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# New Characterization of $\mathcal{D}-$ Focal Curves in Minkowski 3-space

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ABSTRACT: In this paper, we study new $\mathcal{D}-$focal curves in the Minkowski 3-space with Darboux frame. Moreover, we obtain some integral equations which they are characterizations for a space curve to be a $\mathcal{D}-$focal curve. Finally, we give some characterizations about $\mathcal{D}-$focal curves in the Minkowski 3-space $\mathbb{M}_1^3$.

Key Words: Darboux frame, Minkowski 3-space, Focal curve, Normal curvature, Geodesic Curvature, Geodesic torsion.

## Contents



## 1. The basic knowledge of curves and surfaces

In differential geometry, especially the theory of space curves, the Darboux vector is the areal velocity vector of the Frenet frame of a space curve. It is named after Gaston Darboux who discovered it. It is also called angular momentum vector, because it is directly proportional to angular momentum.

Note that the arc-length parameterization $\mathbf{r} : s \to \mathbf{r}(s)$ of a curve satisfies $\|\mathbf{r}'(s)\| = 1$ and $\mathbf{r}'(s) \perp \mathbf{r}''(s)$ for all $s$. However, in this paper, a general parameterization $\mathbf{r} : t \to \mathbf{r}(t)$ is often used in the surface construction problem. The parameters of functions may sometimes be omitted when no confusion arises.

With each point $\mathbf{r}(s)$ of a curve satisfying $\mathbf{r}''(s) \neq 0$, we associate the *Serret–Frenet frame* $(\mathbf{T}(s), \mathbf{N}(s), \mathbf{b}(s))$ where $\mathbf{T}(s) = \mathbf{r}'(s), \mathbf{N}(s) = \mathbf{r}''(s)/\|\mathbf{r}''(s)\|$, and $\mathbf{b}(s) = \mathbf{T}(s) \times \mathbf{N}(s)$ are, respectively, the unit *tangent, principal normal*, and *binormal* vectors of the curve at the point $\mathbf{r}(s)$.

**Case 1.** If $\mathbf{r}$ is a timelike curve, then derivative of the Serret–Frenet frame is governed by the relations

$$\frac{d}{ds}\begin{bmatrix}\mathbf{T}(s)\\ \mathbf{N}(s)\\ \mathbf{b}(s)\end{bmatrix} = \begin{bmatrix}0 & \kappa(s) & 0\\ \kappa(s) & 0 & \tau(s)\\ 0 & -\tau(s) & 0\end{bmatrix}\begin{bmatrix}\mathbf{T}(s)\\ \mathbf{N}(s)\\ \mathbf{b}(s)\end{bmatrix}, \tag{1.2}$$

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where

$$\begin{aligned} \langle \mathbf{T}, \mathbf{T} \rangle &= -1, \langle \mathbf{N}, \mathbf{N} \rangle = 1, \langle \mathbf{b}, \mathbf{b} \rangle = 1, \\ \langle \mathbf{T}, \mathbf{N} \rangle &= \langle \mathbf{T}, \mathbf{b} \rangle = \langle \mathbf{N}, \mathbf{b} \rangle = 0. \end{aligned}$$

**Case 2.** If $\mathbf{r}$ is a spacelike curve with a spacelike binormal $\mathbf{b}$;

$$\frac{d}{ds}\begin{bmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{b}(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ \kappa(s) & 0 & \tau(s) \\ 0 & \tau(s) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{b}(s) \end{bmatrix},$$

where

$$\begin{aligned} \langle \mathbf{T}, \mathbf{T} \rangle &= 1, \langle \mathbf{N}, \mathbf{N} \rangle = -1, \langle \mathbf{b}, \mathbf{b} \rangle = 1, \\ \langle \mathbf{T}, \mathbf{N} \rangle &= \langle \mathbf{T}, \mathbf{b} \rangle = \langle \mathbf{N}, \mathbf{b} \rangle = 0. \end{aligned}$$

