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Approximate Essential Character Amenability of Banach Algebras

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abstract: Let $\mathcal{A}$ be a Banach algebra and φ be a character on $\mathcal{A}$. The notions of approximate essential φ-amenability and approximate essential character amenability of $\mathcal{A}$ are introduced and some properties of such algebras are investigated. Then by means of some examples the distinctions between these new notions and essentially φ-amenability for $\mathcal{A}$ are shown.

Key Words: Approximately inner, Approximate character amenablility, Derivation.

Contents

1. Introduction

Let $\mathcal{A}$ be a Banach algebra and $\mathfrak{X}$ be a Banach $\mathcal{A}$-bimodule then $\mathfrak{X}^*$, the conjugate of $\mathfrak{X}$, has a natural $\mathcal{A}$-bimodule structure defined by

$$\langle a\cdot x^*, x\rangle = \langle x^*, x\cdot a\rangle, \quad \langle x^*\cdot a, x\rangle = \langle x^*, a\cdot x\rangle \quad (a\in\mathcal{A}, x\in\mathfrak{X}, x^*\in\mathfrak{X}^*).$$

Moreover, $\mathfrak{X}$ is called neo-unital if $\mathcal{A}\cdot\mathfrak{X} = \mathfrak{X}\cdot\mathcal{A} = \mathfrak{X}$. A derivation $D:\mathcal{A}\longrightarrow\mathfrak{X}^*$ is a continuous linear map such that $D(ab) = a\cdot Db + Da\cdot b$ for $a,b\in\mathcal{A}$. Given $x^*\in\mathfrak{X}^*$, the inner derivation $ad_{x^*}:\mathcal{A}\longrightarrow\mathfrak{X}^*$ is defined by $ad_{x^*}(a) = a\cdot x^* - x^*\cdot a$. Hence, the derivation $D:\mathcal{A}\longrightarrow\mathfrak{X}^*$ is called approximately inner if there exists a net $(x^*_\alpha)\subseteq\mathfrak{X}^*$ such that $Da = \lim_\alpha ad_{x^*_\alpha}(a) = \lim_\alpha a\cdot x^*_\alpha - x^*_\alpha\cdot a$ for each $a\in\mathcal{A}$. A Banach algebra $\mathcal{A}$ is called amenable if every derivation from $\mathcal{A}$ into each dual Banach $\mathcal{A}$-bimodule $\mathfrak{X}^*$ is inner. Ghahramani and Loy [5] introduced the notions of approximate amenability and approximate essential amenability for the Banach algebras. A Banach algebra $\mathcal{A}$ is said approximately amenable if every derivation $D:\mathcal{A}\longrightarrow\mathfrak{X}^*$ is approximately inner for each Banach $\mathcal{A}$-bimodule $\mathfrak{X}$ and it is approximately essentially amenable, if for every neo-unital Banach $\mathcal{A}$-bimodule $\mathfrak{X}$, every derivation $D:\mathcal{A}\longrightarrow\mathfrak{X}^*$ is approximately inner.

Let $\sigma(\mathcal{A})$ be the set of all non-zero multiplicative linear functionals on $\mathcal{A}$. Suppose that $\varphi\in\sigma(\mathcal{A})$, then $\mathcal{A}$ is said φ-amenable if there exists a $m\in\mathcal{A}^{**}$ such that $m(\varphi)=1$ and $m(a\cdot a^*) = \varphi(a)m(a^*)$ for each $a\in\mathcal{A}$. This concept was introduced by Kaniuth, Lau and Pym in [7], after that they characterize it in terms of certain derivations on $\mathcal{A}$ [7, Theorem 1.1]. At the same time, Monfared introduced the

