.

NEW FORMS OF μ-COMPACTNESS WITH RESPECT TO HEREDITARY CLASSES

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Abstract. A hereditary class on a set X is a nonempty collection of subsets closed under heredity. The aim of this paper is to introduce and study strong forms of μ-compactness in generalized topological spaces with respect to a hereditary class, called $\mathcal{S}\mu\mathcal{H}$-compactness and $\mathbf{S}-\mathcal{S}\mu\mathcal{H}$-compactness. Also several of their properties are presented. Finally some effects of various kinds of functions on them are studied.

1. Introduction

This work is developed around the concept of μ-compactness with respect a hereditary class which was introduced by Carpintero, Rosas, Salas-Brown and Sanabria in [4]. In this research, we use the notions of generalized topology and hereditary class introduced by Császár in [1] and [2], respectively, in order to define and characterize the $\mathcal{S}\mu\mathcal{H}$-compactness and $\mathbf{S}-\mathcal{S}\mu\mathcal{H}$-compactness spaces. Also some properties of these spaces are obtained and the behavior of these spaces under certain kinds of functions also is investigated. The strategy of using generalized topologies and hereditary classes to extend classical topological concepts have been used by many authors such as [2], [6], [9], [14], among others.

2. Preliminaries

Let X be a non-empty set and 2^X denote the power set of X. We call a class $\mu \subseteq 2^X$ a generalized topology [1] (briefly, GT) if $\phi \in \mu$ and arbitrary union of elements of μ belongs to μ. A set X with a GT is called a generalized topological space (briefly, GTS) and is denoted by (X, μ). For a GTS (X, μ), the elements of μ are called μ-open sets and the complement of μ-open sets are called μ-closed sets. For $A \subseteq X$, we denote by $c_\mu(A)$ the intersection of all μ-closed sets containing A, i.e., the smallest μ-closed set containing A and by $i_\mu(A)$ the union of all μ-open sets contained in A, i.e., the largest μ-open set contained in A (see [1], [3]). Let $A \subset X$. A family $\mathcal{C}$ of subsets of X is called a μ-covering of A if $\mathcal{C}$ is a covering of A by μ-open sets [5]. A subset A of X is said to be μ-compact if for every μ-covering $\{V_\alpha : \alpha \in \Lambda\}$ of A there exists a finite subfamily $\{V_\alpha : \alpha \in \Lambda_0\}$ that also covers A. X is said to be μ-compact if X is μ-compact as a subset [5].

A nonempty family $\mathcal{H}$ of subsets of X is called a hereditary class [2] if $A \in \mathcal{H}$ and $B \subset A$ imply that $B \in \mathcal{H}$. Given a generalized topological space (X, μ) with a hereditary class $\mathcal{H}$, for a subset A of X, the generalized local function of A with respect to $\mathcal{H}$ and μ [2] is defined as follows: $A^* = \{x \in X : U \cap A \notin \mathcal{H} \quad \text{for all} \quad U \in \mu_x\}$, where $\mu_x = \{U : x \in U \quad \text{and} \quad U \in \mu\}$. And for A a subset of X, is defined:

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$c^*_\mu(A) = A \cup A^*$. The family $\mu^* = \{A \subset X : X \setminus A = c^\star_\mu(X \setminus A)\}$ is a GT on X. The elements of μ^* are called μ^*-open and the complement of a μ^*-open set is called μ^*-closed set. It is clear that a subset A is μ^*-closed if and only if $A^* \subset A$. If the hereditary class $\mathcal{H}$ satisfies the additional condition: if $A, B \in \mathcal{H}$ implies $A \cup B \in \mathcal{H}$, then $\mathcal{H}$ is called an ideal on X [7]. We call $(X, \mu, \mathcal{H})$ a hereditary generalized topological space and briefly we denote it by HGTS. If $(X, \mu, \mathcal{H})$ is a HGTS, the set $\mathcal{B} = \{V \setminus H : V \in \mu \quad \text{and} \quad H \in \mathcal{H}\}$ is a base for a GT μ^*, finer than μ [2]. If there is no confusion, we simply write A^* instead of $A^*(\mathcal{H}, \mu)$.

Definition 2.1. [1] *Let (X, μ) and (Y, ν) be two GTSs, then a function $f : (X, \mu) \to (Y, \nu)$ is said to be (μ, ν)-continuous if $U \in \nu$ implies $f^{-1}(U) \in \mu$.*

Definition 2.2. [13] *A function $f : (X, \mu) \to (Y, \nu)$ is (μ, ν)-open (or μ-open) if $U \in \mu$ implies $f(U) \in \nu$.*

Definition 2.3. *Let (X, μ) be a GTS. Then a subset A of X is called a μ-generalized closed set (in short, μg-closed set)*[10] *if $c_\mu(A) \subseteq U$ whenever $A \subseteq U$ where U is μ-open in X. The complement of a μg-closed set is called a μg-open set.*

Theorem 2.4. [2] *Let (X, μ) be a GTS and $\mathcal{H}$ be a hereditary class on X and A a subset of X, then $A^* \subset c_\mu(A)$.*

Theorem 2.5. [2] *Let (X, μ) be a GTS, $\mathcal{H}$ a hereditary class on X and A be a subset of X. If A is μ^*-open, then for each $x \in A$ there exist $U \in \mu_x$ and $H \in \mathcal{H}$ such that $x \in U \setminus H \subset A$.*

3. $\mathcal{S}\mu\mathcal{H}$-Compactness Spaces

We recall that a subset A of a HGTS $(X, \mu, \mathcal{H})$ is said to be $\mu\mathcal{H}$-compact [4], if for every μ-open cover $\{V_\alpha : \alpha \in \Lambda\}$ of A by elements of μ, there exists a finite subset Λ_0 of Λ such that $A \setminus \bigcup_{\alpha \in \Lambda_0} V_\alpha \in \mathcal{H}$. The HGTS $(X, \mu, \mathcal{H})$ is said to be $\mu\mathcal{H}$-compact if X is $\mu\mathcal{H}$- compact as a subset.