**Case 3.** If $\mathbf{r}$ is a spacelike curve with a spacelike principal normal $\mathbf{N}$;

$$\frac{d}{ds}\begin{bmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{b}(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & \tau(s) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{b}(s) \end{bmatrix},$$

where

$$\begin{aligned} \langle \mathbf{T}, \mathbf{T} \rangle &= 1, \langle \mathbf{N}, \mathbf{N} \rangle = 1, \langle \mathbf{b}, \mathbf{b} \rangle = -1, \\ \langle \mathbf{T}, \mathbf{N} \rangle &= \langle \mathbf{T}, \mathbf{b} \rangle = \langle \mathbf{N}, \mathbf{b} \rangle = 0. \end{aligned}$$

The osculating plane at each curve point $\mathbf{r}(s)$ is spanned by the two vectors $\mathbf{T}(s), \mathbf{N}(s)$ and does not depend on the curve parameterization. If $\kappa(s) = 0$ for some $s$, then $\mathbf{r}''(s) = 0$ and the normal vector $\mathbf{n}(s)$ and osculating plane are undefined at that point. This condition identifies an *inflection* of the curve, [8].

On a regular oriented surface $(u, v) \to \mathbf{R}(u, v)$, the unit normal is defined at each point in terms of the partial derivatives $\mathbf{R}_u = \partial \mathbf{R}/\partial u, \mathbf{R}_v = \partial \mathbf{R}/\partial v$ by

$$\mathbf{n}(u, v) = \frac{\mathbf{R}_u(u, v) \times \mathbf{R}_v(u, v)}{\|\mathbf{R}_u(u, v) \times \mathbf{R}_v(u, v)\|}.$$

Consider a curve $\mathbf{r}(s) = \mathbf{R}((u(s), v(s))$ on a surface $\mathbf{R}(u, v)$, where $s$ denotes arc length for the space curve $\mathbf{r}(s)$, but not necessarily for the plane curve defined by $s \to ((u(s), v(s))$. With each point $\mathbf{r}(s)$ we associate the *Darboux frame* $(\mathbf{T}(s), \mathbf{P}(s), \mathbf{n}(s))-$ where $\mathbf{T}(s)$ is the unit tangent vector of the curve. $\mathbf{n}(s)$ is the unit normal vector of the surface at the point $\mathbf{R}((u(s), v(s)) = \mathbf{r}(s)$, and

$\mathbf{P}(s) = \mathbf{n}(s) \times \mathbf{T}(s)$. The arc-length derivative of the Darboux frame is given by the relations

In case of $\mathbf{r}(s)$ is a time-like curve, the derivative formula of the Darboux frame of $\mathbf{r}(s)$ is in the following form:

$$\frac{d}{ds}\begin{bmatrix} \mathbf{T}(s) \\ \mathbf{P}(s) \\ \mathbf{n}(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g(s) & \kappa_n(s) \\ \kappa_g(s) & 0 & -\tau_g(s) \\ \kappa_n(s) & \tau_g(s) & 0 \end{bmatrix}\begin{bmatrix} \mathbf{T}(s) \\ \mathbf{P}(s) \\ \mathbf{n}(s) \end{bmatrix},$$

where $\mathbf{T}, \mathbf{P}, \mathbf{n}$ satisfy the following properties:

$$\begin{aligned} &< \mathbf{T}, \mathbf{T} > = -1, \quad < \mathbf{n}, \mathbf{n} > = 1, \quad < \mathbf{P}, \mathbf{P} > = 1, \\ &< \mathbf{T}, \mathbf{n} > = < \mathbf{T}, \mathbf{P} > = < \mathbf{n}, \mathbf{P} > = 0. \end{aligned}$$