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concepts of left and right character amenability [12]. Suppose that $\varphi \in \sigma(\mathcal{A}) \cup \{0\}$, we say that $\mathcal{X}$ is a $(\varphi, \mathcal{A})$-bimodule if its left action is $a \cdot x = \varphi(a)x \quad (a \in \mathcal{A}, x \in \mathcal{X})$. A Banach algebra $\mathcal{A}$ is left φ-amenable or is $(\varphi, \mathcal{A})$-amenable if every derivation from $\mathcal{A}$ into $\mathcal{X}^*$ is inner which $\mathcal{X}$ is a $(\varphi, \mathcal{A})$-bimodule. We should mention that left φ-amenability is equivalent to φ-amenability in the sense of [7]. Moreover, $\mathcal{A}$ is left character amenable if every derivation $D : \mathcal{A} \longrightarrow \mathcal{X}^*$ is inner which $\mathcal{X}$ is a $(\varphi, \mathcal{A})$-bimodule and $\varphi \in \sigma(\mathcal{A}) \cup \{0\}$. Meanwhile, $\mathcal{A}$ is called approximately $(\varphi, \mathcal{A})$-amenable, if every derivation $D : \mathcal{A} \longrightarrow \mathcal{X}^*$ is approximately inner which $\mathcal{X}$ is a $(\varphi, \mathcal{A})$-bimodule. The concepts of approximate left and right character amenability for Banach algebras were introduced in [1] and also another versions of character amenability of Banach algebras, called module character amenability and module approximate character amenability of Banach algebras were introduced and studied in [2,3]. Meanwhile, the concept of essentially φ-amenable for a Banach algebra which $\varphi \in \sigma(\mathcal{A}) \cup \{0\}$, was introduced by Nasr-Isfahani and Nemati in [9]. In this paper, the subjects of [9] will be extended, i.e., we introduce the notions of approximate essential $(\varphi, \mathcal{A})$-amenability and approximate essential character amenability for Banach algebras and obtain some properties of such algebras.

2. Main results

We start this section with a definition, which is variant of the concept of essential character amenability by Nasr-Isfahani and Nemati in [9].

Definition 2.1. Let $\mathcal{A}$ be a Banach algebra and $\varphi \in \sigma(\mathcal{A})$. $\mathcal{A}$ is called approximately essentially $(\varphi, \mathcal{A})$ -amenable if for every neo-unital Banach $(\varphi, \mathcal{A})$-bimodule $\mathcal{X}$, every derivation $D : \mathcal{A} \longrightarrow \mathcal{X}^*$ is approximately inner and it is called approximately essentially 0-amenable if for every Banach $\mathcal{A}$ bimodule $\mathcal{X}$ with the zero left action such that $\mathcal{X} \cdot \mathcal{A} = \mathcal{X}$, every derivation $D : \mathcal{A} \longrightarrow \mathcal{X}^*$ is approximately inner. Moreover, $\mathcal{A}$ is called approximately essentially character amenable if it is approximately essentially $(\varphi, \mathcal{A})$-amenable for each $\varphi \in \sigma(\mathcal{A}) \cup \{0\}$.

Proposition 2.1. *Let $\mathcal{A}$ be a Banach algebra, $\varphi \in \sigma(\mathcal{A})$ and $\mathcal{I}$ be a closed ideal of $\mathcal{A}$ with a bounded approximate identity such that $\mathcal{I} \nsubseteq ker(\varphi)$. Then the followings are equivalent.*

(1) *$\mathcal{A}$ is approximately $(\varphi, \mathcal{A})$-amenable;*
(2) *$\mathcal{A}$ is approximately essentially $(\varphi, \mathcal{A})$-amenable;*
(3) *$\mathcal{I}$ is approximately essentially $(\varphi|_{\mathcal{I}}, \mathcal{I})$-amenable;*
(4) *$\mathcal{I}$ is approximately $(\varphi|_{\mathcal{I}}, \mathcal{I})$-amenable.*

Proof: (1) $\Longrightarrow$ (2) It is clear.

(2) $\Longrightarrow$ (3) Let $\mathcal{X}$ be a neo-unital Banach $(\varphi|_{\mathcal{I}}, \mathcal{I})$-bimodule and $D : \mathcal{I} \longrightarrow \mathcal{X}^*$ be a derivation. Then by cohen'factorozation theorem, $\mathcal{I}\mathcal{A} = \mathcal{I}$, hence $\mathcal{X}\mathcal{A} = (\mathcal{X}\mathcal{I})\mathcal{A} = \mathcal{X}(\mathcal{I}\mathcal{A}) = \mathcal{X}\mathcal{I} = \mathcal{X}$, therefore $\mathcal{X}$ is a neo-unital Banach $(\varphi, \mathcal{A})$-bimodule and D has an extention $\widetilde{D} : \mathcal{A} \longrightarrow \mathcal{X}^*$ in view of [10, proposition 2.1.6]. By assumption $\widetilde{D}$ is approximately inner, so D is also approximately inner. Hence $\mathcal{I}$ is approximately essentially $(\varphi|_{\mathcal{I}}, \mathcal{I})$-amenable.