Definition 3.1. *Let (X, μ) be a GTS and $\mathcal{H}$ be a hereditary class on X. A subset A of X is said to be strong $\mu\mathcal{H}$-compact (briefly $\mathcal{S}\mu\mathcal{H}$-compact) if for every family $\{V_\alpha : \alpha \in \Lambda\}$ of μ-open subsets of X with $A \setminus \bigcup_{\alpha \in \Lambda} V_\alpha \in \mathcal{H}$ then there exists a finite subset Λ_0 of Λ such that $A \setminus \bigcup_{\alpha \in \Lambda_0} V_\alpha \in \mathcal{H}$. The HGTS $(X, \mu, \mathcal{H})$ is said to be strong $\mu\mathcal{H}$-compact (briefly $\mathcal{S}\mu\mathcal{H}$-compact) if X is $\mathcal{S}\mu\mathcal{H}$-compact as a subset.*

Remark 3.2.
1. *It is clear that (X, μ) is μ-compact if and only if $(X, \mu, \{\phi\})$ is $\mathcal{S}\mu\{\phi\}$-compact.*
2. *If $(X, \mu, \mathcal{H})$ is $\mathcal{S}\mu\mathcal{H}$-compact then $(X, \mu, \mathcal{H})$ is $\mu\mathcal{H}$-compact. The converse is not true as shown by the following example.*

Example 3.3. *Let $X = [-5, 5]$, $\mu = \{\phi,\ X\} \cup \left\{\left(\frac{1}{r} - 5,\ 5\right) :\ r \in \mathbb{Z}^+\right\}$ and $\mathcal{H} = \{A : A \subset [-5, 5] \cap \mathbb{Z}\}$, then:*
1. *$(X, \mu, \mathcal{H})$ is $\mu\mathcal{H}$-compact, because if $\{V_\alpha : \alpha \in \Lambda\}$ is an μ-covering of X, then there exists $\alpha_0 \in \Lambda$ with $V_{\alpha_0} = X$, and so $X \setminus V_{\alpha_0} = \phi \in \mathcal{H}$.*
2. *$(X, \mu, \mathcal{H})$ is not $\mathcal{S}\mu\mathcal{H}$-compact, because $X \setminus \bigcup_{r=1}^{\infty} \left(\frac{1}{r} - 5,\ 5\right) = \{-5, 5\} \in \mathcal{H}$, but if k is a positive integer there exist a finite set $n_1, ..., n_k$. If we take*

$N = max\{n_1, ..., n_k\}$ then

$$X \setminus \bigcup_{i=1}^{k} \left(\frac{1}{r_i} - 5,\ 5\right) = X \setminus \left(\frac{1}{N} - 5,\ 5\right) \notin \mathcal{H}.$$

Definition 3.4. *A subset A of a HGTS $(X, \mu, \mathcal{H})$ is said to be $\mu\mathcal{H}_g$-closed if for every $U \in \mu$ with $A \setminus U \in \mathcal{H}$ then $c_\mu(A) \subseteq U$.*

Remark 3.5. *It is clear that A is $\mu\{\phi\}_g$-closed if and only if A is μg-closed. We note that if A is $\mu\mathcal{H}_g$-closed then A is μg-closed. The converse is not true as shown by the following examples.*

Example 3.6. *Let $X = \mathbb{R}$ and $\mu = \{\phi, \mathbb{R}\} \cup \{(r, +\infty) : r \in \mathbb{R}\}$. The hereditary class on $\mathbb{R}$,*

$$\mathcal{H} = \{B : B \subseteq \mathbb{Q} \cap (0, +\infty) \quad or \quad B \subseteq \mathbb{Q} \cap (-\infty, 0]\}.$$

If $A = \mathbb{Q}$, then:

1. *(1) A is μg-closed because if $U \in \mu$ and $A \subseteq U$, then $U = \mathbb{R}$ and so $c_\mu(A) = \mathbb{R} \subseteq U$;*
2. *(2) A is not $\mu\mathcal{H}_g$-closed since $A \setminus (0, +\infty) \in \mathcal{H}$, but $c_\mu(A) = \mathbb{R} \not\subseteq (0, +\infty)$.*

Example 3.7. *If $X = \{a, b, c, d\}$, $\mu = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$, $\mathcal{H} = \{\phi, \{a\}, \{b\}, \{a, b\}\}$ and $A = \{c\}$, then A is $\mu\mathcal{H}g$-closed because if $U \in \mu$ and $A \setminus U \in \mathcal{H}$, we have that $A \subseteq U$, and so $U = X$ and $c_\mu(A) \subseteq U$.*

Proposition 3.8. *Let $(X, \mu, \mathcal{H})$ be a HGTS and $\mathcal{B}$ be a base for μ. Then the following are equivalent:*

1. *(1) $(X, \mu, \mathcal{H})$ is $\mathcal{S}\mu\mathcal{H}$-compact;*
2. *(2) for any family $\{V_\alpha : \alpha \in \Lambda\}$ of μ-open sets in $\mathcal{B}$, if $X \setminus \bigcup_{\alpha \in \Lambda} V_\alpha \in \mathcal{H}$ then there exists $\Lambda_0 \subseteq \Lambda$, finite, with $X \setminus \bigcup_{\alpha \in \Lambda_0} V_\alpha \in \mathcal{H}$.*

Proof. (1) $\Rightarrow$ (2): Let $\{V_\alpha : \alpha \in \Lambda\}$ be a family of non-empty μ-open subsets of X such that $X \setminus \bigcup_{\alpha \in \Lambda} V_\alpha \in \mathcal{H}$. For each $\alpha \in \Lambda$ there exists a family $\{B_{\alpha\beta} : \beta \in \Lambda_\alpha\} \subseteq \mathcal{B}$ such that $V_\alpha = \bigcup_{\beta \in \Lambda_\alpha} B_{\alpha\beta}$. Given that $X \setminus \bigcup_{\alpha \in \Lambda} V_\alpha = X \setminus \bigcup_{\alpha \in \Lambda} \left(\bigcup_{\beta \in \Lambda_\alpha} B_{\alpha\beta} \right) \in \mathcal{H}$ and $(X, \mu, \mathcal{H})$ is $\mathcal{S}\mu\mathcal{H}$-compact there exist $B_{\alpha_1\beta_1}, B_{\alpha_2\beta_2}, ..., B_{\alpha_k\beta_k}$ such that $X \setminus \bigcup_{i=1}^{k} B_{\alpha_i\beta_i} \in \mathcal{H}$. But $X \setminus \bigcup_{i=1}^{k} V_k \subseteq X \setminus \bigcup_{i=1}^{k} B_{\alpha_i\beta_i}$ and so $X \setminus \bigcup_{i=1}^{k} V_k \in \mathcal{H}$.
(2) $\Rightarrow$ (1): It is obvious. □

