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Smarandache Curves for Spherical Indicatrix of the Bertrand Curves Pair

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ABSTRACT: In this paper, we investigate special Smarandache curves with regard to Sabban frame for Bertrand partner curve spherical indicatrix. Some results have been obtained. These results were expressed depending on the Bertrand curve. Besides, we are given examples of our results.

Key Words: Bertrand curves pair, Smarandache curve, Sabban frame, Geodesic curvature.

Contents

1. Introduction and Preliminaries

The Bertrand of the curve is well known by the mathematicians especially the differential geometry scientists. Bertrand curves have been studied by some authors [5,6,7]. Bertrand curves detected by J. Bertrand in 1850 are one of the important and interesting matters of classical special curve theory. A Bertrand curve is defined as a special curve which parallel its principal normals with another special curve, called Bertrand partner curve. If the curve α^* is Bertrand partner of α, then we may write that

$$\alpha^*(s^*) = \alpha(s) + \lambda(s)N(s) \tag{1.1}$$

where λ=constant, [6] . It is proved in most studies on the subject that the characteristic property of Bertrand curve is the existence of a linear relation between its curvature and torsion as [6],

$$\lambda\kappa + \mu\tau = 1, \quad \mu = -\lambda\cot\theta. \tag{1.2}$$

where $\angle(T, T^*) = \theta$. Whose position vector is composed by Frenet frame vectors regular curve is called a Smarandache curve [12]. Special Smarandache curves have been studied by some authors [1,2,4,8]. K. Taşköprü, M. Tosun studied special Smarandache curves according to Sabban frame on S^2 [11]. Şenyurt and Çalışkan investigated special Smarandache curves in terms of Sabban frame of spherical

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indicatrix curves and they gave some characterization of Smarandache curves [3]. Let $\alpha : I \to E^3$ be a unit speed curve, we defined the quantities of the Serret-Frenet apparatus, respectively

$$T(s) = \alpha'(s), \;\; N(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|}, \;\; B(s) = T(s) \wedge N(s), \tag{1.3}$$

$$\kappa(s) = \|T'(s)\|, \;\; \tau(s) = \langle N'(s), B(s) \rangle \tag{1.4}$$

we have an orthonormal frame $\{T, N, B\}$ along α. This frames is called the Frenet frame of α. This curve the Frenet formulae are, respectively, [6]

$$T'(s) = \kappa(s)N(s), \;\; N'(s) = -\kappa(s)T(s) + \tau(s)B(s), \;\; B'(s) = -\tau(s)N(s). \tag{1.5}$$

Let $\alpha : I \to E^3$ unit speed and $\alpha^* : I \to E^3$ be the C^2-class differentiable two curves and the amounts of $\{T(s), N(s), B(s)\}$ and $\{T^*(s^*), N^*(s^*), B^*(s^*)\}$ are entirely Frenet-Serret frames of the curves α Bertrand and the Bertrand partner α^*, respectively, then [5]

$$T^* = \cos\theta T - \sin\theta B, \;\; N^* = N, \;\; B^* = \sin\theta T + \cos\theta B \tag{1.6}$$

For the curvatures and the torsions we have

$$\kappa^* = \frac{\lambda\kappa - \sin^2\theta}{\lambda(1 - \lambda\kappa)}, \;\; \tau^* = \frac{\sin^2\theta}{\lambda^2\tau}. \tag{1.7}$$

In addition to

$$\sin\varphi^* = \sin(\varphi - \theta), \;\; \cos\varphi^* = \cos(\varphi - \theta), \;\; \varphi^{*\prime} = \varphi', \;\; \|W^*\| = \|W\|. \tag{1.8}$$

Let $\gamma : I \to S^2$ be a unit speed spherical curve. We denote s as the arc-length parameter of γ. Let us denote by

$$\gamma(s) = \gamma(s), \;\; t(s) = \gamma'(s), \;\; d(s) = \gamma(s) \wedge t(s) \tag{1.9}$$

$\{\gamma(s), t(s), d(s)\}$ frame is called the Sabban frame of γ on S^2. Then we have the following spherical Frenet formulae of γ

$$\gamma'(s) = t(s), \;\; t'(s) = -\gamma(s) + \kappa_g(s)d(s), \;\; d'(s) = -\kappa_g(s)t(s) \tag{1.10}$$

where κ_g is called the geodesic curvature of the curve γ on S^2 which is

$$\kappa_g(s) = \langle t'(s), d(s), \;\; [11]. \tag{1.11}$$

2. Smarandache Curves for Spherical Indicatrix of the Bertrand Curves Pair

In this section, we investigate special Smarandache curves created by Sabban frame, $\{T^*, T_{T^*}, T^* \wedge T_{T^*}\}$, $\{N^*, T_{N^*}, N^* \wedge T_{N^*}\}$ and $\{B^*, T_{B^*}, B^* \wedge T_{B^*}\}$, that belongs to spherical indicatrix of a α^* curve are defined. We will find some results. These results will be expressed depending on the Bertrand curve. Let $\alpha_{T^*}(s_{T^*}) = T^*(s^*)$, $\alpha_{N^*}(s_{N^*}) = N^*(s^*)$ and $\alpha_{B^*}(s_{B^*}) = B^*(s^*)$ be a regular spherical curves on S^2. Sabban frame for (T^*), (N^*) and (B^*) are, respectively

$$T^* = T^*, \;\; T_{T^*} = N^*, \;\; T^* \wedge T_{T^*} = B^*, \tag{2.1}$$

$$\begin{cases} N^* = N^* \\ T_{N^*} = -\cos\varphi^* T^* + \sin\varphi^* B^* \\ N^* \wedge T_{N^*} = \sin\varphi^* T^* + \cos\varphi^* B^*, \end{cases} \tag{2.2}$$

