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Tarig Projected Differential Transform Method to Solve Fractional Nonlinear Partial Differential Equations

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ABSTRACT: Recent advances in nonlinear analyzes and fractional calculus is to address the challenges arise in the solution methodology of nonlinear fractional partial differential equations. This paper presents a hybrid technique to solve nonlinear fractional partial differential equations, which is a combination of Tarig transform and Projected Differential Transform Method (TPDTM). The effectiveness of the method is examined by solving three numerical examples that arises in the field of heat transfer analyzes. In this proposed scheme the solution obtained as a convergent series and using the result the hyper diffusive process with pre local information regarding the heat transfer for different values of fractional order are analyzed. To validate the results and to analyze the computational efficiency of the proposed method a comparative study has been carried out with the solution obtained by the Laplace Adomian Decomposition Method (LADM) and Homotophy Pertubation Method (HPM) and observed good agreement. Also, the computational time in each method is calculated using CPU and the results are presented. It was observed that the proposed technique provide good results with less computational time than homotophy pertubation technique. Even though there is a uniformity between the solutions obtained by TPDTM and LADM, the proposed hybrid technique overcome the complexity of manupulation of Adomian polynomials in LADM and evaluation of integrals in HPM respectively. The methodology and the results presented in this paper clearly reveals the computational efficiency of the present method. The TPDTM, due to its computational efficiency has the potential to be used as a novel tool, not only for solving nonlinear fractional differential equations but also to analyse the prelocal information of the system.

Key Words: Nonlinear parabolic equations, Fractional derivatives and integrals, Tarig transform, Projected Differential Transform Method.

Contents

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1. Introduction

Linear and nonlinear fractional partial differential equations have attracted many researchers due to their enormous applications in various fields of engineering like vibration analysis, fluid flow and heat transfer analysis etc. In recent years, most of the physical and biological problems are modeled as nonlinear fractional partial differential equations and have been analyzed by different semi analytical solution techniques like Adomian decomposition, homotopy perturbation and variational iteration method etc.

The emergence of new mathematical method is to reduce the computational complexity in solving nonlinear fractional differential equations. In literature, He (1999) introduced topology based homotopy perturbation method (HPM), which has been used to solve all types of differential equations. Varsha Daftardar Gejji and Jafari (2005) proposed Adomian decomposition method for solving linear and nonlinear fractional differential equations, where the computations are based on the Adomian polynomials. Abbasbandy (2006) applied homotopy analysis method (HAM) to solve nonlinear heat transfer problems. Unlike homotopy perturbation method (HPM), homotophy analysis method is applicable for all kind of auxiliary parameter. Eltayeb and Cman (2007) applied double convolution method in Mellin transform to solve the partial differential equations. Soliman (2008) adopted modified extended direct algebraic (MEDA) method to solve different kinds of nonlinear partial differential equations. Elzaki and Ezaki (2011) introduced "Elzaki transform" for solving partial differential equations. Like the Laplace and Fourier transform, the "Elzaki transform" converts a time domain function into a frequency domain function with a different kernel. Moitsheki and Harley (2011) analysed the heat transfer in longitudinal fins of various profiles with temperature-dependent thermal conductivity and heat transfer coefficient and proposed a closed form solution. Elzaki (2012) combined both Elzaki transform and differential transform method (DTM) and applied this hybrid technique to solve nonlinear partial differential equations. Elzaki and Ezaki (2013) developed a new transformation called "Tarig transform" to solve linear system of integro-differential equations. Zhuo-Jia

Fu et al., (2013) developed Laplace transformed boundary particle method for solving time fractional diffusion equation and studied the long time-history of fractional diffusion systems. Butera and Paola (2014) have proposed a solution methodology using complex Mellin transform for solving multi order fractional differential equations.