In case of $\mathbf{r}(s)$ is a spacelike curve, the derivative formula of the Darboux frame of $\mathbf{r}(s)$ is in the following form:

$$\frac{d}{ds}\begin{bmatrix} \mathbf{T}(s) \\ \mathbf{P}(s) \\ \mathbf{n}(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g(s) & \kappa_n(s) \\ -\kappa_g(s) & 0 & \tau_g(s) \\ \kappa_n(s) & \tau_g(s) & 0 \end{bmatrix}\begin{bmatrix} \mathbf{T}(s) \\ \mathbf{P}(s) \\ \mathbf{n}(s) \end{bmatrix},$$

where $\mathbf{T}, \mathbf{P}, \mathbf{n}$ satisfy the following properties:

$$\begin{aligned} &< \mathbf{T}, \mathbf{T} > = 1, \quad < \mathbf{n}, \mathbf{n} > = -1, \quad < \mathbf{P}, \mathbf{P} > = 1, \\ &< \mathbf{T}, \mathbf{n} > = < \mathbf{T}, \mathbf{P} > = < \mathbf{n}, \mathbf{P} > = 0. \end{aligned}$$

In case of $\mathbf{r}(s)$ is a spacelike curve, the derivative formula of the Darboux frame of $\mathbf{r}(s)$ is in the following form:

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where $\mathbf{T}, \mathbf{P}, \mathbf{n}$ satisfy the following properties:

$$\begin{aligned} &< \mathbf{T}, \mathbf{T} > = 1, < \mathbf{n}, \mathbf{n} > = 1, \quad < \mathbf{P}, \mathbf{P} >= -1 \\ &< \mathbf{T}, \mathbf{n} > = < \mathbf{T}, \mathbf{P} > = < \mathbf{n}, \mathbf{P} > = 0. \end{aligned}$$

Define the *normal curvature* $\kappa_n(s)$, the *geodesic curvature* $\kappa_g(s)$, and the *geodesic torsion* $\tau_g(s)$ at each point of the curve $\mathbf{r}(s)$ as

$$\kappa_n = \left\langle \frac{d\mathbf{T}}{ds}, \mathbf{n} \right\rangle, \quad \kappa_g = \left\langle \frac{d\mathbf{T}}{ds}, \mathbf{P} \right\rangle, \quad \tau_g = \left\langle \frac{d\mathbf{T}}{ds}, \mathbf{n} \right\rangle.$$

*A regular curve $t \to \mathbf{r}(t)$ is a geodesic on the surface $\mathbf{R}(u, v)$ if and only if*

($D1$) *the geodesic curvature of* $\mathbf{r}(t)$ *is identically zero;*

($D2$) *the principal normal at each non-inflection point of* $\mathbf{r}(t)$ *is orthogonal to the surface tangent plane at the point* $\mathbf{R}((u(t), v(t)) = \mathbf{r}(t)$;

($D3$) *the osculating plane at each non-inflection point of* $\mathbf{r}(t)$ *is orthogonal to the surface tangent plane at the point* $\mathbf{R}((u(t), v(t)) = \mathbf{r}(t)$.

## 2. $\mathcal{D}-$Focal Curves According To Darboux Frame In $\mathbb{M}_1^3$

Denoting the focal curve by $\mathfrak{D}_\gamma$, we can write

$$\mathfrak{D}_\gamma(s) = (\gamma + \mathfrak{f}_1^{\mathcal{D}}\mathbf{P} + \mathfrak{f}_2^{\mathcal{D}}\mathbf{n})(s), \tag{2.1}$$

where the coefficients $\mathfrak{f}_1^{\mathcal{D}}, \mathfrak{f}_2^{\mathcal{D}}$ are smooth functions of the parameter of the curve $\gamma$, called the first and second focal curvatures of $\gamma$, respectively.

To separate a focal curve according to Darboux frame from that of Frenet-Serret frame, in the rest of the paper, we shall use notation for the focal curve defined above as $\mathcal{D}$-focal curve.