(3) $\Longrightarrow$ (4) It follows from [13, proposition 2.1].

(4) $\Longrightarrow$ (1) Let $\mathcal{X}$ be a Banach $(\varphi, \mathcal{A})$-bimodule and $D : \mathcal{A} \longrightarrow \mathcal{X}^*$ be a derivation, then the map $D|_{\mathcal{I}} : \mathcal{I} \longrightarrow \mathcal{X}^*$ is a derivation. By assumption there exists a net $(m_\alpha) \subseteq \mathcal{X}^*$ such that

$$D|_{\mathcal{I}}(b) = \lim_\alpha (b \cdot m_\alpha - m_\alpha \cdot b) \quad (b \in \mathcal{I}).$$

Consider $b_0 \in \mathcal{I}$ such that $\varphi|_{\mathcal{I}}(b_0) = 1$. Now set $M_\alpha = b_0 \cdot m_\alpha \in \mathcal{X}^*$, so for each $a \in \mathcal{A}$,

$$\begin{aligned}
\lim_\alpha (a \cdot M_\alpha - M_\alpha \cdot a) &= \lim_\alpha (ab_0 \cdot m_\alpha - \varphi(a) b_0 \cdot m_\alpha) \\
&= \lim_\alpha (ab_0 \cdot m_\alpha - \varphi(ab_0) m_\alpha + \varphi(ab_0) m_\alpha - \varphi(a) b_0 \cdot m_\alpha) \\
&= \lim_\alpha (ab_0 \cdot m_\alpha - \varphi(ab_0) m_\alpha) - \varphi(a) \lim_\alpha (b_0 \cdot m_\alpha - m_\alpha) \\
&= D(ab_0) - \varphi(a) D(b_0) \\
&= D(a) \cdot b_0 + a \cdot D(b_0) - \varphi(a) D(b_0) \\
&= Da + \lim_\alpha (ab_0 \cdot m_\alpha - a \cdot m_\alpha) - \lim_\alpha (\varphi(a) b_0 \cdot m_\alpha - \varphi(a) m_\alpha).
\end{aligned}$$

Hence $Da = \lim_\alpha (a \cdot m_\alpha - \varphi(a) m_\alpha)$. □

Remark According to Proposition 2.1 or [13, propostion 2.1] it is seen that for a Banach algebra $\mathcal{A}$ with a bounded approximate identity and $\varphi \in \sigma(\mathcal{A})$, $\mathcal{A}$ is approximately $(\varphi, \mathcal{A})$-amenable if and only if it is approximately essentially $(\varphi, \mathcal{A})$-amenable. But in general it is not true, i.e. in the following example we show that there are approximately essentially $(\varphi, \mathcal{A})$-amenable Banach algebras which are not approximately $(\varphi, \mathcal{A})$-amenable.

Example 1 Let $\mathcal{X}$ be a Banach space and take $\varphi \in \mathcal{X}^* - \{0\}$ with $\|\varphi\| \leq 1$. We are following [9, example 2.4]. Define a product on $\mathcal{X}$ by $ab = \varphi(b)a \quad (a, b \in \mathcal{X})$. Therefore $\mathcal{X}$ is a Banach algebra, which is denoted by $\mathcal{A}_\varphi(\mathcal{X})$. It is clear that $\sigma(\mathcal{A}_\varphi(\mathcal{X})) = \{\varphi\}$. We show that $\mathcal{A}_\varphi(\mathcal{X})$ is approximately $(\varphi, \mathcal{A})$-amenable if and only if $\mathcal{X}$ is one dimensional. Suppose that $\mathcal{A}_\varphi(\mathcal{X})$ is approximately $(\varphi, \mathcal{A})$-amenable. Notice that, the module actions of $\mathcal{A}_\varphi(\mathcal{X})$ on $\mathcal{A}_\varphi(\mathcal{X})^*$ are as follows:

$$a \cdot f = \varphi(a) f, \quad f \cdot a = f(a)\varphi \quad (a \in \mathcal{A}_\varphi(\mathcal{X}), f \in \mathcal{A}_\varphi(\mathcal{X})^*).$$