Sobhan Mosayebidorcheha et al., (2014) applied differential transform method (DTM) to solve PDE for the heat transfer analysis along fins. Rawashdeh and Maitama (2014, 2015) proposed natural decomposition method (NDM) to solve coupled system of nonlinear partial differential equations and ordinary differential equations, which is a combination of natural transform method (NTM) and Adomian decomposition method. Elzaki and Alamri (2014) combined Elzaki transform and projected DTM to solve nonlinear partial differential equations. Rabie and Elzaki (2014) used Adomian with modified decomposition method for solving systems of nonlinear partial differential equations. Hilal and Elzaki (2014) applied Laplace and variational iteration method for the solution of system of nonlinear partial differential equations. Deshna Loonker (2014) adopted Tarig transform to solve nonhomogeneous fractional order differential equations. Elzaki (2015) applied projected differential transform method to solve nonlinear, space and time fractional partial differential equations. DiMatteoa and Pirrottaa (2015) applied differential transform method to solve linear and nonlinear boundary value problems of fractional order. Wei et al., (2015) developed a mesh free local radial basis function method for solving two dimensional time fractional diffusion equations, where the spatial and temporal discretization is based on the local collocation nodes with implicit time-marching. Zhuo-Jia Fu et al., (2015) proposed method of approximate particular solutions to analyze the constant and variable order fractional diffusion models, which is based on the linear combination of the particular solutions of the non homogeneous equations and radial basis functions. Pang et al (2015) applied Kansa method to solve space fractional advection dispersion equations. Pang et al (2016) carried out a comparative study between the Finite Element and Finite Difference Methods to solve two Dimensional Space Fractional Advection Dispersion Equation. Recently, Chen and Pang (2016) newly-defined fractional Laplacian for modeling and to analyse the power law behaviors of three-dimensional nonlocal heat conduction.

Many of the researchers have focused on the solution methodology of either fractional differential equations or nonlinear differential equations. This research work, makes a useful contribution in fractional calculus and in the nonlinear analyzes of physical and biological problems. In particular, this paper proposes a hybrid technique to solve nonlinear fractional partial differential equations (NLFPDE) called Tarig projected differential transform method (TPDTM). This method has been applied to solve linear, nonlinear fractional partial differential equations in heat transfer analysis and the temperature distribution functions are obtained as a convergent series for different values of fractional order and nonlinearity. The solution obtained by TPDTM has been compared with HPM, LADM solution and the results are validated with the integer order solution and are represented graphically. The TPDTM is an effective tool for solving nonlinear fractional differential equations and more adaptable than the other methods.

The paper is organized as follows. The basic definitions and useful mathematical results which are necessary for the present analysis are discussed in section 2. A detail description of the present hybrid methodology to solve NLFPDE is proposed in section 3. The proposed method is clearly illustrated by solving few example problems and the solutions are presented in section 4. The results are briefly discussed in section 5 with the conclusion in section 6.

2. Preliminaries

In this section we provide some basic definitions of fractional calculus, which will be used in this study.

Definition 1. The Riemann-Liouville fractional integrals [Samko et al. (1993)] of the left and right sided are defined for any function $\varphi(x) \in L_1(a,b)$ as,

$$(I^{\alpha}_{a+}\varphi)(x) = \frac{1}{\Gamma(\alpha)}\int_a^{\infty}(x-t)^{\alpha-1}\varphi(t)dt,\ x>a \tag{2.1}$$

$$(I^{\alpha}_{b-}\varphi)(x) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{b}(t-x)^{\alpha-1}\varphi(t)dt,\ x<b \tag{2.2}$$

where the order $\alpha > 0$.