**Case 1.** If $\gamma$ is a timelike curve, then we have

**Theorem 2.1.** *Let* $\gamma : I \longrightarrow \mathbb{M}_1^3$ *be a unit speed timelike curve and* $\mathfrak{D}_\gamma$ *its focal curve on* $\mathbb{M}_1^3$. *Then,*

$$\begin{aligned} \mathfrak{D}_\gamma^{\mathcal{D}}(s) &= \gamma(s) + e^{-\int \frac{\tau_g \kappa_g}{\kappa_n} ds}[\mathfrak{C} - \int \frac{\tau_g}{\kappa_n} e^{\int \frac{\tau_g \kappa_g}{\kappa_n} ds} ds]\mathbf{P} \\ &\quad + [\frac{1}{\kappa_n} - \frac{\kappa_g}{\kappa_n} e^{-\int \frac{\tau_g \kappa_g}{\kappa_n} ds}[\mathfrak{C} + \int \frac{\tau_g}{\kappa_n} e^{\int \frac{\tau_g \kappa_g}{\kappa_n} ds} ds]]\mathbf{n}, \end{aligned} \tag{2.2}$$

*where* $\mathfrak{C}$ *is a constant of integration.*

**Proof.** Assume that $\gamma$ is a unit speed curve and $\mathfrak{D}_\gamma$ its focal curve on $\mathbb{M}_1^3$. By differentiating the formula (2.1), we get

$$\mathfrak{D}_\gamma^{\mathcal{D}}(s)' = (1 + \mathfrak{f}_1^{\mathcal{D}}\kappa_g + \mathfrak{f}_2^{\mathcal{D}}\kappa_n)\mathbf{T} + ((\mathfrak{f}_1^{\mathcal{D}})' + \mathfrak{f}_2^{\mathcal{D}}\tau_g)\mathbf{P} + ((\mathfrak{f}_2^{\mathcal{D}})' - \mathfrak{f}_1^{\mathcal{D}}\tau_g)\mathbf{n}, \tag{2.3}$$

where the coefficients $\mathfrak{f}_1^{\mathcal{D}}, \mathfrak{f}_2^{\mathcal{D}}$ are smooth functions of the parameter of the curve $\gamma$.
Using above equation, the first 2 components vanish, we get

$$\begin{aligned} \mathfrak{f}_1^{\mathcal{D}}\kappa_g + \mathfrak{f}_2^{\mathcal{D}}\kappa_n &= -1, \\ (\mathfrak{f}_1^{\mathcal{D}})' + \mathfrak{f}_2^{\mathcal{D}}\tau_g &= 0. \end{aligned}$$

Considering first equation of above system, we have

$$\mathfrak{f}_1^{\mathcal{D}} = \frac{-1 - \mathfrak{f}_2^{\mathcal{D}}\kappa_n}{\kappa_g} \text{ and } \mathfrak{f}_2^{\mathcal{D}} = \frac{-1 - \mathfrak{f}_1^{\mathcal{D}}\kappa_g}{\kappa_n}$$

Putting in the second equation we have

$$\begin{aligned} \left(\mathfrak{f}_1^{\mathfrak{D}}\right)' - \tau_g\left(\frac{-1-\mathfrak{f}_1^{\mathfrak{D}}\kappa_g}{\kappa_n}\right) &= 0, \\ \left(\mathfrak{f}_1^{\mathfrak{D}}\right)' + \mathfrak{f}_1^{\mathfrak{D}}\left(\frac{\tau_g\kappa_g}{\kappa_n}\right) &= -\frac{\tau_g}{\kappa_n}. \end{aligned}$$

By means of obtained equations, we express (2.2). This completes the proof.