Consider the quotient Banach $\mathcal{A}$-bimodule $\mathcal{Y} = \mathcal{A}_\varphi(\mathcal{X})^* / \mathbb{C}\varphi$. Let $F_0 \in \mathcal{A}_\varphi(\mathcal{X})^{**}$ be such that $F_0(\varphi) = 1$. Since $ad_{F_0}(\mathcal{A}_\varphi(\mathcal{X})) \subseteq \mathcal{Y}^* = \{F \in \mathcal{A}_\varphi(\mathcal{X})^{**} : \langle F, \varphi \rangle = 0\}$, then by the assumption there exists a net $(F_\alpha) \subseteq \mathcal{Y}^*$ such that

$$ad_{F_0}(a) = \lim_\alpha (a \cdot F_\alpha - F_\alpha \cdot a) = \lim_\alpha \varphi(a)(-F_\alpha)$$

Set $F = \lim_\alpha F_\alpha \in \mathcal{A}_\varphi(\mathcal{X})^{**}$, so $a = \varphi(a)(F_0 - F)$. It follows that $\mathcal{A}_\varphi(\mathcal{X})$ is one dimensional and the converse is also trivial. The rest of the proof is following from [9, example 2.4].

Theorem 2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras, $\varphi \in \sigma(\mathcal{A}) \cup \{0\}$, $\theta : \mathcal{A} \longrightarrow \mathcal{B}$ a continuous epimorphism and $\mathcal{A}$ be approximately essentially $(\varphi \circ \theta, \mathcal{A})$-amenable. Then $\mathcal{B}$ is approximately essentially $(\varphi, \mathcal{B})$-amenable.*

Proof: Suppose that $\mathcal{A}$ is approximately essentially $(\varphi \circ \theta, \mathcal{A})$-amenable and $D : \mathcal{B} \longrightarrow \mathcal{X}^*$ is a derivation in which $\mathcal{X}$ is a neo-unital $(\varphi, \mathcal{B})$-bimodule. We can consider $\mathcal{X}$ as a neo-unital $(\varphi \circ \theta, \mathcal{A})$-bimodule as follows:

$$a \cdot x = \varphi \circ \theta(a) \cdot x, \quad \mathcal{X} \cdot \mathcal{A} = \mathcal{X} \cdot \theta(\mathcal{A}) = \mathcal{X} \cdot \mathcal{B} = \mathcal{X}.$$

Since $D \circ \theta : \mathcal{A} \longrightarrow \mathcal{X}^*$ is a derivations, by assumption $D \circ \theta$ is approximately inner, hence D is approximately inner. □

Let $\mathcal{A}$ be a Banach algebra and $\varphi \in \sigma(\mathcal{A})$. Consider $\mathbb{C}$ as a Banach $\mathcal{A}$-bimodule with the module actions

$$a \cdot \lambda = \lambda \cdot a = \varphi(a)\lambda \quad (a \in \mathcal{A}, \lambda \in \mathbb{C}).$$

Thus $\mathbb{C}$ is a neo-unital Banach $\mathcal{A}$-bimodule which is denoted by $\mathbb{C}_\varphi$. A derivation $d : \mathcal{A} \longrightarrow \mathbb{C}_\varphi$ is called a point derivation at φ.

Proposition 2.3. *Let $\mathcal{A}$ be a approximately essentially $(\varphi, \mathcal{A})$-amenable. Then $\mathcal{A}$ does not has a non-zero point derivation at φ.*

Proof: Since $\mathbb{C} = \mathbb{C}_\varphi$ is a neo-unital $(\varphi, \mathcal{A})$-bimodule, then by assumption every bounded point derivation at φ is approximately inner, therefore it is trivial. □

Recall that for a Banach algebra and $\varphi \in \sigma(\mathcal{A})$, a net $(m_\alpha) \subseteq \mathcal{A}^{**}$ is called an approximate φ-mean, if $m_\alpha(\varphi) = 1$ and $a.m_\alpha - \varphi(a)m_\alpha \longrightarrow 0$ for all $a \in \mathcal{A}$. It is shown that $\mathcal{A}$ is approximately $(\varphi, \mathcal{A})$-amenable if and only if it has an approximate φ-mean by [1, proposition 2.2].