Theorem 3.9. *If $(X, \mu, \mathcal{H})$ is a HGTS then the following are equivalent:*

1. *(1) $(X, \mu, \mathcal{H})$ is $\mathcal{S}\mu\mathcal{H}$-compact;*
2. *(2) For any family $\{F_\alpha : \alpha \in \Lambda\}$ of μ-closed subsets of X such that $\cap\{F_\alpha : \alpha \in \Lambda\} \in \mathcal{H}$, there exists a finite subset Λ_0 of Λ such that $\cap\{F_\alpha : \alpha \in \Lambda_0\} \in \mathcal{H}$.*

Proof. (1) $\Rightarrow$ (2): Let $\{F_\alpha : \alpha \in \Lambda\}$ be a family of μ-closed subsets of X such that $\cap\{F_\alpha : \alpha \in \Lambda\} \in \mathcal{H}$. Then $\{X \setminus F_\alpha : \alpha \in \Lambda\}$ is a family of μ-open subsets of X. Let $\cap\{F_\alpha : \alpha \in \Lambda\} = H \in \mathcal{H}$. Then $X \setminus \cap\{F_\alpha : \alpha \in \Lambda\} = \cup\{X \setminus F_\alpha : \alpha \in \Lambda\} = X \setminus H$. By (1) since $(X, \mu, \mathcal{H})$ is $\mathcal{S}\mu\mathcal{H}$-compact, $X \setminus \cup\{X \setminus F_\alpha : \alpha \in \Lambda\} \in \mathcal{H}$ and there exists a finite subset Λ_0 of Λ, such that $X \setminus \cup\{X \setminus F_\alpha : \alpha \in \Lambda_0\} \in \mathcal{H}$. This implies

that $\cap\{F_\alpha : \alpha \in \Lambda_0\} \in \mathcal{H}$.
(2) $\Rightarrow$ (1): Let $\{V_\alpha : \alpha \in \Lambda\}$ be any family of μ-open subsets of X. Then $\{X \setminus V_\alpha : \alpha \in \Lambda\}$ is a family of μ-closed subsets of X. By (2) we have $\cap\{X \setminus V_\alpha : \alpha \in \Lambda\} \in \mathcal{H}$. Thus $X \setminus \cup\{V_\alpha : \alpha \in \Lambda\} \in \mathcal{H}$. Since, $\cap\{X \setminus V_\alpha : \alpha \in \Lambda_0\} \in \mathcal{H}$, then $X \setminus \cup\{V_\alpha : \alpha \in \Lambda_0\} \in \mathcal{H}$. This shows that $(X, \mu, \mathcal{H})$ is $\mathcal{S}\mu\mathcal{H}$-compact. □

Proposition 3.10. *If $(X, \mu, \mathcal{H})$ is a HGTS and $\mathcal{H}$ is an ideal, then the following are equivalent:*

(1) $(X, \mu, \mathcal{H})$ is $\mathcal{S}\mu\mathcal{H}$-compact;
(2) $(X, \mu^*, \mathcal{H})$ is $\mathcal{S}\mu\mathcal{H}$-compact.

Proof. (1) $\Rightarrow$ (2): The set $\mathcal{B} = \{U \setminus H : U \in \mu \quad \text{and} \quad H \in \mathcal{H}\}$ is a base for μ^*. Let $\{V_\alpha : \alpha \in \Lambda\}$ be a family of μ^*-open subsets of X. For some $x \in X$, there exists $\alpha_x \in \Lambda$ such that $x \in V_{\alpha_x}$. Then there exist $U_{\alpha_x} \in \mu_x$ and $H_{\alpha_x} \in \mathcal{H}$ such that $x \in U_{\alpha_x} \setminus H_{\alpha_x} \subset V_{\alpha_x}$. Now $\{U_{\alpha_x} : \alpha_x \in \Lambda\}$ is a family of μ-open subsets of X. Since $X \setminus \bigcup\limits_{\alpha_x \in \Lambda} U_{\alpha_x} \in \mathcal{H}$ then there exists a finite subset Λ_0 of Λ such that $X \setminus \bigcup\limits_{\alpha_x \in \Lambda_0} U_{\alpha_x} \in \mathcal{H}$. Hence, $H \cup \bigcup\limits_{\alpha_x \in \Lambda_0} H_{\alpha_x} \in \mathcal{H}$.
Observe that $X \setminus \bigcup\limits_{\alpha_x \in \Lambda_0} V_{\alpha_x} \subseteq H \cup \bigcup\limits_{\alpha_x \in \Lambda_0} H_{\alpha_x} \in \mathcal{H}$. By the heredity property of the class $\mathcal{H}$ we have $X \setminus \bigcup\limits_{\alpha_x \in \Lambda_0} V_{\alpha_x} \in \mathcal{H}$ and therefore $(X, \mu^*, \mathcal{H})$ is $\mathcal{S}\mu\mathcal{H}$-compact.
(2) $\Rightarrow$ (1): It is obvious. □

Next we study the behavior of some types of subspaces of a $\mathcal{S}\mu\mathcal{H}$-compact space relative to X.

Theorem 3.11. *If $(X, \mu, \mathcal{H})$ is $\mathcal{S}\mu\mathcal{H}$-compact and $A \subseteq X$ is $\mu\mathcal{H}_g$-closed, then A is $\mathcal{S}\mu\mathcal{H}$-compact.*

Proof. Let $\{V_\alpha : \alpha \in \Lambda\}$ be a family of μ-open subsets of X such that $A \setminus \bigcup\limits_{\alpha \in \Lambda} V_\alpha \in \mathcal{H}$. Since A is is $\mu\mathcal{H}_g$-closed, $c_\mu(A) \subseteq \bigcup\limits_{\alpha \in \Lambda} V_\alpha$. Then $(X \setminus c_\mu(A)) \cup \bigcup\limits_{\alpha \in \Lambda} V_\alpha$ is a μ-covering of X and so $X \setminus \left[X \setminus c_\mu(A) \cup (\bigcup\limits_{\alpha \in \Lambda} V_\alpha)\right] = \phi \in \mathcal{H}$. Given that X is $\mathcal{S}\mu\mathcal{H}$-compact, there exists a finite subset Λ_0 of Λ, such that $X \setminus \left[X \setminus c_\mu(A) \cup (\bigcup\limits_{\alpha \in \Lambda_0} V_\alpha)\right] \in \mathcal{H}$. Since,

$$X \setminus \bigcup_{\alpha \in \Lambda_0} V_\alpha \subseteq X \setminus \left[X \setminus c_\mu(A) \cup (\bigcup_{\alpha \in \Lambda_0} V_\alpha)\right], \quad \text{then} \quad X \setminus \bigcup_{\alpha \in \Lambda_0} V_\alpha \in \mathcal{H}.$$