and

$$B^* = B^*, \ \ T_{B^*} = -N^*, \ \ B^* \wedge T_{B^*} = T^*. \tag{2.3}$$

From the equation (1.10), we have the following spherical Frenet formulae of (T^*), (N^*) and (B^*) are, respectively,

$$T^{*\prime} = T_{T^*}, \ \ {T_{T^*}}' = -T^* + \frac{\tau^*}{\kappa^*} T^* \wedge T_{T^*}, \ \ (T^* \wedge T_{T^*})' = -\frac{\tau^*}{\kappa^*} T_{T^*}, \tag{2.4}$$

$$N^{*\prime} = T_{N^*}, \ \ {T_{N^*}}' = -N^* + \frac{\varphi^{*\prime}}{\|W^*\|} N^* \wedge T_{N^*}, \ \ (N^* \wedge T_{N^*})' = -\frac{\varphi^{*\prime}}{\|W^*\|} T_{N^*}, \tag{2.5}$$

and

$$B^{*\prime} = T_{B^*}, \ \ {T_{B^*}}' = -B^* + \frac{\kappa^*}{\tau^*} B^* \wedge T_{B^*}, \ \ (B^* \wedge T_{B^*})' = -\frac{\kappa^*}{\tau^*} T_{B^*}. \tag{2.6}$$

From the equation (1.11) we have the following geodesic curvatures of (T^*), (N^*) and (B^*) are, respectively,

$$\kappa_g^{T^*} = \frac{\tau^*}{\kappa^*}, \ \ \kappa_g^{N^*} = \frac{\varphi^{*\prime}}{\|W^*\|} \text{ and } \kappa_g^{B^*} = \frac{\kappa^*}{\tau^*}. \tag{2.7}$$

β_1**-Smarandache Curve**, can be defined by,

$$\beta_1(s_{T^*}) = \frac{1}{\sqrt{2}}(T^* + T_{T^*}), \tag{2.8}$$

or from the equation (2.1), we can write

$$\beta_1(s^*) = \frac{1}{\sqrt{2}}(T^* + N^*). \tag{2.9}$$

Differentiating (2.8), we reach

$$T_{\beta_1}(s^*) = \frac{1}{\sqrt{2 + (\frac{\tau^*}{\kappa^*})^2}}(-T^* + N^* + \frac{\tau^*}{\kappa^*} B^*). \tag{2.10}$$

Considering the equations (2.9) and (2.10), with ease seen that

$$(\beta_1 \wedge T_{\beta_1})(s^*) = \frac{1}{\sqrt{4 + 2(\frac{\tau^*}{\kappa^*})^2}}(\frac{\tau^*}{\kappa^*} T^* - \frac{\tau^*}{\kappa^*} N^* + 2B^*). \tag{2.11}$$

Differentiating (2.10),

$$\begin{cases} \lambda_1 = -2 - (\frac{\tau^*}{\kappa^*})^2 + (\frac{\tau^*}{\kappa^*})'(\frac{\tau^*}{\kappa^*}), \\ \lambda_2 = -2 - 3(\frac{\tau^*}{\kappa^*})^2 - (\frac{\tau^*}{\kappa^*})^4 - (\frac{\tau^*}{\kappa^*})'(\frac{\tau^*}{\kappa^*}), \\ \lambda_3 = 2(\frac{\tau^*}{\kappa^*}) + (\frac{\tau^*}{\kappa^*})^3 + (\frac{\tau^*}{\kappa^*})', \end{cases} \tag{2.12}$$

including we can write,

$$T'_{\beta_1}(s^*) = \frac{\sqrt{2}}{\left(2 + (\frac{\tau^*}{\kappa^*})^2\right)^2}(\lambda_1 T^* + \lambda_2 N^* + \lambda_3 B^*). \tag{2.13}$$

From the equation (1.11), (2.11) and (2.13), $\kappa_g^{\beta_1}$ geodesic curvature of $\beta_1(s^*)$ is

$$\kappa_g^{\beta_1} = \frac{1}{(2+(\frac{\tau^*}{\kappa^*})^2)^{\frac{5}{2}}}(\frac{\tau^*}{\kappa^*}\lambda_1 + \frac{\tau^*}{\kappa^*}\lambda_2 + 2\lambda_3). \tag{2.14}$$

Substituting the equations (1.6) and (1.7) into equation (2.9), (2.10), (2.11) and (2.13), Sabban apparataus of the β_1-Smarandache curve for Bertrand curve are

$$\begin{aligned}
\beta_1(s) &= \frac{1}{\sqrt{2}}(\cos\theta T + N - \sin\theta B),\\
T_{\beta_1}(s) &= \frac{\tan(\varphi-\theta)\sin\theta - \cos\theta}{\sqrt{2+\tan^2(\varphi-\theta)}}T + \frac{1}{\sqrt{2+\tan^2(\varphi-\theta)}}N\\
&\quad + \frac{\tan(\varphi-\theta)\cos\theta + \sin\theta}{\sqrt{2+\tan^2(\varphi-\theta)}}B,\\
(\beta_1 \wedge T_{\beta_1})(s) &= \frac{2\sin\theta + \tan(\varphi-\theta)\cos\theta}{\sqrt{4+2\tan^2(\varphi-\theta)}}T - \frac{\tan(\varphi-\theta)}{\sqrt{4+2\tan^2(\varphi-\theta)}}N\\
&\quad + \frac{2\cos\theta - \tan(\varphi-\theta)\sin\theta}{\sqrt{4+2\tan^2(\varphi-\theta)}}B,\\
T'_{\beta_1}(s) &= \frac{(\overline{\lambda}_3\sin\theta + \overline{\lambda}_1\cos\theta)\sqrt{2}}{(2+\tan^2(\varphi-\theta))^2}T + \frac{\overline{\lambda}_2\sqrt{2}}{(2+\tan^2(\varphi-\theta))^2}N\\
&\quad + \frac{(\overline{\lambda}_3\cos\theta - \overline{\lambda}_1\sin\theta)\sqrt{2}}{(2+\tan^2(\varphi-\theta))^2}B
\end{aligned}$$