Definition 2. Subjected to variable limit the Riemann integral on the half axis can be expressed as

$$(I^{\alpha}_{0+}\varphi)(x) = \frac{1}{\Gamma(\alpha)}\int_0^{x}(x-t)^{\alpha-1}\varphi(t)dt,\ 0<x<\infty \tag{2.3}$$

Definition 3. Similarly, the left and right handed Riemann-Liouville fractional derivatives of order $\alpha,\ 0<\alpha<1$, in the interval $[a,b]$ are defined as

$$(D^{\alpha}_{a+}f)(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^{\infty}(x-t)^{-\alpha}f(t)dt, \tag{2.4}$$

$$(D^{\alpha}_{b-}f)(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_x^{b}(t-x)^{-\alpha}f(t)dt. \tag{2.5}$$

Definition 4. Caputo fractional derivative of order α is defined as

$$D^{\alpha}_{a}f(x) = \frac{1}{\Gamma(m-\alpha)}\int_a^{x}\frac{f^{(m)}\xi}{(x-\xi)^{\alpha-m+1}}d\xi, \tag{2.6}$$

where $m-1<\alpha\leq m, m \in N$.

Definition 5. The Mittag-Leffler function which is a generalization of exponential function is defined as

$$E_\alpha(Z) = \sum_{n=0}^{\infty} \frac{Z^n}{\Gamma(\alpha n+1)}, (\alpha \in C, R(\alpha) > 0). \tag{2.7}$$

The fundamental definition of Tarig transform, Projected differential transform method and few theorems are provided in the next consecutive subsections.

2.1. Tarig transform

If $f(t)$ is any time domain function then, the Tarig transform of $f(t)$ is defined as

$$T[f(t)] = \frac{1}{v}\int_0^\infty \exp(\frac{-t}{v^2})f(t)dt, v \neq 0 \tag{2.8}$$

where v is the frequency domain variable. Let $f(t)$, $g(t)$ be any two time domain functions and its frequency domain functions under Tarig transform are $F(v)$, $G(v)$ respectively. The Tarig transform of standard functions and few basic properties are given in Appendix [A].

As in [Deshna Loonker and Banerji, 2014], if $F(v)$ is the Tarig transform of $f(t)$, then the Tarig transform of Fractional integral of $f(t)$ of order α is

$$T[(I_{0+}^{\alpha} f)(t)] = v^{2\alpha}F(v) = v^{2\alpha}T[f(t)]. \tag{2.9}$$

Similarly, the Tarig transform of fractional derivative of $f(t)$ of order α is

$$T[D^\alpha f(t)] = F^\alpha(v) = \frac{1}{v^{2\alpha}}F(v) - \sum_{i=1}^{n} v^{2(i-\alpha)-1} f^{(i-1)}(0). \tag{2.10}$$

2.2. Projected differential transform method

The projected differential transform method is a modified technique of the differential transform method [Elzaki and Alamri (2014)]. If $f(x_1, x_2, ..., x_n)$ is a multivariable function then, the projected differential transform of $f(x_1, x_2, ..., x_n)$ is defined as,

$$f(x_1, x_2, ..., x_{n-1}, k) = \frac{1}{k!}\left[\frac{\partial^k f(x_1, x_2, ..., x_n)}{\partial x_n^k}\right]_{x_n=0}, \tag{2.11}$$

where, $f(x_1, x_2, ..., x_n)$ is the original function and $f(x_1, x_2, ..., x_{n-1}, k)$ is the projected differential transform function. The differential inverse transform of $f(x_1, x_2, ..., x_{n-1}, k)$ is defined as,

$$f(x_1, x_2, ..., x_n) = \sum_{k=0}^{\infty} f(x_1, x_2, ..., x_{n-1}, k)(x - x_0)^k. \tag{2.12}$$

The basic theorems obtained by the PDTM, which are useful for our study are listed below. Consider $u(x_1, x_2, ..., x_n)$, $v(x_1, x_2, ..., x_n)$ be any two multi variable functions and $u(x_1, x_2, ..., x_{n-1}, k)$, $v(x_1, x_2, ..., x_{n-1}, k)$ are the transformed functions of u and v respectively. Let c be a constant.

