| \\ &= \left\| \sum_{k=0}^{m-1} \frac{y^{(k)}(0)}{\Gamma(k+1)} t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_s) ds - y(t) \right\| \\ &= \| \int_0^t (t-s)^{\alpha-1} \sigma_y(s) ds \| \\ &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|\sigma_y(s)\| ds \\ &\le \frac{\epsilon}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \psi(s) ds \\ &\le \lambda \, \epsilon \, \psi(t) \end{aligned}$$

Therefore,

$$\|x^1(t) - x^0(t)\| \le \frac{\epsilon}{L} (\lambda L) \psi(t), \; t \in [0, b],$$

which is the inequality (4.17) for $j = 1$.

Let us suppose that the inequality (4.17) hold for $j = r \in \mathbb{N}$. Then by definition of successive approximations and Lipschitz condition of f, for any $t \in [0, b]$ we obtain,

$$\begin{aligned} \|x^{r+1}(t) - x^r(t)\| &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\| f(s, x_s^r) - f(s, x_s^{r-1}) \right\| ds \\ &\le \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\| x_s^r - x_s^{r-1} \right\|_C ds. \end{aligned} \tag{4.18}$$

Since (4.17) hold for $j = r$, we have

$$\|x^r(t) - x^{r-1}(t)\| \le \frac{\epsilon}{L} (\lambda L)^r \psi(t), \; t \in [0, b].$$

By an application of Lemma 2.2 to the above inequality gives

$$\|x_t^r - x_t^{r-1}\|_C \le \frac{\epsilon}{L}(\lambda L)^r \psi(t), \ \ t \in [0,b]$$

Thus the inequality (4.18) reduces to

$$\begin{aligned}\|x^{r+1}(t) - x^r(t)\| &\le \frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\frac{\epsilon}{L}(\lambda L)^r\psi(t)ds \\ &= \epsilon(\lambda L)^r \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\psi(s)ds \\ &\le \epsilon(\lambda L)^r \lambda\, \psi(t) \\ &= \frac{\epsilon}{L}(\lambda L)^{r+1}\, \psi(t)\end{aligned}$$

Therefore,

$$\|x^{r+1}(t) - x^r(t)\| \le \frac{\epsilon}{L}\frac{(Lt^\alpha)^{r+1}}{\Gamma((r+1)\alpha+1)}, \ \ t \in [0,b].$$

which is the inequality (4.17) for $j = r+1$. The proof of the inequality (4.17) is completed by principle of mathematical induction.

Using the inequality (4.17) and the fact $0 < \lambda L < 1$, for any $t \in [0,b]$,

$$\sum_{j=1}^{\infty}\|x^j(t) - x^{j-1}(t)\| \le \frac{\epsilon}{L}\sum_{j=1}^{\infty}(\lambda L)^j\psi(t) = \frac{\epsilon}{L}\frac{\lambda L}{1-\lambda L}\psi(t). \tag{4.19}$$

Thus

$$\sum_{j=1}^{\infty}\|x^j(t) - x^{j-1}(t)\| \le \frac{\lambda}{(1-\lambda L)}\,\epsilon\,\psi(t), \ \ t \in [0,b]. \tag{4.20}$$

Hence the series

$$x^0(t) + \sum_{j=1}^{\infty}[x^j(t) - x^{j-1}(t)]$$

converges absolutely and uniformly on $[0,b]$, say to $\widehat{x}(t)$ in the norm $\|\cdot\|$. Define,

$$x(t) = \begin{cases} y(t), & t \in [-r,0], \\ \widehat{x}(t), & t \in [0,b]. \end{cases} \tag{4.21}$$

Proceeding as in the proof of Theorem 4.1 one can show that $x(t)$ is a solution of (1.1) with $x^{(k)}(0) = y^{(k)}(0), k = 0,1,2,...,m-1$, that satisfies

$$\|y(t) - x(t)\| \le \frac{\lambda}{(1-\lambda L)}\,\epsilon\,\psi(t), \ \ t \in [-r,b].$$

Therefore equation (1.1) is Ulam–Hyers–Rassias stable. □

5. $\mathbb{E}_\alpha$–Ulam–Hyers stability

We consider the following definitions of $\mathbb{E}_\alpha$–Ulam–Hyers stabilities introduced by Wang and Li [3].