In any case $X \setminus \left[X \setminus c_\mu(A) \cup (\bigcup\limits_{\alpha \in \Lambda_0} V_\alpha)\right] \in \mathcal{H}$. But $X \setminus \left[X \setminus c_\mu(A) \cup (\bigcup\limits_{\alpha \in \Lambda} V_\alpha)\right] = c_\mu(A) \cap (X \setminus \bigcup\limits_{\alpha \in \Lambda} V_\alpha)$ and since $A \setminus \bigcup\limits_{\alpha \in \Lambda_0} V_\alpha \subseteq c_\mu(A) \setminus \bigcup\limits_{\alpha \in \Lambda_0} V_\alpha$ we have that $A \setminus \bigcup\limits_{\alpha \in \Lambda_0} V_\alpha \in \mathcal{H}$. Thus A is $\mathcal{S}\mu\mathcal{H}$-compact. □

Theorem 3.12. *If A and B are $\mathcal{S}\mu\mathcal{H}$-compact subsets of a HGTS $(X, \mu, \mathcal{H})$, and $\mathcal{H}$ is an ideal then $A \cup B$ is $\mathcal{S}\mu\mathcal{H}$-compact.*

Proof. Let $\{V_\alpha : \alpha \in \Lambda\}$ be a family of μ-open subsets of X such that $A \cup B \setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{H}$. Since, $A \setminus \bigcup_{\alpha\in\Lambda} V_\alpha \subseteq A \cup B \setminus \bigcup_{\alpha\in\Lambda} V_\alpha$ and $B \setminus \bigcup_{\alpha\in\Lambda} V_\alpha \subseteq A \cup B \setminus \bigcup_{\alpha\in\Lambda} V_\alpha$ then $A \setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{H}$ and $B \setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{H}$. Since A and B are $\mathcal{S}\mu\mathcal{H}$-compact, then there exists finite subsets Λ_0 and Λ_1 of Λ with $A \setminus \bigcup_{\alpha\in\Lambda_0} V_\alpha \in \mathcal{H}$ and $B \setminus \bigcup_{\alpha\in\Lambda_1} V_\alpha \in \mathcal{H}$. This implies that $A \setminus \bigcup_{\alpha\in\Lambda_0\cup\Lambda_1} V_\alpha \in \mathcal{H}$ and $B \setminus \bigcup_{\alpha\in\Lambda_0\cup\Lambda_1} V_\alpha \in \mathcal{H}$ and since $\mathcal{H}$ is an ideal we have that $\left(A \setminus \bigcup_{\alpha\in\Lambda_0\cup\Lambda_1} V_\alpha\right) \cup \left(B \setminus \bigcup_{\alpha\in\Lambda_0\cup\Lambda_1} V_\alpha\right) \in \mathcal{H}$. Thus $A \cup B \setminus \bigcup_{\alpha\in\Lambda_0\cup\Lambda_1} V_\alpha \in \mathcal{H}$. So $A \cup B$ is $\mathcal{S}\mu\mathcal{H}$-compact. □

The following example shows that the previous theorem does not hold when $\mathcal{H}$ is just a hereditary class, not an ideal.

Example 3.13. *Let $\mathbb{R}$ be the set of real numbers, μ the usual topology, $\mathcal{H} = \{A \subset \mathbb{R} : A \subset (1,2) \quad or \quad A \subset (2,3)\}$ and if $A = (1,2)$ and $B = (2,3)$, then:*
(1) It is clear that $A = (1,2)$ and $B = (2,3)$ are $\mathcal{S}\mu\mathcal{H}$-compact subsets.
(2) $A \cup B$ is not $\mathcal{S}\mu\mathcal{H}$-compact if $\{(1 + \frac{1}{n}, 3 - \frac{1}{n}) : n \in \mathbb{Z}^+\}$ is a family of μ-open subsets of X, $A \cup B \setminus \bigcup_{n=1}^{\infty} \left(1 + \frac{1}{n},\ \ 3 - \frac{1}{n}\right) = A \cup B \setminus (1,3) = \phi \in \mathcal{H}$, but if we choose a finite set $n_1, ..., n_k$ and take $N = max\{n_1, ..., n_k\}$, follows that $A \cup B \setminus \bigcup_{i=1}^{k} \left(1 + \frac{1}{n_i},\ \ 3 - \frac{1}{n_i}\right) = A \cup B \setminus \left(1 + \frac{1}{N},\ \ 3 - \frac{1}{N}\right) = \left(1,\ \ 1 + \frac{1}{N}\right] \cup \left[3 - \frac{1}{N},\ \ 3\right) \notin \mathcal{H}$.

Theorem 3.14. *Let $(X, \mu, \mathcal{H})$ be a HGTS and $A \subseteq X$. If $A \setminus U \in \mathcal{H}$ for every $U \in \mu$ then there exists $B \subseteq X$ such that B is $\mathcal{S}\mu\mathcal{H}$-compact, $A \subseteq B$ and $B \setminus U \in H$. Then A is $\mathcal{S}\mu\mathcal{H}$-compact.*

Proof. Let $\{V_\alpha : \alpha \in \Lambda\}$ be a family of μ-open subsets of X such that $A \setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{H}$. There exists $B \subseteq X$ such that B is $\mathcal{S}\mu\mathcal{H}$-compact, $A \subseteq B$ and $B \setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{H}$. There exists a finite subset Λ_0 of Λ with $B \setminus \bigcup_{\alpha\in\Lambda_0} V_\alpha \in \mathcal{H}$. Since, $A \setminus \bigcup_{\alpha\in\Lambda_0} V_\alpha \subseteq B \setminus \bigcup_{\alpha\in\Lambda_0} V_\alpha$ we have that $A \setminus \bigcup_{\alpha\in\Lambda_0} V_\alpha \in \mathcal{H}$. □

Theorem 3.15. *If $(X, \mu, \mathcal{H})$ is a HGTS, $A \subseteq B \subseteq X$, $B \subseteq c_\mu(A)$ and A is $\mu\mathcal{H}_g$-closed then the following statements equivalent:*
(1) A is $\mathcal{S}\mu\mathcal{H}$-compact;
(2) B is $\mathcal{S}\mu\mathcal{H}$-compact.