(i). $z(x_1, x_2, ..., x_n) = u(x_1, x_2, ..., x_n) \pm v(x_1, x_2, ..., x_n)$ then $z(x_1, x_2, ..., x_{n-1}, k) = u(x_1, x_2, ..., x_{n-1}, k) \pm v(x_1, x_2, ..., x_{n-1}, k)$

(ii). $z(x_1, x_2, ..., x_n) = cu(x_1, x_2, ..., x_n)$ then $z(x_1, x_2, ..., x_{n-1}, k) = cu(x_1, x_2, ..., x_{n-1}, k)$

(iii). $z(x_1, x_2, ..., x_n) = \frac{d^n u(x_1, x_2, ..., x_n)}{dx_n^n}$ then $z(x_1, x_2, ..., x_{n-1}, k) = \frac{k+n}{k!} u(x_1, x_2, ..., x_{n-1}, k+n)$

(iv). $z(x_1, x_2, ..., x_n) = u(x_1, x_2, ..., x_n)v(x_1, x_2, ..., x_n)$ then $z(x_1, x_2, ..., x_{n-1}, k) = \sum_{m=0}^{k} u(x_1, x_2, ..., x_{n-1}, m)v(x_1, x_2, ..., x_{n-1}, k-m)$

(v). $z(x_1, x_2, ..., x_n) = u_1(x_1, x_2, ..., x_n)u_2(x_1, x_2, ..., x_n)...u_n(x_1, x_2, ..., x_n)$ then $z(x_1, x_2, ..., x_{n-1}, k) = \sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} \cdots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} u_1(x_1, x_2, ..., x_{n-1}, k_1)$ $u_2(x_1, x_2, ..., x_{n-1}, k_2 - k_1)...u_{n-1}(x_1, x_2, ..., x_{n-1}, k_{n-1} - k_{n-2})u_n(x_1, x_2, ..., x_{n-1}, k - k_{n-1})$

The basic idea of the hybrid technique to solve fractional nonlinear partial differential equation is described clearly in the next section.

3. Tarig Projected Differential Transform Method

Consider a fractional time nonlinear partial differential equation with initial condition as,

$$D^\alpha u(x,t) + Ru(x,t) + Nu(x,t) = g(x,t),$$
$$u(x,0) = f(x), \tag{3.1}$$

where D^α is the fractional order differential operator $D^\alpha = \frac{\partial^\alpha}{\partial t^\alpha}$, R is the linear differential operator, N is the nonlinear differential operator and $g(x,t)$ is the source term. By applying the Tarig transform (denoted throughout this paper by T) on both sides

$$T[D^\alpha u(x,t)] + T[Ru(x,t)] + T[Nu(x,t)] = T[g(x,t)]. \tag{3.2}$$

Using the differentiation property of Tarig transform 2.10, on equation 3.1, we have

$$T[u(x,t)] = vf(x) + v^{2\alpha}\left[T[g(x,t)] - T[Ru(x,t)] - T[Nu(x,t)]\right]. \tag{3.3}$$

Applying the inverse of the Tarig transform on both sides of equation 3.3 imply

$$u(x,t) = G(x,t) - T^{-1}\left[v^{2\alpha}\left[T[Ru(x,t)] + T[Nu(x,t)]\right]\right], \tag{3.4}$$

where $G(x,t)$ represent the term arise from the source term and the prescribed initial conditions. Using PDTM as in Elsaki and Alamri (2014), the nonlinear terms can be easily decomposed as

$$u(x,m+1) = -T^{-1}\left[v^{2\alpha}\left[T[Ru(x,m)] + T[Nu(x,m)]\right]\right],\ m > 0$$

$$u(x,0) = f(x). \tag{3.5}$$

The exact solution of equation 3.1 can be computed in series form as

$$u(x,t) = \sum_{m=0}^{\infty} u(x,m), \tag{3.6}$$

where each term $u(x,m)$ is obtained as a function of x and t.

Unlike the discretization of derivative and complex computation of nonlinear terms the exact or accurate solution can be easily obtained in series form for nonlinear fractional partial differential equations using the present approach.