**Corollary 2.2.** *Let $\gamma : I \longrightarrow \mathbb{M}_1^3$ be a unit speed timelike curve and $\mathfrak{D}_\gamma$ its focal curve on $\mathbb{M}_1^3$. Then, the focal curvatures of $\mathfrak{F}_\gamma$ are*

$$\mathfrak{f}_1^{\mathfrak{D}} = e^{-\int \frac{\tau_g\kappa_g}{\kappa_n}ds}[\mathfrak{C} - \int \frac{\tau_g}{\kappa_n}e^{\int \frac{\tau_g\kappa_g}{\kappa_n}ds}ds],$$

$$\mathfrak{f}_2^{\mathfrak{D}} = [-\frac{1}{\kappa_n} - \frac{\kappa_g}{\kappa_n}e^{-\int \frac{\tau_g\kappa_g}{\kappa_n}ds}[\mathfrak{C} - \int \frac{\tau_g}{\kappa_n}e^{\int \frac{\tau_g\kappa_g}{\kappa_n}ds}ds]],$$

*where $\mathfrak{C}$ is a constant of integration.*

In the light of Theorem 2.1, we express the following corollary without proof:

**Lemma 2.3.** *Let $\gamma : I \longrightarrow \mathbb{M}_1^3$ be a unit speed curve and $\mathfrak{F}_\gamma$ its focal curve on $\mathbb{M}_1^3$. If $\kappa_n$ and $\kappa_g$ are constants then, the focal curvatures of $\mathfrak{F}_\gamma$ are*

$$\mathfrak{f}_1^{\mathfrak{D}} = -\frac{1}{\kappa_g} - \mathfrak{Q}e^{\frac{\tau_g\kappa_g}{\kappa_n}s}$$

$$\mathfrak{f}_2^{\mathfrak{D}} = -\frac{1}{\kappa_n} - \frac{\kappa_g}{\kappa_n}[-\frac{1}{\kappa_g} - \mathfrak{Q}e^{\frac{\tau_g\kappa_g}{\kappa_n}s}],$$

*where $\mathfrak{Q}$ is a constant of integration.*

**Theorem 2.4.** *Let $\gamma : I \longrightarrow \mathbb{M}_1^3$ be a unit speed curve and $\mathfrak{F}_\gamma$ its focal curve on $\mathbb{M}_1^3$. If $\kappa_n$ and $\kappa_g$ are constants then,*

$$\mathfrak{D}_\gamma^{\mathfrak{D}}(s) = \gamma(s) + [-\frac{1}{\kappa_g} - \mathfrak{Q}e^{\frac{\tau_g\kappa_g}{\kappa_n}s}]\mathbf{P} + [-\frac{1}{\kappa_n} - \frac{\kappa_g}{\kappa_n}[-\frac{1}{\kappa_g} - \mathfrak{Q}e^{\frac{\tau_g\kappa_g}{\kappa_n}s}]]\mathbf{n},$$

*where $\mathfrak{Q}$ is a constant of integration.*

**Corollary 2.5.** *Let $\gamma : I \longrightarrow \mathbb{M}_1^3$ be a unit speed curve and $\mathfrak{F}_\gamma$ its focal curve on $\mathbb{M}_1^3$. If $\gamma$ is a principal line then,*

$$\mathfrak{D}_\gamma^{\mathfrak{D}}(s) = \gamma(s) + \mathfrak{A}\mathbf{P} + [\frac{-1-\mathfrak{A}\kappa_g}{\kappa_n}]\mathbf{n},$$

*where $\mathfrak{A}$ is a constant of integration.*

**Case 2.** If $\gamma$ is a spacelike curve with timelike $\mathbf{n}$, then we have

**Theorem 2.6.** *Let $\gamma : I \longrightarrow \mathbb{M}_1^3$ be a unit speed spacelike curve with timelike $\mathbf{n}$ and $\mathfrak{D}_\gamma$ its focal curve on $\mathbb{M}_1^3$. Then,*