Proof. (1) ⇒ (2): Suppose that A is $\mathcal{S}\mu\mathcal{H}$-compact and $\{V_\alpha : \alpha \in \Lambda\}$ be a family of μ-open subsets of X such that $B \setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{H}$. By the heredity property, $A \setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{H}$ and given that A is $\mathcal{S}\mu\mathcal{H}$-compact there exists $\Lambda_0 \subseteq \Lambda$, finite, such that $A \setminus \bigcup_{\alpha\in\Lambda_0} V_\alpha \in \mathcal{H}$. Since A is $\mu\mathcal{H}_g$-closed, $c_\mu(A) \subseteq \bigcup_{\alpha\in\Lambda_0} V_\alpha$ and so $c_\mu(A) \setminus \bigcup_{\alpha\in\Lambda_0} V_\alpha \in \mathcal{H}$. This implies that $B \setminus \bigcup_{\alpha\in\Lambda_0} V_\alpha \in \mathcal{H}$.
(2) ⇒ (1): Suppose that B is $\mathcal{S}\mu\mathcal{H}$-compact and $\{V_\alpha : \alpha \in \Lambda\}$ be a family of μ-open subsets of X such $A \setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{H}$. Given that A is $\mu\mathcal{H}_g$-closed, $c_\mu(A) \setminus \bigcup_{\alpha\in\Lambda} V_\alpha =$

$\phi \in \mathcal{H}$ and this implies $B \setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{H}$. Since B is $\mathcal{S}\mu\mathcal{H}$-compact, there exists $\Lambda_0 \subseteq \Lambda$, finite, with $B \setminus \bigcup_{\alpha\in\Lambda_0} V_\alpha \in \mathcal{H}$. Hence $A \setminus \bigcup_{\alpha\in\Lambda_0} V_\alpha \in \mathcal{H}$. □

A GTS (X, μ) is said to be μ-Hausdroff [11] for each pair of distinct points x and y in X, there exist μ-open sets U_x and V_y containing x and y, respectively, such that $U_x \cap V_y = \phi$.

Theorem 3.16. [8] *Every $\mu\mathcal{H}$-compact subset of a μ-Hausdroff HGTS $(X, \mu, \mathcal{H})$ is μ^*-closed.*

The following theorem is consequence of the above theorem

Theorem 3.17. *Let $(X, \mu, \mathcal{H})$ be a HGTS such that (X, μ) is μ-Hausdroff. If A is a $\mathcal{S}\mu\mathcal{H}$-compact subset of X, then A is closed in (X, μ^*).*

Now we study the behavior of $\mathcal{S}\mu\mathcal{H}$compactness under certain types of functions.

Theorem 3.18. *If $(X, \mu, \mathcal{H})$ is $\mathcal{S}\mu\mathcal{H}$-compact, $f : (X, \mu) \to (Y, \nu)$ is a (μ, ν)-continuous function and if $\mathcal{G} = \{B \subseteq Y : f^{-1}(B) \in \mathcal{H}\}$ then:*

(1) $\mathcal{G}$ is a hereditary class on Y.
(2) $(Y, \nu, \mathcal{G})$ is $\mathcal{S}\mu\mathcal{H}$-compact.

Proof. (1) Suppose that $A \subseteq B \subseteq Y$ and $B \in \mathcal{G}$. Since $f^{-1}(A) \subseteq f^{-1}(B) \in \mathcal{H}$, then $f^{-1}(A) \in \mathcal{H}$, and so $A \in \mathcal{G}$.
(2) Let $\{V_\alpha : \alpha \in \Lambda\}$ be a family of μ-open subsets of Y such that $Y \setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{G}$. Since $X \setminus \bigcup_{\alpha\in\Lambda} f^{-1}(V_\alpha) = f^{-1}\left(Y \setminus \bigcup_{\alpha\in\Lambda} V_\alpha\right) \in \mathcal{H}$ and $(X, \mu, \mathcal{H})$ is $\mathcal{S}\mu\mathcal{H}$-compact, there exists a finite subset Λ_0 of Λ with $f^{-1}\left(Y \setminus \bigcup_{\alpha\in\Lambda_0} V_\alpha\right) = X \setminus \bigcup_{\alpha\in\Lambda_0} f^{-1}(V_\alpha) \in \mathcal{H}$. Thus $Y \setminus \bigcup_{\alpha\in\Lambda_0} V_\alpha \in \mathcal{G}$. □

The following lemma is very useful in studying the preservation of $\mathcal{S}\mu\mathcal{H}$-compact by certain classes of functions

Lemma 3.19. [4] *Let $f : (X, \mu) \to (Y, \nu)$ be a function. If $\mathcal{H}$ is a hereditary class on X, then $f(\mathcal{H}) = \{f(H) : H \in \mathcal{H}\}$ is a hereditary class on Y.*

Theorem 3.20. *If $(X, \mu, \mathcal{H})$ is $\mathcal{S}\mu\mathcal{H}$-compact and $f : (X, \mu) \to (Y, \nu)$ is a bijective (μ, ν)-continuous function, then $(Y, \nu, f(\mathcal{H}))$ is $\mathcal{S}\upsilon f(\mathcal{H})$-compact.*

Proof. Let $\{V_\alpha : \alpha \in \Lambda\}$ be a family of μ-open subsets of Y such that $Y \setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in f(\mathcal{H})$. There exists $H \in \mathcal{H}$ with $Y \setminus \bigcup_{\alpha\in\Lambda} V_\alpha = f(H)$. Then $H = f^{-1}(f(H)) = X \setminus \bigcup_{\alpha\in\Lambda} f^{-1}(V_\alpha) \in \mathcal{H}$. Given that $(X, \mu, \mathcal{H})$ is $\mathcal{S}\mu\mathcal{H}$-compact, there exists a finite subset Λ_0 of Λ, with $f^{-1}\left(Y \setminus \bigcup_{\alpha\in\Lambda_0} V_\alpha\right) = X \setminus \bigcup_{\alpha\in\Lambda_0} f^{-1}(V_\alpha) \in \mathcal{H}$. Thus $Y \setminus \bigcup_{\alpha\in\Lambda_0} V_\alpha = f(f^{-1}(Y \setminus \bigcup_{\alpha\in\Lambda_0} V_\alpha)) \in f(\mathcal{H})$. □