Proposition 2.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras, $\theta : \mathcal{A} \longrightarrow \mathcal{B}$ be a continuous epimorphism and $\varphi \in \sigma(\mathcal{B})$. If $\Lambda : \mathcal{A}^* \longrightarrow \mathcal{B}^*$ is a continuous liner map such that $\Lambda(\varphi \circ \theta) \in \sigma(\mathcal{B})$ and $\Lambda(f \cdot a) = \Lambda(f) \cdot \theta(a)$ for all $a \in \mathcal{A}$ and $f \in \mathcal{A}^*$. Then $\mathcal{A}$ is approximately $(\varphi \circ \theta, \mathcal{A})$-amenable if and only if $\mathcal{B}$ is approximately $(\varphi, \mathcal{B})$-amenable.*

Proof: Suppose that $\mathcal{B}$ is approximately $(\varphi, \mathcal{B})$-amenable, so there exists a net $(m_\alpha) \subseteq \mathcal{B}^{**}$ such that

$$m_\alpha(\varphi) = 1, \quad b \cdot m_\alpha - \varphi(b)m_\alpha \longrightarrow 0 \quad (b \in \mathcal{B}). \quad (1)$$

Since $\Lambda(\varphi \circ \theta) \in \sigma(\mathcal{B})$ and $\Lambda(\varphi \circ \theta) \cdot \theta(a) = \varphi(\theta(a))\Lambda(\varphi \circ \theta)$ for all $a \in \mathcal{A}$, then $\Lambda(\varphi \circ \theta) = \varphi$. Consider the net $(m_\alpha \circ \Lambda) \in \mathcal{A}^{**}$, therefore we get

$$(m_\alpha \circ \Lambda)(\varphi \circ \theta) = m_\alpha(\Lambda(\varphi \circ \theta)) = m_\alpha(\varphi) = 1.$$

Now, with respect to relation (1) and being onto θ , for each $a \in \mathcal{A}$, we have

$$\begin{aligned}\lim_\alpha(a \cdot (m_\alpha \circ \Lambda) - \varphi \circ \theta(a) m_\alpha \circ \Lambda) &= \lim_\alpha((\theta(a) \cdot m_\alpha) \circ \Lambda - \varphi(\theta(a)) m_\alpha \circ \Lambda) \\ &= \lim_\alpha((\theta(a) \cdot m_\alpha) - \varphi(\theta(a)) m_\alpha) \circ \Lambda) = 0.\end{aligned}$$

It implies that $\mathcal{A}$ is approximately $(\varphi \circ \theta, \mathcal{A})$-amenable. □

If $\mathcal{A}$ has a bounded approximate identity and the conditions of above proposition are held, then by Proposition 2.2, $\mathcal{A}$ is approximately essentially $(\varphi\circ\theta, \mathcal{A})$-amenable if and only if $\mathcal{B}$ is approximately essentially $(\varphi, \mathcal{B})$-amenable.

Proposition 2.5. *Let $\mathcal{A}$ be an approximately essentially $(\varphi, \mathcal{A})$-amenable Banach algebra with $\varphi \in \sigma(\mathcal{A})$ and $\mathcal{A}^* \cdot \mathcal{A} = \mathcal{A}^*$. Then $\mathcal{A}$ is approximately $(\varphi, \mathcal{A})$-amenable.*

Proof: Suppose that $\mathcal{A}$ is approximately essentially $(\varphi, \mathcal{A})$-amenable. Consider $\mathcal{A}^*$ as an $\mathcal{A}$-bimodule that its left module action is $a \cdot a^* = \varphi(a) a^*$ for all $a \in \mathcal{A}, a^* \in \mathcal{A}^*$. Since $\mathcal{A}^* \cdot \mathcal{A} = \mathcal{A}^*$, then $\mathcal{A}^*$ is neo-unital $(\varphi, \mathcal{A})$-bimodule and $\mathcal{X} = \mathcal{A}^*/\mathbb{C}\varphi$ is also a neo-unital $(\varphi, \mathcal{A})$-bimodule. Choose $m_0 \in \mathcal{A}^{**}$ such that $m_0(\varphi) = 1$. It is easy seen that $ad_{m_0}(\mathcal{A}) \subseteq \mathcal{X}^* = \{m \in \mathcal{A}^{**}, \langle m, \varphi \rangle = 0\}$, so by assumption there exists a net $(m_\alpha) \subseteq \mathcal{X}^*$ such that

$$ad_{m_0}(a) = \lim_\alpha(a \cdot m_\alpha - m_\alpha \cdot a) \quad (a \in \mathcal{A}).$$