$$\begin{aligned}\mathfrak{D}_\gamma^{\mathcal{D}}(s) &= \gamma(s) + e^{-\int \frac{\tau_g \kappa_g}{\kappa_n} ds}[\mathfrak{C} + \int \frac{\tau_g}{\kappa_n} e^{\int \frac{\tau_g \kappa_g}{\kappa_n} ds} ds]\mathbf{P} \\ &\quad + [\frac{1}{\kappa_n} - \frac{\kappa_g}{\kappa_n} e^{-\int \frac{\tau_g \kappa_g}{\kappa_n} ds}[\mathfrak{C} + \int \frac{\tau_g}{\kappa_n} e^{\int \frac{\tau_g \kappa_g}{\kappa_n} ds} ds]]\mathbf{n},\end{aligned}$$

*where $\mathfrak{C}$ is a constant of integration.*

**Corollary 2.7.** *Let $\gamma : I \longrightarrow \mathbb{M}^3$ be a unit speed curve and $\mathfrak{D}_\gamma$ its focal curve on $\mathbb{M}^3$. Then, the focal curvatures of $\mathfrak{F}_\gamma$ are*

$$\mathfrak{f}_1^{\mathcal{D}} = e^{-\int \frac{\tau_g \kappa_g}{\kappa_n} ds}[\mathfrak{C} + \int \frac{\tau_g}{\kappa_n} e^{\int \frac{\tau_g \kappa_g}{\kappa_n} ds} ds],$$

$$\mathfrak{f}_2^{\mathcal{D}} = [\frac{1}{\kappa_n} - \frac{\kappa_g}{\kappa_n} e^{-\int \frac{\tau_g \kappa_g}{\kappa_n} ds}[\mathfrak{C} + \int \frac{\tau_g}{\kappa_n} e^{\int \frac{\tau_g \kappa_g}{\kappa_n} ds} ds]],$$

*where $\mathfrak{C}$ is a constant of integration.*

In the light of Theorem 2.6, we express the following corollary without proof:

**Lemma 2.8.** *Let $\gamma : I \longrightarrow \mathbb{M}^3$ be a unit speed spacelike curve with timelike $\mathbf{n}$ and $\mathfrak{F}_\gamma$ its focal curve on $\mathbb{M}^3$. If $\kappa_n$ and $\kappa_g$ are constant then, the focal curvatures of $\mathfrak{F}_\gamma$ are*

$$\mathfrak{f}_1^{\mathcal{D}} = -\frac{1}{\kappa_g} + \mathfrak{Q} e^{\frac{\tau_g \kappa_g}{\kappa_n} s}$$

$$\mathfrak{f}_2^{\mathcal{D}} = \frac{1}{\kappa_n} - \frac{\kappa_g}{\kappa_n}[-\frac{1}{\kappa_g} + \mathfrak{Q} e^{\frac{\tau_g \kappa_g}{\kappa_n} s}],$$

*where $\mathfrak{Q}$ is a constant of integration.*

**Theorem 2.9.** *Let $\gamma : I \longrightarrow \mathbb{M}^3$ be a unit speed curve and $\mathfrak{F}_\gamma$ its focal curve on $\mathbb{M}^3$. If $\kappa_n$ and $\kappa_g$ are constants then,*

$$\mathfrak{D}_\gamma^{\mathcal{D}}(s) = \gamma(s) + [-\frac{1}{\kappa_g} + Q e^{\frac{\tau_g \kappa_g}{\kappa_n} s}]\mathbf{P} + [\frac{1}{\kappa_n} - \frac{\kappa_g}{\kappa_n}[-\frac{1}{\kappa_g} + Q e^{\frac{\tau_g \kappa_g}{\kappa_n} s}]]\mathbf{n},$$