Corollary 3.21. *If $f : (X, \mu) \to (Y, \nu)$ is a bijective μ-open function and $(Y, \nu, \mathcal{G})$ is $\mathcal{S}\upsilon\mathcal{G}$- compact, then $(X, \mu, f^{-1}(\mathcal{G}))$ is $\mathcal{S}\mu f^{-1}(\mathcal{G})$-compact*

Proof. Let $\{V_\alpha : \alpha \in \Lambda\}$ be a family of μ-open subsets of X such that $X \setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in f^{-1}(\mathcal{G})$. There exists $G \in \mathcal{G}$ with $X \setminus \bigcup_{\alpha\in\Lambda} V_\alpha = f^{-1}(G)$. Then $Y \setminus \bigcup_{\alpha\in\Lambda} f(V_\alpha) = f(f^{-1}(G)) = G \in \mathcal{G}$, and given that $(Y, \nu, \mathcal{G})$ is $\mathcal{S}v\mathcal{G}$-compact then there exists a finite subset Λ_0 of Λ with $f(X \setminus \bigcup_{\alpha\in\Lambda_0} V_\alpha) = Y \setminus \bigcup_{\alpha\in\Lambda_0} f(V_\alpha) \in \mathcal{G}$. This implies that $X \setminus \bigcup_{\alpha\in\Lambda_0} V_\alpha \in f^{-1}(\mathcal{G})$.

□

4. $\mathbf{S} - \mathcal{S}\mu\mathcal{H}$-Compactness Spaces

In this section we present a strong form of $\mathcal{S}\mu\mathcal{H}$-compact. Next, we study some properties of these spaces.

Definition 4.1. *If $(X, \mu, \mathcal{H})$ is a HGTS and $A \subseteq X$, A is said to be strong $\mathcal{S}\mu\mathcal{H}$-compact (briefly $\mathbf{S} - \mathcal{S}\mu\mathcal{H}$-compact) if for every family $\{V_\alpha : \alpha \in \Lambda\}$ of μ-open subsets of X with $A \setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{H}$ then there exists a finite subset Λ_0 of Λ, such that $A \subseteq \bigcup_{\alpha\in\Lambda_0} V_\alpha$. The HGTS $(X, \mu, \mathcal{H})$ is said to be $\mathbf{S} - \mu\mathcal{H}$-compact if X is $\mathbf{S} - \mathcal{S}\mu\mathcal{H}$-compact.*

Clearly, the following diagram follows immediately from the definitions and facts.

$$\begin{array}{ccccc} & \nearrow & \mathcal{S}\mu\mathcal{H} - compact & \searrow & \\ \mathbf{S} - \mathcal{S}\mu\mathcal{H} - compact & & & & \mu\mathcal{H} - compact \\ & \searrow & \mu - compact & \nearrow & \end{array}$$

Remark 4.2. *We note that if $(X, \mu, \mathcal{H})$ is a HGTS and $(X, \mu^*, \mathcal{H})$ is $\mathbf{S} - \mathcal{S}\mu\mathcal{H}$-compact, then $(X, \mu, \mathcal{H})$ is $\mathbf{S}-\mathcal{S}\mu\mathcal{H}$-compact, and that $(X, \mu, \mathcal{H})$ is $\mathbf{S}-\mathcal{S}\mu\mathcal{H}$-compact if and only if for any family $\{F_\alpha : \alpha \in \Lambda\}$ of μ-closed subsets of X, if $\bigcap_{\alpha\in\Lambda} F_\alpha \in \mathcal{H}$ then there exists $\Lambda_0 \subseteq \Lambda$, finite, such that $\bigcap_{\alpha\in\Lambda_0} F_\alpha = \phi$.*

Remark 4.3.
1. *It is clear that the GT (X, μ) is μ-compact if and only if $(X, \mu, \{\phi\})$ is $\mathbf{S} - \mathcal{S}\mu\{\phi\}$-compact.*
2. *If $(X, \mu, \mathcal{H})$ is $\mathbf{S}-\mathcal{S}\mu\mathcal{H}$-compact then $(X, \mu, \mathcal{H})$ is $\mathcal{S}\mu\mathcal{H}$-compact, and (X, μ) is μ-compact. The converse is not true as shown by the following examples.*

Example 4.4. *Let $\mathbb{N}$ be the set of natural numbers, the GT defined as:*

$$\mu = \{A \subset \mathbb{N} : \mathbb{N} \setminus A \quad is \quad finite\} \cup \{\phi\},$$

and the hereditary class on $\mathbb{N}$, $\mathcal{H} = \{\mathbb{N} \setminus A : A \in \mu\}$. Then:
(1) The HGTS $(X, \mu, \mathcal{H})$ is $\mathcal{S}\mu\mathcal{H}$-compact, because if $\{V_\alpha : \alpha \in \Lambda\}$ is a family of μ-open subsets of $\mathbb{N}$ then $\mathbb{N} \setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{H}$. If Λ_0 is any finite subset of Λ we have that $\mathbb{N} \setminus \bigcup_{\alpha\in\Lambda_0} V_\alpha \in \mathcal{H}$.
(2) The HGTS $(X, \mu, \mathcal{H})$ is not $\mathbf{S} - \mathcal{S}\mu\mathcal{H}$-compact , because if $F_n = \mathbb{N} \setminus (\mathbb{N} \setminus \{1, 2, ..., n\})$ then F_n is a μ-closed subset of X and $\bigcap_{n=1}^{\infty} F_n = \bigcap_{n=1}^{\infty} \mathbb{N} \setminus (\mathbb{N} \setminus \{1, 2, ..., n\}) = \mathbb{N}\setminus(\mathbb{N}\setminus\{1\}) \in \mathcal{H}$ but if $n_1, n_2, ..., n_r \in \mathbb{N}$ then $\bigcap_{k=1}^{r} F_{n_k} = \bigcap_{k=1}^{r} \mathbb{N} \setminus (\mathbb{N} \setminus \{1, 2, ..., n_k\}) \neq \phi$.