Set $M_\alpha = m_0 - m_\alpha$, therefore $\langle M_\alpha, \varphi \rangle = 1$ and $a \cdot M_\alpha - \varphi(a) M_\alpha \longrightarrow 0$. It implies that $\mathcal{A}$ is approximately $(\varphi, \mathcal{A})$-amenable. □

Proposition 2.6. *Let $\mathcal{A}$ be a Banach algebra without identity, $\varphi \in \sigma(\mathcal{A}) \cup \{0\}$ and $\mathcal{A}^\#$ be approximately essentially $(\widetilde{\varphi}, \mathcal{A}^\#)$-amenable. Then $\mathcal{A}$ is approximately essentially $(\varphi, \mathcal{A})$-amenable.*

Proof: Suppose that $\mathcal{X}$ is a neo-unital Banach $(\varphi, \mathcal{A})$-bimodule and $D : \mathcal{A} \longrightarrow \mathcal{X}^*$ is a derivation. Consider $\mathcal{X}$ as a neo-unital Banach $(\widetilde{\varphi}, \mathcal{A}^\#)$-bimodule as follows:

$$(a+\lambda e)\cdot x = (\varphi(a)+\lambda)x, \quad x\cdot e = x, \quad x\cdot(a+\lambda e) = x\cdot a+\lambda x \quad (a \in \mathcal{A}, x \in \mathcal{X}, \lambda \in \mathbb{C}).$$

Moreover we define the map

$$\widetilde{D} : \mathcal{A}^\# \longrightarrow \mathcal{X}^*, \quad \widetilde{D}(a + \lambda e) = D(a) \quad (a \in \mathcal{A}, \lambda \in \mathbb{C}).$$

Since $\widetilde{D}$ is a derivation so there exists a net $(x^*_\alpha) \subseteq \mathcal{X}^*$ such that

$$\begin{aligned}Da = \widetilde{D}(a + \lambda e) &= \lim_\alpha(a + \lambda e) \cdot x^*_\alpha - (\varphi(a) + \lambda) x^*_\alpha \\ &= \lim_\alpha a \cdot x^*_\alpha - \varphi(a) x^*_\alpha.\end{aligned}$$

It implies that D is approximately inner. □

According to [1, Theorem 3.8], $\mathcal{A}$ is approximately character amenable if and only if $\mathcal{A}^{\#}$ is so. Now, suppose that $\mathcal{A}$ is a Banach algebra which is approximately essentially character amenable, which it is not approximately charater amenable. Therefore by Proposition 2.1 , $\mathcal{A}^{\#}$ can not be approximately essentially amenable, hence the converse of the above proposition is not true. Consequently, for two Banach algebras $\mathcal{A}$ and $\mathcal{B}$ which are approximately essentially character amenable, its ℓ^1-direct sum, i.e. $\mathcal{A} \oplus_1 \mathcal{B}$ does not necessary is so.

Theorem 2.7. *Let $\mathcal{A}$ be a Banach algebra with identity and $\varphi \in \sigma(\mathcal{A})$. The following are equivalent.*

(1) *$\mathcal{A}$ is approximately $(\varphi, \mathcal{A})$-contractible;*
(2) *$\mathcal{A}$ is approximately essentially $(\varphi, \mathcal{A})$-amenable;*
(3) *$\mathcal{A}$ is approximately $(\varphi, \mathcal{A})$-amenable;*
(4) *$\mathcal{A}$ is approximately essentially $(\varphi, \mathcal{A})$-contractible.*

Proof: (1) $\Longrightarrow$ (2) It is clear.

(2) $\Longrightarrow$ (3) It follows from [9, Corollary 2.3].

(3) $\Longrightarrow$ (4) It follows from Proposition 2.1 and [1, Theorem 5.2].