3.1. Error calculation and Convergence of TPDTM

It is essential to test the convergence of the series solution obtained in equation 3.6 by TPDTM. The approximate solution of equation 3.1 can be obtained as $u_{app(k)}(x,t) = \sum_{m=0}^{k} u(x,m)$ from equation 3.6 by truncating the terms for $m = k+1, k+2, ...\infty$. Then, the exact solution of equation 3.1 is represented as

$$u(x,t) = u_{app(k)}(x,t) + eu_k(x,t), \tag{3.7}$$

where $eu_k(x,t)$ is the error function.
Generally, the absolute error is defined as $eu_k(x,t) = |u(x,t) - u_{app(k)}(x,t)|$. But in most of the practical cases the exact solution $u(x,t)$ is not known. So, define the approximate absolute error as $Eu_k(x,t) = |u_{app(k)}(x,t) - u_{app(k+1)}(x,t)|$. To establish the convergence of equation 3.6, it is necessary to show that the sequence $\{Eu_k(x,t)\}$ is a convergent sequence. Since the sequence is bounded below, it is enough to prove that the sequence $\{Eu_k(x,t)\}$ is monotonically decreasing.
Hence, the convergence criteria is $\left|\frac{Eu_p(x,t)}{Eu_k(x,t)}\right| < 1$ for $k < p$.

Using the following algorithm, the convergence of the iterative solution $u_{app(k)}(x,t)$ to the exact solution $u(x,t)$ can be shown as follows.

- Compute $u_{app(k)}(x,t)$, $u_{app(k+1)}(x,t)$,
- Compute $u_{app(p)}(x,t)$, $u_{app(p+1)}(x,t)$, $k \leq p$.
- Define $Eu_k(x,t) = |u_{app(k)}(x,t) - u_{app(k+1)}(x,t)|$
  $Eu_p(x,t) = |u_{app(p)}(x,t) - u_{app(p+1)}(x,t)|$ for some x and t.
- If $Eu_k(x,t) \geq Eu_p(x,t)$, we can conclude that $u_{app(k)}(x,t)$ converges to the exact solution $u(x,t)$, when $k \to \infty$.

This algorithm is applied in this paper to prove the convergence of the series solution obtained by TPDTM.

4. Numerical Examples

In this section, few example problems are solved to illustrate the proposed hybrid technique and to show the computational efficiency.

4.1. Heat conduction in the absence of source term with constant thermal conductivity

Consider a fractional time linear heat conduction problem along a rod of length l with sinusoidal initial temperature and constant thermal conductivity.

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \kappa \frac{\partial^2 u}{\partial x^2},\ \kappa = \frac{k}{c\rho}, 0 < \alpha \leq 1,$$

$$u(x,0) = 100 sin\frac{\pi x}{l}, 0 < x < l, 0 < \alpha \leq 1. \tag{4.1}$$

Here $k,\ c$ and ρ are the thermal conductivity, specific heat and density of the rod respectively. By applying Tarig transform on both sides of equation 4.1 yield

$$T[u(x,t)] = v100 sin\frac{\pi x}{l} + \kappa v^{2\alpha} T[u_{xx}], \tag{4.2}$$

The inverse of the Tarig transform imply that

$$u(x,t) = 100 sin\frac{\pi x}{l} + \kappa T^{-1}[v^{2\alpha} T[u_{xx}]]. \tag{4.3}$$

Using PDTM, we express

$$u(x,m+1) = \kappa T^{-1}[v^{2\alpha} T\left[\frac{\partial^2 u(x,m)}{\partial x^2}]\right], u(x,0) = 100 sin\frac{\pi x}{l}. \tag{4.4}$$

From the relation in equation 4.4, we obtain

$$u(x,1) = -100\kappa \left(\frac{\pi}{l}\right)^2 sin\frac{\pi x}{l}\frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$u(x,2) = 100\kappa^2 \left(\frac{\pi}{l}\right)^4 sin\frac{\pi x}{l}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$u(x,3) = -100\kappa^3 \left(\frac{\pi}{l}\right)^6 sin\frac{\pi x}{l}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, etc.$$