*where $\mathfrak{Q}$ is a constant of integration.*

**Corollary 2.10.** *Let $\gamma : I \longrightarrow \mathbb{M}^3$ be a unit speed spacelike curve with timelike $\mathbf{n}$ and $\mathfrak{F}_\gamma$ its focal curve on $\mathbb{M}^3$. If $\gamma$ is a principal line then,*

$$\mathfrak{D}_\gamma^{\mathcal{D}}(s) = \gamma(s) + \mathfrak{A}\mathbf{P} + [\frac{1 - \mathfrak{A}\kappa_g}{\kappa_n}]\mathbf{n},$$

*where* $\mathfrak{A}$ *is a constant of integration.*

**Case 3.** If $\gamma$ is a spacelike curve with timelike $\mathbf{P}$, then we have

**Theorem 2.11.** *Let* $\gamma : I \longrightarrow \mathbb{M}^3$ *be a unit speed spacelike curve with timelike* $\mathbf{P}$ *and* $\mathfrak{D}_\gamma$ *its focal curve on* $\mathbb{M}^3$*. Then,*

$$\begin{aligned}\mathfrak{D}_\gamma^{\mathcal{D}}(s) &= \gamma(s) + e^{-\int \frac{\tau_g \kappa_g}{\kappa_n} ds}[\mathfrak{C} - \int \frac{\tau_g}{\kappa_n} e^{\int \frac{\tau_g \kappa_g}{\kappa_n} ds} ds]\mathbf{P} \\ &\quad + [-\frac{1}{\kappa_n} - \frac{\kappa_g}{\kappa_n} e^{-\int \frac{\tau_g \kappa_g}{\kappa_n} ds}[\mathfrak{C} - \int \frac{\tau_g}{\kappa_n} e^{\int \frac{\tau_g \kappa_g}{\kappa_n} ds} ds]]\mathbf{n},\end{aligned}$$

*where* $\mathfrak{C}$ *is a constant of integration.*

**Corollary 2.12.** *Let* $\gamma : I \longrightarrow \mathbb{M}^3$ *be a unit speed spacelike curve with timelike* $\mathbf{P}$ *and* $\mathfrak{D}_\gamma$ *its focal curve on* $\mathbb{M}^3$*. Then, the focal curvatures of* $\mathfrak{F}_\gamma$ *are*

$$\mathfrak{f}_1^{\mathcal{D}} = e^{-\int \frac{\tau_g \kappa_g}{\kappa_n} ds}[\mathfrak{C} - \int \frac{\tau_g}{\kappa_n} e^{\int \frac{\tau_g \kappa_g}{\kappa_n} ds} ds],$$

$$\mathfrak{f}_2^{\mathcal{D}} = [-\frac{1}{\kappa_n} - \frac{\kappa_g}{\kappa_n} e^{-\int \frac{\tau_g \kappa_g}{\kappa_n} ds}[\mathfrak{C} - \int \frac{\tau_g}{\kappa_n} e^{\int \frac{\tau_g \kappa_g}{\kappa_n} ds} ds]],$$

*where* $\mathfrak{C}$ *is a constant of integration.*

In the light of Theorem 2.11, we express the following corollary without proof:

**Lemma 2.13.** *Let* $\gamma : I \longrightarrow \mathbb{M}^3$ *be a unit speed spacelike curve with timelike* $\mathbf{P}$ *and* $\mathfrak{F}_\gamma$ *its focal curve on* $\mathbb{M}^3$*. If* $\kappa_n$ *and* $\kappa_g$ *are constants then, the focal curvatures of* $\mathfrak{F}_\gamma$ *are*

$$\mathfrak{f}_1^{\mathcal{D}} = -\frac{1}{\kappa_g} - \mathfrak{Q} e^{\frac{\tau_g \kappa_g}{\kappa_n} s}$$

$$\mathfrak{f}_2^{\mathcal{D}} = -\frac{1}{\kappa_n} - \frac{\kappa_g}{\kappa_n}[-\frac{1}{\kappa_g} - \mathfrak{Q} e^{\frac{\tau_g \kappa_g}{\kappa_n} s}],$$

*where* $\mathfrak{Q}$ *is a constant of integration.*

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