and

$$\kappa_g^{\beta_1} = \frac{1}{\big(2+\tan^2(\varphi-\theta)\big)^{\frac{5}{2}}}\big(\tan(\varphi-\theta)\overline{\lambda}_1 + \tan(\varphi-\theta)\overline{\lambda}_2 + 2\overline{\lambda}_3\big),$$

where

$$\begin{cases}
\overline{\lambda}_1 = -2 - \tan^2(\varphi-\theta) + \tan'(\varphi-\theta)\tan(\varphi-\theta)\\
\overline{\lambda}_2 = -2 - 3\tan^2(\varphi-\theta) - \tan^4(\varphi-\theta) - \tan'(\varphi-\theta)\tan(\varphi-\theta)\\
\overline{\lambda}_3 = 2\tan(\varphi-\theta) + \tan^3(\varphi-\theta) + 2\tan'(\varphi-\theta).
\end{cases}$$

β_2-Smarandache Curve can be defined by

$$\beta_2(s_{T^*}) = \frac{1}{\sqrt{2}}(T_{T^*} + T^* \wedge T_{T^*}) \tag{2.15}$$

or from the equations (1.6) and (2.1), we can write

$$\beta_2(s) = \frac{1}{\sqrt{2}}\big(\sin\theta T + N + \cos\theta B\big). \tag{2.16}$$

Differentiating (2.16), we can write

$$T_{\beta_2}(s) = \frac{\tan(\varphi-\theta)\sin\theta - \cos\theta}{1+2\tan^2(\varphi-\theta)}T - \frac{\tan(\varphi-\theta)}{1+2\tan^2(\varphi-\theta)^2}N + \frac{\tan(\varphi-\theta)\cos\theta - \sin\theta}{\sqrt{1+2\tan^2(\varphi-\theta)}}B. \tag{2.17}$$

Considering the equations (2.16) and (2.17), with ease seen that

$$\begin{aligned}\left(\beta_2 \wedge T_{\beta_2}\right)(s) &= \frac{\sin\theta + 2\tan(\varphi-\theta)\cos\theta}{\sqrt{2+4\tan^2(\varphi-\theta)}}T - \frac{1}{\sqrt{2+4\tan^2(\varphi-\theta)}}N \\ &+ \frac{\cos\theta - \tan(\varphi-\theta)\sin\theta}{\sqrt{2+4\tan^2(\varphi-\theta)}}B \end{aligned} \tag{2.18}$$

Differentiating (2.17), where

$$\begin{cases}\bar{\varepsilon}_1 = \tan(\varphi-\theta) + 2\tan^3(\varphi-\theta) + 2\tan'(\varphi-\theta)\tan(\varphi-\theta) \\ \bar{\varepsilon}_2 = -1 - 3\tan^2(\varphi-\theta) - 2\tan^4(\varphi-\theta) - \tan'(\varphi-\theta) \\ \bar{\varepsilon}_3 = \tan^2(\varphi-\theta) - 2\tan^4(\varphi-\theta) + \tan'(\varphi-\theta),\end{cases} \tag{2.19}$$

including we have

$$T'_{\beta_2}(s) = \frac{(\bar{\varepsilon}_3\sin\theta + \bar{\varepsilon}_1\cos\theta)\sqrt{2}}{(1+2\tan^2(\varphi-\theta))^2}T + \frac{\bar{\varepsilon}_2\sqrt{2}}{(1+2\tan^2(\varphi-\theta))^2}N + \frac{(\bar{\varepsilon}_3\cos\theta - \bar{\varepsilon}_1\sin\theta)\sqrt{2}}{(1+2\tan^2(\varphi-\theta))^2}B. \tag{2.20}$$

$\kappa_g^{\beta_2}$ geodesic curvature of β_2-Smarandache curve according to Bertrand curve is

$$\kappa_g^{\beta_2} = \frac{1}{\left(1+2\tan^2(\varphi-\theta)\right)^{\frac{5}{2}}}\left(2\tan(\varphi-\theta)\bar{\varepsilon}_1 - \bar{\varepsilon}_2 + \bar{\varepsilon}_3\right). \tag{2.21}$$

β_3**-Smarandache Curve**, can be defined by

$$\beta_3(s_{T^*}) = \frac{1}{\sqrt{3}}(T^* + T_{T^*} + T^* \wedge T_{T^*}) \tag{2.22}$$

or from the equations (1.6) and (2.1), we can write

$$\beta_3(s) = \frac{1}{\sqrt{3}}\Big(\big(\sin\theta + \cos\theta\big)T + N + \big(\cos\theta - \sin\theta\big)B\Big). \tag{2.23}$$