Remark 4.5. *$\mathcal{S}\mu\mathcal{H}$-compactness and μ-compactness are independent of each other as the following examples show.*

Example 4.6. *Consider $X = (0,1)$, μ is the usual topology, and $\mathcal{H} = \{A : A \subseteq (0,1)\}$ then (X,μ) is not μ-compact, but $(X,\mu,\mathcal{H})$ is, evidently, $\mathcal{S}\mu\mathcal{H}$-compact.*

Example 4.7. *Let $X = [1,2]$, $\mu = \{X \cap (a,b) : a < b, a, b \in \mathbb{R}\}$, and $\mathcal{H} = \{\phi, \{1\}, \{2\}\}$. Observe that (X,μ) is μ-compact but $(X,\mu,\mathcal{H})$ is not $\mathcal{S}\mu\mathcal{H}$-compact. In fact, if $V_n = \left(1 + \frac{1}{n},\ 2\right]$, for all integer number $n \geq 1$, then $X \setminus \bigcup\limits_{n\geq 1} V_n = \{1\} \in \mathcal{H}$. If we take $N = max\{n_1, ..., n_k\}$, $k \in \mathbb{Z}^+$ and $n_1, n_2, ..., n_k$ are integer numbers then $X \setminus \bigcup\limits_{i=1}^{k} V_{n_i} = X \setminus \left(1 + \frac{1}{N},\ \ 2\right] = \left[1,\ 1 + \frac{1}{N}\right] \notin \mathcal{H}$.*

Proposition 4.8. *Let $(X,\mu,\mathcal{H})$ be a HGTS and $\mathcal{B}$ is a base for μ. Then the following are equivalent:*

1. *$(X,\mu,\mathcal{H})$ is $\mathbf{S} - \mathcal{S}\mu\mathcal{H}$-compact;*
2. *for any family $\{V_\alpha : \alpha \in \Lambda\}$ of μ-open sets in $\mathcal{B}$, if $X \setminus \bigcup\limits_{\alpha\in\Lambda} V_\alpha \in \mathcal{H}$ then there exists $\Lambda_0 \subseteq \Lambda$, finite, with $X = \bigcup\limits_{\alpha\in\Lambda_0} V_\alpha$.*

Proof. (1) $\Rightarrow$ (2): Let $\{V_\alpha : \alpha \in \Lambda\}$ be a family of non-empty μ-open subsets of X such that $X \setminus \bigcup\limits_{\alpha\in\Lambda} V_\alpha \in \mathcal{H}$. For all $\alpha \in \Lambda$ there exists a family $\{B_{\alpha\beta} : \beta \in \Lambda_\alpha\} \subseteq \mathcal{B}$ such that $V_\alpha = \bigcup\limits_{\beta\in\Lambda_\alpha} B_{\alpha\beta}$. Given that $X \setminus \bigcup\limits_{\alpha\in\Lambda} V_\alpha = X \setminus \bigcup\limits_{\alpha\in\Lambda}\left(\bigcup\limits_{\beta\in\Lambda_\alpha} B_{\alpha\beta}\right) \in \mathcal{H}$ and $(X,\mu,\mathcal{H})$ is $\mathbf{S} - \mathcal{S}\mu\mathcal{H}$-compact there exist $B_{\alpha_1\beta_1}, B_{\alpha_2\beta_2}, ..., B_{\alpha_k\beta_k}$ such that $X = \bigcup\limits_{i=1}^{k} B_{\alpha_i\beta_i}$. But $X = \bigcup\limits_{i=1}^{k} B_{\alpha_i\beta_i} \subseteq \bigcup\limits_{i=1}^{k} V_i$ and so $X = \bigcup\limits_{i=1}^{k} V_i$.
(2) $\Leftarrow$ (1): It is obvious. □

Next we study the behavior of some types of subspaces of a $\mathbf{S} - \mathcal{S}\mu\mathcal{H}$-compact space relative to X.

Theorem 4.9. *Every $\mu\mathcal{H}_g$-closed subset of a $\mathbf{S} - \mathcal{S}\mu\mathcal{H}$-compact space is $\mathbf{S} - \mathcal{S}\mu\mathcal{H}$-compact.*

Proof. Let A be any $\mu\mathcal{H}_g$-closed of $(X,\mu,\mathcal{H})$ and $\{V_\alpha : \alpha \in \Lambda\}$ be a family of μ-open subsets of X such that $A \setminus \bigcup\limits_{\alpha\in\Lambda} V_\alpha \in \mathcal{H}$. Since A is $\mu\mathcal{H}_g$-closed, $c_\mu(A) \subseteq \bigcup\limits_{\alpha\in\Lambda} V_\alpha$. Then $(X \setminus c_\mu(A)) \cup (\bigcup\limits_{\alpha\in\Lambda} V_\alpha)$ is a μ-covering of X and so $X \setminus \left[X \setminus c_\mu(A) \cup (\bigcup\limits_{\alpha\in\Lambda} V_\alpha)\right] = \phi \in \mathcal{H}$. Given that X is $\mathbf{S} - \mathcal{S}\mu\mathcal{H}$-compact there exists s finite subset Λ_0 of Λ such that $X = (X \setminus c_\mu(A) \cup \bigcup\limits_{\alpha\in\Lambda_0} V_\alpha)$. Then $A = A \cap [(X \setminus c_\mu(A)) \cup \bigcup\limits_{\alpha\in\Lambda_0} V_\alpha] = A \cap \bigcup\limits_{\alpha\in\Lambda_0} V_\alpha \subseteq \bigcup\limits_{\alpha\in\Lambda_0} V_\alpha$. □

Theorem 4.10. *If A and B are $\mathbf{S} - \mathcal{S}\mu\mathcal{H}$-compact subsets of a HGTS $(X,\mu,\mathcal{H})$, then $A \cup B$ is $\mathbf{S} - \mathcal{S}\mu\mathcal{H}$-compact.*

Proof. Let $\{V_\alpha : \alpha \in \Lambda\}$ be a family of μ-open subsets of X such that $A \cup B \setminus \bigcup\limits_{\alpha\in\Lambda} V_\alpha \in \mathcal{H}$. Since, $A \setminus \bigcup\limits_{\alpha\in\Lambda} V_\alpha \subseteq A \cup B \setminus \bigcup\limits_{\alpha\in\Lambda} V_\alpha$ and $B \setminus \bigcup\limits_{\alpha\in\Lambda} V_\alpha \subseteq A \cup B \setminus \bigcup\limits_{\alpha\in\Lambda} V_\alpha$

then $A\setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{H}$ and $B\setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{H}$ and so there exist finite subsets Λ_0 and Λ_1 of Λ such that $A \subseteq \bigcup_{\alpha\in\Lambda_0} V_\alpha$ and $B \subseteq \bigcup_{\alpha\in\Lambda_1} V_\alpha$. This implies that $A \subseteq \bigcup_{\alpha\in\Lambda_0\cup\Lambda_1} V_\alpha$ and $B \subseteq \bigcup_{\alpha\in\Lambda_0\cup\Lambda_1} V_\alpha$ and so $A\cup B \subseteq \bigcup_{\alpha\in\Lambda_0\cup\Lambda_1} V_\alpha$. Hence $A\cup B$ is $\mathbf{S}-\mathcal{S}\mu\mathcal{H}$-compact. □