Definition 5.1. *We say that equation (1.1) has $\mathbb{E}_\alpha$–Ulam–Hyers stability if there exists a real number $K > 0$ such that for each $\epsilon > 0$, if $y : [-r, b] \to \mathbb{R}^n$ in B satisfies*

$$||^c D^\alpha y(t) - f(t, y_t)|| \le \epsilon,\ \ t \in [0, b],$$

then there exists a solution $x : [-r, b] \to \mathbb{R}^n$ of equation (1.1) in B with

$$||y(t) - x(t)|| \le K\mathbb{E}_\alpha(\gamma_f t^\alpha)\epsilon,\ \gamma_f \ge 0,\ \ t \in [-r, b].$$

Definition 5.2. *We say that equation (1.1) has generalized $\mathbb{E}_\alpha$–Ulam–Hyers stability if there exists a nonnegative function $\psi \in C([0, b], \mathbb{R}_+), \psi(0) = 0$ such that for each $\epsilon > 0$, if $y : [-r, b] \to \mathbb{R}^n$ in B satisfies*

$$||^c D^\alpha y(t) - f(t, y_t)|| \le \epsilon,\ \ t \in [0, b],$$

there exists a solution $x : [-r, b] \to \mathbb{R}^n$ of equation (1.1) in B with

$$||y(t) - x(t)|| \le \psi(\epsilon)\mathbb{E}_\alpha(\gamma_f t^\alpha),\ \gamma_f \ge 0,\ \ t \in [-r, b].$$

Remark 5.1. *Definition 5.1 $\Rightarrow$ Definition 5.2.*

Theorem 5.1. *Let $f : [0, b] \times C \to \mathbb{R}^n$ satisfies Lipschitz condition*

$$\|f(t, u) - f(t, v)\|| \le L\|u - v\|_C,\ \ u, v \in C,\ \ t \in [0, b].$$

For every $\epsilon > 0$, if $y : [-r, b] \to \mathbb{R}^n$ in B satisfies,

$$||^c D^\alpha y(t) - f(t, y_t)|| \le \epsilon,\ \ t \in [0, b],$$

then there exists unique solution $x : [-r, b] \to \mathbb{R}^n$ of equation (1.1) in B with $x^k(0) = y^k(0), k = 0, 1, 2, ..., m - 1$, that satisfies

$$\|y(t) - x(t)\| \le \frac{1}{L} E_\alpha(Lt^\alpha)\epsilon,\ t \in [-r, b]$$

Proof: We define the sequence of approximations as in Theorem 4.1. Noting that $x^0(t) = y(t)$, we write from (4.4) (4.7) and (4.9)

$$\|y(t) - x(t)\| \le \sum_{j=1}^{\infty} \|x^j(t) - x^{j-1}(t)\| \le \sum_{j=0}^{\infty} \frac{\epsilon}{L} \frac{(Lt^\alpha)^j}{\Gamma(j\alpha + 1)} \le \frac{1}{L} E_\alpha(Lt^\alpha)\epsilon,\ t \in [0, b] \tag{5.1}$$

Showing that the FDDE (1.1) is $\mathbb{E}_\alpha$–Ulam–Hyers stable with initial conditions. □

Remark 5.2. *If we set $\psi(\epsilon) = \dfrac{\epsilon}{L}$ in result (5.1) then $\psi(0) = 0$, showing that FDDE (1.1) is generalized $\mathbb{E}_\alpha$–Ulam–Hyers stable with initial conditions.*

6. Examples

We remark that the argument " without initial condition we cannot obtain best and unique approximate solution " of Huang et al. [13] about the best and unique approximate solution for integer order delay differential equations is also hold for fractional order delay differential equations.

To illustrate this we consider an example in the space $\mathbb{R}^1$.

Example 6.1: Consider the fractional delay differential equation

$$^{c}D^{\frac{1}{2}}x(t) = x(t-1),\ t \in [0,1]. \tag{6.1}$$

Then the function $y(t) = 1,\ t \in [-1,1]$ satisfies the inequality

$$|^{c}D^{\frac{1}{2}}y(t) - y(t-1)| \leq 1,\ t \in [0,1].$$

Further note that $y(t)$ is not a solution of equation (6.1) as for any $t \in [0,1]$,

$$y(t-1) = 1 \text{ and } {}^{c}D^{\frac{1}{2}}y(t) = {}^{c}D^{\frac{1}{2}}(1) = 0.$$

By using successive approximations defined in Theorem 4.1 we obtain first approximate solution to (6.1) as

$$\begin{aligned} x^1(t) &= y(0) + \frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-s)^{\frac{1}{2}-1} f(s, x_s^0)ds \\ &= 1 + \frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-s)^{\frac{1}{2}-1} f(s, y_s)ds \\ &= 1 + \frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-s)^{\frac{1}{2}-1} ds \\ &= 1 + \frac{1}{\Gamma(\frac{1}{2})}\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2})}t^{\frac{1}{2}} = 1 + \frac{1}{\Gamma(\frac{3}{2})}t^{\frac{1}{2}} \end{aligned}$$

Define,

$$\psi(t) = \begin{cases} 1(=y(t)), & t \in [-1,0], \\ 1 + \frac{1}{\Gamma(\frac{3}{2})}t^{\frac{1}{2}}, & t \in [0,1]. \end{cases}$$

Then it is easy to verify that the function $\psi(t)$ forms a solution of (6.1). Also we find

$$|y(t) - \psi(t)| = |1 - (1 + \frac{1}{\Gamma(\frac{3}{2})}t^{\frac{1}{2}})| \leq \frac{1}{\Gamma(\frac{3}{2})} = 1.1283$$