Differentiating (2.23), we reach

$$\begin{aligned} T_{\beta_3}(s) &= \frac{\tan(\varphi-\theta)\sin\theta - \cos\theta}{\sqrt{2(1-\tan(\varphi-\theta)+\tan^2(\varphi-\theta))}}T + \frac{1}{\sqrt{2(1-\tan(\varphi-\theta)+\tan^2(\varphi-\theta))}}N \\ &+ \frac{\tan(\varphi-\theta)\cos\theta + \sin\theta}{\sqrt{2(1-\tan(\varphi-\theta)+\tan^2(\varphi-\theta))}}B. \end{aligned} \tag{2.24}$$

Considering the equations (2.23) and (2.24), it is easily seen

$$\left(\beta_3 \wedge T_{\beta_3}\right)(s) = \frac{(2\tan(\varphi-\theta)-1)\cos\theta + (2-\tan(\varphi-\theta))\sin\theta}{\sqrt{6-6\tan(\varphi-\theta)+6\tan^2(\varphi-\theta)}}T \tag{2.25}$$

$$\begin{aligned} &- \frac{1+\tan(\varphi-\theta)}{\sqrt{6-6\tan(\varphi-\theta)+6\tan^2(\varphi-\theta)}}N \\ &+ \frac{(2-\tan(\varphi-\theta))\cos\theta - (2\tan(\varphi-\theta)-1)\sin\theta}{\sqrt{6-6\tan(\varphi-\theta)+6\tan^2(\varphi-\theta)}}B. \end{aligned} \tag{2.26}$$

Differentiating (2.24), where

$$\begin{cases}\overline{\phi}_1=-2+4\tan(\varphi-\theta)-4\tan^2(\varphi-\theta)+2\tan^3(\varphi-\theta)+\tan'(\varphi-\theta)(1-2\tan(\varphi-\theta))\\ \overline{\phi}_2=-2+2\tan(\varphi-\theta)-4\tan^2(\varphi-\theta)+2\tan^3(\varphi-\theta)-2\tan^4(\varphi-\theta)\\ \quad -\tan'(\varphi-\theta)(1+\tan(\varphi-\theta))\\ \overline{\phi}_3=2\tan(\varphi-\theta)-4\tan^2(\varphi-\theta)+4\tan^3(\varphi-\theta)-2\tan^4(\varphi-\theta)+\tan'(\varphi-\theta)(2-\tan(\varphi-\theta)),\end{cases} \tag{2.27}$$

including we reach

$$\begin{aligned}T'_{\beta_3}(s) &= \frac{(\overline{\phi}_3\sin\theta+\overline{\phi}_1\cos\theta)\sqrt{3}}{4(1-\tan(\varphi-\theta)+\tan^2(\varphi-\theta))^2}T+\frac{\overline{\phi}_2\sqrt{3}}{(4(1-\tan(\varphi-\theta)+\tan^2(\varphi-\theta))^2}N\\ &\quad+\frac{(\overline{\phi}_3\cos\theta-\overline{\phi}_1\sin\theta)\sqrt{3}}{4(1-\tan(\varphi-\theta)+\tan^2(\varphi-\theta))^2}B.\end{aligned} \tag{2.28}$$

$\kappa_g^{\beta_3}$ geodesic curvature of β_3-Smarandache curve according to Bertrand curve is

$$\kappa_g^{\beta_3}=\frac{(2\tan(\varphi-\theta)-1)\overline{\phi}_1-(1+\tan(\varphi-\theta))\overline{\phi}_2+(2-\tan(\varphi-\theta))\overline{\phi}_3}{4\sqrt{2}\big(1-\tan(\varphi-\theta)+\tan^2(\varphi-\theta)\big)^{\frac{5}{2}}}. \tag{2.29}$$

β_4**-Smarandache Curve**, can be defined by

$$\beta_4(s_{N^*})=\frac{1}{\sqrt{2}}(N^*+T_{N^*}) \tag{2.30}$$

or from the equations (1.6), (1.8) and (2.2), we can write

$$\beta_4(s) = \frac{1}{\sqrt{2}}\big(-\cos\varphi T+N+\sin\varphi B\big). \tag{2.31}$$

Differentiating (2.31), we can write

$$T_{\beta_4}(s)=\frac{\varphi'\sin\varphi-\|W\|\cos\varphi}{\sqrt{2\|W\|^2+\varphi'^2}}T-\frac{\|W\|}{\sqrt{2\|W\|^2+\varphi'^2}}N+\frac{\varphi'\cos\varphi+\|W\|\sin\varphi}{\sqrt{2\|W\|^2+\varphi'^2}}B. \tag{2.32}$$

Considering the equations (2.31) and (2.32), it is easily seen

$$(\beta_4\wedge T_{\beta_4})(s)=\frac{\varphi'\cos\varphi+2\|W\|\sin\varphi}{\sqrt{4\|W\|^2+2\varphi'^2}}T-\frac{\varphi'}{\sqrt{4\|W\|^2+2\varphi'^2}}N+\frac{2\|W\|\cos\varphi-\varphi'\sin\varphi}{\sqrt{4\|W\|^2+2\varphi'^2}}B. \tag{2.33}$$

Differentiating (2.32), where

$$\begin{cases}\overline{\chi}_1=-2-(\frac{\varphi'}{\|W\|})^2+(\frac{\varphi'}{\|W\|})'(\frac{\varphi'}{\|W\|})\\ \overline{\chi}_2=-2-3(\frac{\varphi'}{\|W\|})^2-(\frac{\varphi'}{\|W\|})^4-(\frac{\varphi'}{\|W\|})'(\frac{\varphi'}{\|W\|})\\ \overline{\chi}_3=2(\frac{\varphi'}{\|W\|})+(\frac{\varphi'}{\|W\|})^3+(\frac{\varphi'}{\|W\|})',\end{cases} \tag{2.34}$$

including we can write

$$\begin{aligned}T'_{\beta_4}(s) &= \frac{\|W\|^4\sqrt{2}(\overline{\chi}_3\sin\varphi-\overline{\chi}_2\cos\varphi)}{\big(2\|W\|^2+\varphi'^2\big)^2}T\\ &\quad+\frac{\overline{\chi}_1\|W\|^4\sqrt{2}}{\big(2\|W\|^2+\varphi'^2\big)^2}N\\ &\quad+\frac{\|W\|^4\sqrt{2}(\overline{\chi}_2\sin\varphi+\overline{\chi}_3\cos\varphi)}{\big(2\|W\|^2+\varphi'^2\big)^2}B.\end{aligned} \tag{2.35}$$