Theorem 4.11. *If $(X, \mu, \mathcal{H})$ is a HGTS, $A \subseteq B \subseteq X$ and $B \subseteq c_\mu(A)$ then the following statements holds.*

(1) If A is μg-closed and $\mathbf{S}-\mathcal{S}\mu\mathcal{H}$-compact, then B is $\mathbf{S}-\mathcal{S}\mu\mathcal{H}$-compact;
(2) If A is $\mu\mathcal{H}g$-closed and B is $\mathbf{S}-\mathcal{S}\mu\mathcal{H}$-compact, then A is $\mathbf{S}-\mathcal{S}\mu\mathcal{H}$-compact.

Proof. (1) Let $\{V_\alpha : \alpha \in \Lambda\}$ be a family of μ-open subsets of X such that $B \setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{H}$. Since, $A\setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{H}$ and A is $\mathbf{S}-\mathcal{S}\mu\mathcal{H}$-compact, there exists a finite subset Λ_0 of Λ such that $A \subseteq \bigcup_{\alpha\in\Lambda_0} V_\alpha$. Since A is μg-closed, $c_\mu(A) \subseteq \bigcup_{\alpha\in\Lambda_0} V_\alpha$ and this implies $B \subseteq \bigcup_{\alpha\in\Lambda_0} V_\alpha$.
(2) Let $\{V_\alpha : \alpha \in \Lambda\}$ be a family of μ-open subsets of X such that $A \setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{H}$. Given that A is $\mu\mathcal{H}g$-closed, $c_\mu(A)\setminus \bigcup_{\alpha\in\Lambda} V_\alpha = \phi \in \mathcal{H}$ and this implies $B\setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{H}$. Since B is $\mathbf{S}-\mathcal{S}\mu\mathcal{H}$-compact, there exists a finite subset $\Lambda_0 \subseteq \Lambda$, finite, with $B \subseteq \bigcup_{\alpha\in\Lambda_0} V_\alpha$. Hence $A \subseteq \bigcup_{\alpha\in\Lambda_0} V_\alpha$. □

Now we study the behavior of $\mathbf{S}-\mathcal{S}\mu\mathcal{H}$-compactness under certain types of functions.

Theorem 4.12. *If $(X, \mu, \mathcal{H})$ is $\mathbf{S}-\mathcal{S}\mu\mathcal{H}$-compact, $f : (X, \mu) \to (Y, \nu)$ is a (μ, ν)-continuous surjective function and if $\mathcal{G} = \{B \subseteq Y : f^{-1}(B) \in \mathcal{H}\}$ then $(Y, \nu, \mathcal{G})$ is $\mathbf{S}-\mathcal{S}\mu\mathcal{H}$-compact.*

Proof. Let $\{V_\alpha : \alpha \in \Lambda\}$ be a family of μ-open subsets of Y such that $Y \setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{G}$. Since $X \setminus \bigcup_{\alpha\in\Lambda} f^{-1}(V_\alpha) = f^{-1}\left(Y \setminus \bigcup_{\alpha\in\Lambda} V_\alpha\right) \in \mathcal{H}$ and $(X, \mu, \mathcal{H})$ is $\mathbf{S}-\mathcal{S}\mu\mathcal{H}$-, there exists a finite subset Λ_0 of Λ, such that $X = \bigcup_{\alpha\in\Lambda_0} f^{-1}(V_\alpha)$. Given that f is surjective we have $Y = \bigcup_{\alpha\in\Lambda_0} V_\alpha$. □

Theorem 4.13. *If $(X, \mu, \mathcal{H})$ is $\mathbf{S}-\mathcal{S}\mu\mathcal{H}$-compact and $f : (X, \mu) \to (Y, \nu)$ is a bijective (μ, ν)-continuous function, then $(Y, \nu, f(\mathcal{H}))$ is $\mathbf{S}-\mathcal{S}\mu f(\mathcal{H})$-compact.*

Proof. Let $\{V_\alpha : \alpha \in \Lambda\}$ be a family of μ-open subsets of Y such that $Y \setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in f(\mathcal{H})$. There exists $H \in \mathcal{H}$ with $Y \setminus \bigcup_{\alpha\in\Lambda} V_\alpha = f(H)$. Then $H = f^{-1}(f(H)) = X\setminus \bigcup_{\alpha\in\Lambda} f^{-1}(V_\alpha) \in \mathcal{H}$. Given that $(X, \mu, \mathcal{H})$ is $\mathbf{S}-\mathcal{S}\mu\mathcal{H}$-compact, there exists a finite subset Λ_0 of Λ such that $X = \bigcup_{\alpha\in\Lambda_0} f^{-1}(V_\alpha)$. Since f is surjective, $Y = \bigcup_{\alpha\in\Lambda_0} V_\alpha$. □

Corollary 4.14. *If $f : (X, \mu) \to (Y, \nu)$ is a bijective and μ-open function and $(Y, \nu, \mathcal{G})$ is $\mathbf{S}-\mathcal{S}\upsilon\mathcal{G}$-compact, then $(X, \mu, f^{-1}(\mathcal{G}))$ is $\mathbf{S}-\mathcal{S}\mu f^{-1}(\mathcal{G})$-compact.*

Proof. Let $\{V_\alpha : \alpha \in \Lambda\}$ be a family of μ-open subsets of X such that $X \setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in f^{-1}(\mathcal{G})$. There exists $G \in \mathcal{G}$ with $X \setminus \bigcup_{\alpha\in\Lambda} V_\alpha = f^{-1}(G)$. Then $Y \setminus \bigcup_{\alpha\in\Lambda} f(V_\alpha) = f(f^{-1}(G))$, and given that $(Y, \nu, \mathcal{G})$ is $\mathbf{S} - \mathcal{S}\upsilon\mathcal{G}$- compact then there exists a finite subset Λ_0 of Λ with $Y = \bigcup_{\alpha\in\Lambda_0} f(V_\alpha)$. This implies that $X = \bigcup_{\alpha\in\Lambda_0} V_\alpha$. □

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