Next, we see that the function $\psi^*(t)$ defined by

$$\psi^*(t) = \begin{cases} \Gamma(\frac{3}{2}), & t \in [-1,0], \\ \Gamma(\frac{3}{2})\left(1 + \frac{1}{\Gamma(\frac{3}{2})}t^{\frac{1}{2}}\right), & t \in [0,1]. \end{cases}$$

is also a solution of equation (6.1) and is such that

$$|y(t)-\psi^*(t)| = \begin{cases} |1-\Gamma(\frac{3}{2})| \le 0.1283, & t \in [-1,0], \\ |1-\Gamma(\frac{3}{2}) - t^{\frac{1}{2}}| \le |1-\Gamma(\frac{3}{2})| \le 0.1283, & t \in [0,1]. \end{cases}$$

Hence,

$$|y(t)-\psi^*(t)| \le 0.1283 \text{ for all } t \in [-1,1].$$

This shows that $\psi^*(t)$ is better approximate solution than $\psi(t)$.

Remark 6.1. *The substitution $r=0$, reduces the equation (1.1)-(1.2) to the ordinary fractional differential equation*

$$^cD^{\alpha}x(t) = f(t,x(t)),\ t \in [0,b] \tag{6.2}$$

$$x(0) = \phi(0). \tag{6.3}$$

Let f satisfies the Lipschitz condition with the Lipschitz constant L. Then the initial value problem (6.2)-(6.3) has a unique solution. Further, with the similar assumptions in the Theorem 4.1 the ordinary fractional differential equation (6.2) is Ulam-Hyers stable with Ulam-Hyers stability constant $\frac{(E_\alpha(Lb^\alpha)-1)}{L}$. Other stability results for (6.2) can be obtained similarly.

To illustrate existence and stability results for fractional delay differential equation obtained in this paper we give the following example. Since any two norms on a finite dimensional linear spaces are equivalent here we consider the example in $\mathbb{R}^2$ with the norm

$$\|x\| = |x_1| + |x_2|,\ x = (x_1,x_2) \in \mathbb{R}^2.$$

Example 6.2: Consider the fractional delay differential equation of the form:

$$^cD^{\frac{1}{2}}x(t) = f(t,x_t) = \left(\frac{x_1(t-1)}{1+x_1(t-1)}, x_2(t-1)\right),\ t \in [0,1], \tag{6.4}$$

$$x(t) = (1,t),\ t \in [-1,0] \tag{6.5}$$

where $x:[-1,1] \to \mathbb{R}^2$ and $f:[-1,0]\times C([-1,1],\mathbb{R}^2) \to \mathbb{R}^2$ is a nonlinear function. Let

$$f(t,\phi) = f(t,(\phi_1,\phi_2)) = \left(\frac{\phi_1}{1+\phi_1}, \phi_2\right).$$

Then for any $\phi, \psi \in C([-1,0], \mathbb{R}^2)$, we find

$$\begin{aligned}
\|f(t,\phi) - f(t,\psi)\| &= \|f(t,(\phi_1,\phi_2)) - f(t,(\psi_1,\psi_2))\| \\
&= \left\| \left(\frac{\phi_1}{1+\phi_1}, \phi_2 \right) - \left(\frac{\psi_1}{1+\psi_1}, \psi_2 \right) \right\| \\
&= \left\| \left(\frac{\phi_1}{1+\phi_1} - \frac{\psi_1}{1+\psi_1}, \phi_2 - \psi_2 \right) \right\| \\
&= \left| \frac{\phi_1}{1+\phi_1} - \frac{\psi_1}{1+\psi_1} \right| + |\phi_2 - \psi_2| \\
&= \frac{|\phi_1 - \psi_1|}{|1+\phi_1|\,|1+\psi_1|} + |\phi_2 - \psi_2| \\
&\leq |\phi_1 - \psi_1| + |\phi_2 - \psi_2| \\
&= \|\phi - \psi\|
\end{aligned}$$

Therefore, $\|f(t,\phi) - f(t,\psi)\| \leq \|\phi - \psi\|, \ \forall\, \phi, \psi \in C([-1,0], \mathbb{R}^2)$. This implies f satisfies Lipschitz condition with Lipschitz constant $L = 1$. Hence by Theorem 3.1 FDDEs (6.4)-(6.5) has unique solution.

Further, if $y \in B = C([-1,1], \mathbb{R}^2)$ satisfies

$$\|{}^cD^{\frac{1}{2}} y(t) - f(t, y_t)\| \leq \epsilon, \ t \in [0,1]$$

then as shown in Theorem 4.1, there exist a solution $x \in B$ of (6.4), such that

$$\|y(t) - x(t)\| \leq \left(\frac{E_{\frac{1}{2}}(1) - 1}{1} \right) \epsilon, \ t \in [-1,1].$$

Other stability results for the equation (6.4) can discussed similarly.

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