$\kappa_g^{\beta_4}$ geodesic curvature of β_4-Smarandache curve according to Bertrand curve is

$$\kappa_g^{\beta_4} = \frac{1}{(2+(\frac{\varphi'}{\|W\|})^2)^{\frac{5}{2}}}\big((\frac{\varphi'}{\|W\|})\overline{\chi}_1 - (\frac{\varphi'}{\|W\|})\overline{\chi}_2 + 2\overline{\chi}_3\big). \tag{2.36}$$

β_5**-Smarandache Curve**, can be defined by

$$\beta_5(s_{N^*}) = \frac{1}{\sqrt{2}}(T_{N^*} + N^* \wedge T_{N^*}) \tag{2.37}$$

or from the equations (1.6), (1.8) and (2.2), we can write

$$\beta_5(s) = \frac{1}{\sqrt{2}}\Big(\big(\sin\varphi - \cos\varphi\big)T + \big(\sin\varphi + \cos\varphi\big)B\Big). \tag{2.38}$$

Differentiating (2.38), we can write

$$T_{\beta_5}(s) \quad = \quad \frac{\varphi'(\sin\varphi + \cos\varphi)T - \|W\|N + \varphi'(\cos\varphi - \sin\varphi)B}{\sqrt{\|W\|^2 + 2\varphi'^2}}. \tag{2.39}$$

From the equations (2.38) and (2.39), we can write

$$(\beta_5 \wedge T_{\beta_5})(s) = \frac{(\cos\varphi + \sin\varphi)T + 2\varphi' N + (\cos\varphi - \sin\varphi)B}{\sqrt{2\|W\|^2 + 4\varphi'^2}}. \tag{2.40}$$

Differentiating (2.39), where

$$\begin{cases} \overline{\delta}_1 = (\frac{\varphi'}{\|W\|}) + 2(\frac{\varphi'}{\|W\|})^3 + 2(\frac{\varphi'}{\|W\|})'(\frac{\varphi'}{\|W\|}) \\ \overline{\delta}_2 = -1 - 3(\frac{\varphi'}{\|W\|})^2 - 2(\frac{\varphi'}{\|W\|})^4 - (\frac{\varphi'}{\|W\|})' \\ \overline{\delta}_3 = -(\frac{\varphi'}{\|W\|})^2 - 2(\frac{\varphi'}{\|W\|})^4 + (\frac{\varphi'}{\|W\|})', \end{cases} \tag{2.41}$$

including we have,

$$\begin{aligned} T'_{\beta_5}(s) \quad &= \quad \frac{\|W\|^4\sqrt{2}(\overline{\delta}_3\sin\varphi - \overline{\delta}_2\cos\varphi)}{\big(\|W\|^2 + 2\varphi'^2\big)^2}T + \frac{\overline{\delta}_1\|W\|^4\sqrt{2}}{\big(\|W\|^2 + 2\varphi'^2\big)^2}N \\ &\quad + \frac{\|W\|^4\sqrt{2}(\overline{\delta}_2\sin\varphi + \overline{\delta}_3\cos\varphi)}{\big(\|W\|^2 + 2\varphi'^2\big)^2}B. \end{aligned} \tag{2.42}$$

$\kappa_g^{\beta_5}$ geodesic curvature β_5-Smarandache curve according to Bertrand curve is

$$\kappa_g^{\beta_5} = \frac{1}{(2+(\frac{\varphi'}{\|W\|})^2)^{\frac{5}{2}}}\big(2\frac{\varphi'}{\|W\|}\overline{\delta}_1 - \overline{\delta}_2 + \overline{\delta}_3\big). \tag{2.43}$$

β_6**-Smarandache Curve**, can be defined by

$$\beta_6(s_{N^*}) = \frac{1}{\sqrt{3}}(N^* + T_{N^*} + N^* \wedge T_{N^*}) \tag{2.44}$$

or from the equations (1.6), (1.8) and (2.2), we can write

$$\beta_6(s) = \frac{1}{\sqrt{3}}\Big(\big(\sin\varphi - \cos\varphi\big)T + N + \big(\sin\varphi + \cos\varphi\big)B\Big). \tag{2.45}$$

Differentiating (2.45), we have

$$T_{\beta_6}(s) = \frac{\varphi'\sin\varphi - (\|W\|-\varphi')\cos\varphi}{\sqrt{2(\|W\|^2-\|W\|\varphi'+2\varphi'^2)}}T - \frac{\|W\|}{\sqrt{2(\|W\|^2-\|W\|\varphi'+2\varphi'^2)}}N + \frac{\varphi'\cos\varphi + (\|W\|-\varphi')\sin\varphi}{\sqrt{2(\|W\|^2-\|W\|\varphi'+2\varphi'^2)}}B. \tag{2.46}$$