(4) $\Longrightarrow$ (1) Suppose that $\mathcal{X}$ is a $(\varphi, \mathcal{A})$-bimodule and $D : \mathcal{A} \longrightarrow \mathcal{X}$ is a derivation. Set $\mathcal{X} = \mathcal{X} \cdot e + \mathcal{X} \cdot (1 - e)$. So we have derivations $D_1 : \mathcal{A} \longrightarrow \mathcal{X} \cdot e$ and $D_2 : \mathcal{A} \longrightarrow \mathcal{X} \cdot (1 - e)$ such that $D = D_1 + D_2$. On the other hand, $\mathcal{X} \cdot e$ is a neo-unital $(\varphi, \mathcal{A})$-bimodule, therefore by assumption D_1 is approximately inner. Moreover,

$$D_2(a) = D_2(ae) = D_2 a \cdot e + a \cdot D_2 e = a \cdot D_2 e = a \cdot D_2 e - D_2 e \cdot a = ad_{D_2 e}(a).$$

Hence D_2 is also inner, therefore D is approximately inner. □

Corollary 2.8. *Let $\mathcal{A}$ be a Banach algebra with identity, $\varphi \in \sigma(\mathcal{A})$. and $\mathcal{A}^{**}$ is approximately essentially $(\varphi^{**}, \mathcal{A}^{**})$-contractible. Then $\mathcal{A}$ is approximately essentially $(\varphi, \mathcal{A})$-amenable*

Proof: It follows immediately from Theorem 2.7 and [13, Theorem 2.6]. □

Let $\mathcal{A}$ be a Banach algebra. Define $\mathcal{A}.\mathcal{A}$=$\{ab | a \in \mathcal{A}, b \in \mathcal{B}\}$

Proposition 2.9. *Let $\mathcal{A}$ be a Banach algebra, $\mathcal{I}$ a closed subalgebra of $\mathcal{A}$ that contains $\mathcal{A}.\mathcal{A}$, $\varphi \in \sigma(\mathcal{A})$ such that $\varphi|_{\mathcal{I}} \neq 0$ and $\mathcal{I}$ is approximately essentially $(\varphi|_{\mathcal{I}}, \mathcal{I})$-amenable. Then $\mathcal{A}$ is approximately essentially $(\varphi, \mathcal{A})$-amenable.*

Proof: Let $\mathcal{X}$ be a $(\varphi, \mathcal{A})$ neo-unital Banach $\mathcal{A}$-bimodule and $D : \mathcal{A} \longrightarrow \mathcal{X}^*$ be a derivation. Since

$$\mathcal{X} = \mathcal{X}.\mathcal{A} = (\mathcal{X}.\mathcal{A}).\mathcal{A} = \mathcal{X}.(\mathcal{A}.\mathcal{A}) \subseteq \mathcal{X}.\mathcal{I} \subseteq \mathcal{X},$$

hence $\mathcal{X}$ can be considered as a neo-unital Banach $(\varphi|_{\mathcal{I}}, \mathcal{I})$-bimodule. Moreover, Obviously $D|_I : \mathcal{I} \longrightarrow \mathcal{X}^*$ is a derivation. So by assumption there exists a net $(x_\alpha) \subseteq \mathcal{X}^*$ such that

$$D|_I(a) = \lim_\alpha ad_{x_\alpha}(a) = \lim_\alpha a \cdot x_\alpha - \varphi(a)x_\alpha \quad (a \in \mathcal{I}).$$

Now, set $\overline{D}(a) = D(a) - \lim_\alpha ad_{x_\alpha}(a) \quad (a \in \mathcal{A})$. Obviously, $\overline{D} : \mathcal{A} \longrightarrow \mathcal{X}^*$ is a derivation and $\overline{D}(\mathcal{I}) = \{0\}$. By [11, proposition 4.1], $\overline{D} = 0$, so $D(a) = \lim_\alpha ad_{x_\alpha}(a) \quad (a \in \mathcal{A})$. Hence $\mathcal{A}$ is approximately essentially $(\varphi, \mathcal{A})$-amenable. □

Recall that the concepts of approximate [character] amenability and [character] amenability are the same for commutative Banach algebras. In the following we present an example that shows distingtion between concepts of approximate character amenability and approximate essential character amenability for Banach algebras.

Example 2 Let $\mathcal{A} \neq 0$ be a Banach algebra with zero algebra product. Since $\mathcal{I} = \mathcal{A} \cdot \mathcal{A} = \{0\}$ is amenable so $\mathcal{A}$ is approximately essentially character amenable. On the other hand $\mathcal{A}$ is a commutative Banach algebra without [approximate] identity, moreover $\mathcal{A}$ has no non-zero character. Therefore $\mathcal{A}$ is neither approximately amenable [5, lemma 2.2] nor approximately character amenable [1, proposition 2.8].

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