From the equations (2.45) and (2.46), we can write

$$(\beta_6 \wedge T_{\beta_6})(s) = \frac{(2\|W\|-\varphi')\sin\varphi + (\|W\|+\varphi')\cos\varphi}{\sqrt{6\|W\|^2-6\|W\|\varphi'+6\varphi'^2}}T + \frac{2\varphi'-\|W\|}{\sqrt{6\|W\|^2-6\|W\|\varphi'+6\varphi'^2}}N + \frac{(2\|W\|-\varphi')\cos\varphi - (\|W\|+\varphi')\sin\varphi}{\sqrt{6\|W\|^2-6\|W\|\varphi'+6\varphi'^2}}B. \tag{2.47}$$

Differentiating (2.46), where

$$\begin{cases} \overline{\rho}_1 = -2 + 4(\frac{\varphi'}{\|W\|}) - 4(\frac{\varphi'}{\|W\|})^2 + 2(\frac{\varphi'}{\|W\|})^3 + 2(\frac{\varphi'}{\|W\|})'(2(\frac{\varphi'}{\|W\|}) - 1) \\ \overline{\rho}_2 = -2 + 2(\frac{\varphi'}{\|W\|}) - 4(\frac{\varphi'}{\|W\|})^2 + 2(\frac{\varphi'}{\|W\|})^3 - 2(\frac{\varphi'}{\|W\|})^4 - (\frac{\varphi'}{\|W\|})'(1 + (\frac{\varphi'}{\|W\|})) \\ \overline{\rho}_3 = 2(\frac{\varphi'}{\|W\|}) - 4(\frac{\varphi'}{\|W\|})^2 + 4(\frac{\varphi'}{\|W\|})^3 - 2(\frac{\varphi'}{\|W\|})^4 + (\frac{\varphi'}{\|W\|})'(2 - (\frac{\varphi'}{\|W\|})), \end{cases} \tag{2.48}$$

including we can write

$$T'_{\beta_6}(s) = \frac{\|W\|^4\sqrt{3}(\overline{\rho}_3\sin\varphi - \overline{\rho}_2\cos\varphi)}{4\big(\|W\|^2-\|W\|\varphi'+\varphi'^2\big)^2}T + \frac{\overline{\rho}_1\|W\|^4\sqrt{3}}{4\big(\|W\|^2-\|W\|\varphi'+\varphi'^2\big)^2}N + \frac{\|W\|^4\sqrt{3}(\overline{\rho}_2\sin\varphi + \overline{\rho}_3\cos\varphi)}{4\big(\|W\|^2-\|W\|\varphi'+\varphi'^2\big)^2}B. \tag{2.49}$$

$\kappa_g^{\beta_6}$ geodesic curvature β_6-Smarandache curve according to Bertrand curve is

$$\kappa_g^{\beta_6} = \frac{(2\frac{\varphi'}{\|W\|}-1)\overline{\rho}_1 + (-1-\frac{\varphi'}{\|W\|})\overline{\rho}_2 + (2-\frac{\varphi'}{\|W\|})\overline{\rho}_3}{4\sqrt{2}(1-(\frac{\varphi'}{\|W\|})+(\frac{\varphi'}{\|W\|})^2)^{\frac{5}{2}}}. \tag{2.50}$$

β_7**-Smarandache Curve**, can be defined by

$$\beta_7(s_{B^*}) = \frac{1}{\sqrt{2}}(B^* + T_{B^*}) \tag{2.51}$$

or from the equations (1.6) and (2.3), we can write

$$\beta_7(s) = \frac{1}{\sqrt{2}}\big(\sin\theta T + N + \cos\theta B\big). \tag{2.52}$$

Differentiating (2.52), we reach

$$T_{\beta_7}(s) = \frac{\cot(\varphi-\theta)\cos\theta - \sin\theta}{\sqrt{2+\cot^2(\varphi-\theta)}}T - \frac{1}{\sqrt{2+\cot^2(\varphi-\theta)}}N - \frac{\cot(\varphi-\theta)\sin\theta - \cos\theta}{\sqrt{2+\cot^2(\varphi-\theta)}}B. \tag{2.53}$$

From the equations (2.52) and (2.53), we have

$$(\beta_7 \wedge T_{\beta_7})(s) = \frac{\cot(\varphi-\theta)\sin\theta + 2\cos\theta}{\sqrt{4+2\cot^2(\varphi-\theta)}}T + \frac{\cot(\varphi-\theta)}{\sqrt{4+2\cot^2(\varphi-\theta)}}N + \frac{\cot(\varphi-\theta)\cos\theta - 2\sin\theta}{\sqrt{4+2\cot^2(\varphi-\theta)}}B. \quad (2.54)$$

Differentiating (2.53), where

$$\begin{cases} \overline{\omega}_1 = -2-\cot^2(\varphi-\theta)+\cot'(\varphi-\theta)\cot(\varphi-\theta) \\ \overline{\omega}_2 = -2-3\cot^2(\varphi-\theta)-\cot^4(\varphi-\theta)-\cot'(\varphi-\theta)\cot(\varphi-\theta) \\ \overline{\omega}_3 = 2\cot(\varphi-\theta)+\cot^3(\varphi-\theta)+2\cot'(\varphi-\theta), \end{cases} \quad (2.55)$$

including we can write

$$T'_{\beta_7}(s) = \frac{(\overline{\omega}_1\sin\theta+\overline{\omega}_3\cos\theta)\sqrt{2}}{(2+\cot^2(\varphi-\theta))^2}T - \frac{\overline{\omega}_2\sqrt{2}}{(2+\cot^2(\varphi-\theta))^2}N + \frac{(\overline{\omega}_1\cos\theta-\overline{\omega}_3\sin\theta)\sqrt{2}}{(2+\cot^2(\varphi-\theta))^2}B. \quad (2.56)$$

$\kappa_g^{\beta_7}$ geodesic curvature β_7-Smarandache curve according to Bertrand curve is

$$\kappa_g^{\beta_7} = \frac{1}{\big(2+\cot^2(\varphi-\theta)\big)^{\frac{5}{2}}}\Big(\cot(\varphi-\theta)\overline{\omega}_1 - \cot(\varphi-\theta)\overline{\omega}_2 + 2\overline{\omega}_3\Big). \quad (2.57)$$

β_8**-Smarandache Curve**, can be defined by

$$\beta_8(s_{B^*}) = \frac{1}{\sqrt{2}}(T_{B^*} + B^* \wedge T_{B^*}) \quad (2.58)$$

or from the equations (1.6) and (2.3), we can write

$$\beta_8(s) = \frac{1}{\sqrt{2}}\big(\cos\theta T - N - \sin\theta B\big). \quad (2.59)$$

Differentiating (2.59), we reach

$$T_{\beta_8}(s) = \frac{\cot(\varphi-\theta)\cos\theta - \sin\theta}{\sqrt{1+2\cot^2(\varphi-\theta)}}T + \frac{\cot(\varphi-\theta)}{\sqrt{1+2\cot^2(\varphi-\theta)}}N - \frac{\cot(\varphi-\theta)\sin\theta+\cos\theta}{\sqrt{1+2\cot^2(\varphi-\theta)}}B. \quad (2.60)$$

From the equations (2.59) and (2.60), we can write

$$(\beta_8 \wedge T_{\beta_8})(s) = \frac{2\cot(\varphi-\theta)\sin\theta + \cos\theta}{\sqrt{2+4\cot^2(\varphi-\theta)}}T + \frac{1}{\sqrt{2+4\cot^2(\varphi-\theta)}}N + \frac{2\cot(\varphi-\theta)\cos\theta - \sin\theta}{\sqrt{2+4\cot^2(\varphi-\theta)}}B. \quad (2.61)$$

Differentiating (2.60), where

$$\begin{cases} \overline{\psi}_1 = \cot(\varphi-\theta)+2\cot^3(\varphi-\theta)+2\cot'(\varphi-\theta)\cot(\varphi-\theta) \\ \overline{\psi}_2 = -1-3\cot^2(\varphi-\theta)-2\cot^4(\varphi-\theta)-\cot'(\varphi-\theta) \\ \overline{\psi}_3 = -\cot^2(\varphi-\theta)-2\cot^4(\varphi-\theta)+\cot'(\varphi-\theta), \end{cases} \quad (2.62)$$

including we have

$$T'_{\beta_8}(s) = \frac{(\overline{\psi}_1 \sin\theta + \overline{\psi}_3 \cos\theta)\sqrt{2}}{(1+2\cot^2(\varphi-\theta))^2}T + \frac{\overline{\psi}_2\sqrt{2}}{(1+2\cot^2(\varphi-\theta))^2}N + \frac{(\overline{\psi}_1 \cos\theta - \overline{\psi}_3 \sin\theta)\sqrt{2}}{(1+2\cot^2(\varphi-\theta))^2}B. \quad (2.63)$$

$\kappa_g^{\beta_8}$ geodesic curvature β_8-Smarandache curve according to Bertrand curve is

$$\kappa_g^{\beta_8} = \frac{1}{(1+2\cot^2(\varphi-\theta))^{\frac{5}{2}}}\big(2\cot(\varphi-\theta)\overline{\psi}_1 - \overline{\psi}_2 + \overline{\psi}_3\big). \quad (2.64)$$

β_9**-Smarandache Curve**, can be defined by

$$\beta_9(s_{B^*}) = \frac{1}{\sqrt{3}}(B^* + T_{B^*} + B^* \wedge T_{B^*}) \quad (2.65)$$

or from the equations (1.6) and (2.3), we can write

$$\beta_9(s) = \frac{1}{\sqrt{3}}\Big(\big(\sin\theta + \cos\theta\big)T - N + \big(\cos\theta - \sin\theta\big)B\Big). \quad (2.66)$$

Differentiating (2.66), we have

$$\begin{aligned} T_{\beta_9}(s) \quad = \quad & \frac{\cot(\varphi-\theta)\cos\theta - \sin\theta}{\sqrt{2\big(1-\cot(\varphi-\theta)+\cot^2(\varphi-\theta)\big)}}T - \frac{1-\cot(\varphi-\theta)}{\sqrt{2\big(1-\cot(\varphi-\theta)+\cot^2(\varphi-\theta)\big)}}N \\ & - \frac{\cos\theta + \cot(\varphi-\theta)\sin\theta}{\sqrt{2\big(1-\cot(\varphi-\theta)+\cot^2(\varphi-\theta)\big)}}B. \end{aligned} \quad (2.67)$$

From the equations (2.66) and (2.67), we reach

$$\begin{aligned} (\beta_9 \wedge T_{\beta_9})(s) \quad = \quad & \frac{(2\cot(\varphi-\theta)-1)\sin\theta + (2-\cot(\varphi-\theta))\cos\theta}{\sqrt{6-6\cot(\varphi-\theta)+6\cot^2(\varphi-\theta)}}T \\ & + \frac{1+\cot(\varphi-\theta)}{\sqrt{6-6\cot(\varphi-\theta)+6\cot^2(\varphi-\theta)}}N \\ & + \frac{(2\cot(\varphi-\theta)-1)\cos\theta + (\cot(\varphi-\theta)-2)\sin\theta}{\sqrt{6-6\cot(\varphi-\theta)+6\cot^2(\varphi-\theta)}}B. \end{aligned} \quad (2.68)$$

Differentiating (2.67), where

$$\begin{cases} \overline{\zeta}_1 = -2 + 4(\frac{\|W\|}{\varphi'}) + 4(\frac{\|W\|}{\varphi'}) - (\frac{\|W\|}{\varphi'})^2 + 2(\frac{\|W\|}{\varphi'})^3 + (\frac{\|W\|}{\varphi'})'(2\frac{\|W\|}{\varphi'} - 1) \\ \overline{\zeta}_2 = -2 + 2(\frac{\|W\|}{\varphi'}) - 4(\frac{\|W\|}{\varphi'})^2 + (\frac{\|W\|}{\varphi'})^3 - 2(\frac{\|W\|}{\varphi'})^4 - (\frac{\|W\|}{\varphi'})'(1 + \frac{\|W\|}{\varphi'}) \\ \overline{\zeta}_3 = 2(\frac{\|W\|}{\varphi'}) - 4(\frac{\|W\|}{\varphi'})^2 + 4(\frac{\|W\|}{\varphi'})^3 - 2(\frac{\|W\|}{\varphi'})^4 + (\frac{\|W\|}{\varphi'})'(2 - \frac{\|W\|}{\varphi'}), \end{cases} \quad (2.69)$$

including we can write

$$\begin{aligned} T'_{\beta_9}(s) \quad = \quad & \frac{\varphi'^4(\overline{\zeta}_1 \sin\varphi + \overline{\zeta}_2 \cos\varphi)\sqrt{3}}{4(\|W\|^2 - \|W\|\varphi' + \varphi'^2)^2}T + \frac{\varphi'^4\overline{\zeta}_3\sqrt{3}}{4(\|W\|^2 - \|W\|\varphi' + \varphi'^2)^2}N \\ & + \frac{\varphi'^4(\overline{\zeta}_1 \cos\varphi - \overline{\zeta}_2 \sin\varphi)\sqrt{3}}{4(\|W\|^2 - \|W\|\varphi' + \varphi'^2)^2}B. \end{aligned} \quad (2.70)$$

$\kappa_g^{\beta_9}$ geodesic curvature β_9-Smarandache curve according to Bertrand curve is

$$\kappa_g^{\beta_9} = \frac{\left(2\cot(\varphi-\theta)-1\right)\overline{\zeta}_1 - \left(1+\cot(\varphi-\theta)\right)\overline{\zeta}_2 + \left(2-\cot(\varphi-\theta)\right)\overline{\zeta}_3}{4\sqrt{2}\left(1+\cot(\varphi-\theta)+\left(\cot(\varphi-\theta)\right)^2\right)^{\frac{5}{2}}}. \tag{2.71}$$

Example. Let us consider the unit speed spherical curve:

$$\alpha(s) = \{\frac{2}{5}\sin(2\,s) - \frac{1}{40}\sin(8\,s)\,, -\frac{2}{5}\cos(2\,s) + \frac{1}{40}\cos(8\,s)\,, \frac{4}{15}\sin(3\,s)\}$$

in the context of definitions, we reach Spherical indicatrix curves (T^*), (N^*) and (B^*) (see Figure 1) and Smarandache curves according to Sabban frame on S^2. β_1, β_2, β_3, β_4, β_5, β_6, β_7, β_8 and β_9 (see Figure 2,3,4).

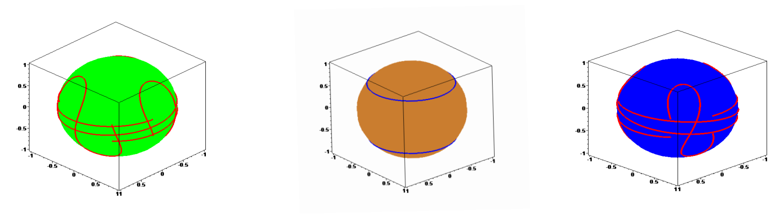

Figure 1: (T^*)-curve (N^*)-curve (B^*)-curve

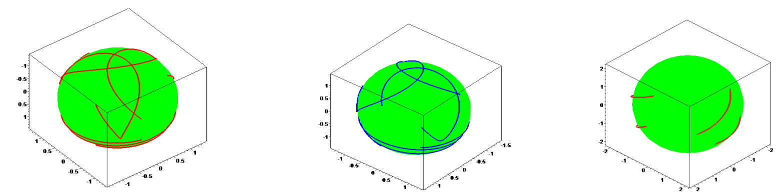

Figure 2: β_1-curve β_2-curve β_3-curve

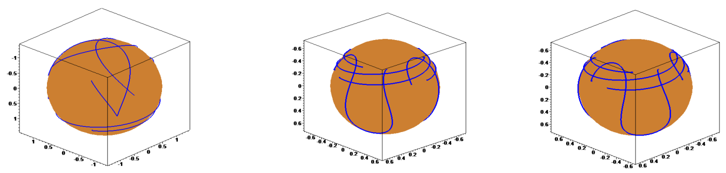

Figure 3: β_4-curve β_5-curve β_6-curve

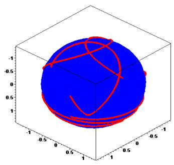
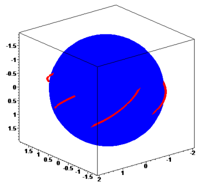
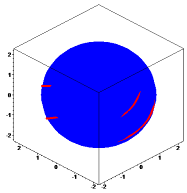

Figure 4: β_7-curve β_8-curve β_9-curve

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