Now, the exact solution of equation 4.1 is

$$u(x,t) = 100 sin\frac{\pi x}{l} - 100\kappa \left(\frac{\pi}{l}\right)^2 sin\frac{\pi x}{l}\frac{t^\alpha}{\Gamma(\alpha+1)}$$

$$+100\kappa^2 \left(\frac{\pi}{l}\right)^4 sin\frac{\pi x}{l}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - 100\kappa^3 \left(\frac{\pi}{l}\right)^6 sin\frac{\pi x}{l}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + ... \tag{4.5}$$

Using Mittag Leffler function the solution can be expressed as,

$$u(x,t) = 100 sin\frac{\pi x}{l} E_\alpha\left(-\kappa\left(\frac{\pi}{l}\right)^2 t^\alpha\right),$$

$$\text{where } E_\alpha\left(-\kappa\left(\frac{\pi}{l}\right)^2 t^\alpha\right) = \sum_{k=0}^{\infty}\frac{\left[-\kappa\left(\frac{\pi}{l}\right)^2 t^\alpha\right]^k}{\Gamma(\alpha k+1)}. \tag{4.6}$$

The series solution of fractional time equation 4.1 obtained in terms Mittag Leffler function in equation 4.6 approaches to the exact solution shown in equation 4.7, when $\alpha = 1$.

$$u(x,t) = 100sin\frac{\pi x}{l}\exp\left(-\kappa\left(\frac{\pi}{l}\right)^2 t\right). \tag{4.7}$$

Using TPDTM, the temperature distributions are predicted at different points along the rod for different values of the fractional order $\alpha = 0.5,\ 0.75,\ 1$ and is shown in Table 1.

$\alpha = 0.5$	$x = 0$	$x = 10$	$x = 20$	$x = 30$	$x = 40$
$t = 0$	0	38.2683	70.7107	92.3880	100.0000
$t = 80$	0	37.5884	69.4543	90.7464	98.2232
$t = 160$	0	37.3125	68.9445	90.0803	97.5022
$t = 240$	0	37.1030	68.5574	89.5746	96.9549
$t = 320$	0	36.9280	68.2340	89.1520	96.4974
$\alpha = 0.75$	$x = 0$	$x = 10$	$x = 20$	$x = 30$	$x = 40$
$t = 0$	0	38.2683	70.7107	92.3880	100.0000
$t = 80$	0	36.3450	67.1568	87.7446	94.9741
$t = 160$	0	35.1090	64.8730	84.7607	91.7443
$t = 240$	0	34.0757	62.9636	82.2660	89.0440
$t = 320$	0	33.1680	61.2866	80.0747	86.6723
$\alpha = 1$	$x = 0$	$x = 10$	$x = 20$	$x = 30$	$x = 40$
$t = 0$	0	38.2683	70.7107	92.3880	100.0000
$t = 80$	0	33.1918	61.3304	80.1320	86.7342
$t = 160$	0	28.8962	53.3932	69.7616	75.5094
$t = 240$	0	25.3817	46.8993	61.2769	66.3256
$t = 320$	0	22.6483	41.8485	54.6777	59.1828

$\alpha = 0.5$	$x = 50$	$x = 60$	$x = 70$	$x = 80$
$t = 0$	92.3880	70.7107	38.2683	0.0000
$t = 80$	90.7464	69.4543	37.5884	0.0000
$t = 160$	90.0803	68.9445	37.3125	0.0000
$t = 240$	89.5746	68.5574	37.1030	0.0000
$t = 320$	89.1520	68.2340	36.9280	0.0000
$\alpha = 0.75$	$x = 50$	$x = 60$	$x = 70$	$x = 80$
$t = 0$	92.3880	70.7107	38.2683	0.0000
$t = 80$	87.7446	67.1568	36.3450	0.0000
$t = 160$	84.7607	64.8730	35.1090	0.0000
$t = 240$	82.2660	62.9636	34.0757	0.0000
$t = 320$	80.0747	61.2866	33.1680	0.0000
$\alpha = 1$	$x = 50$	$x = 60$	$x = 70$	$x = 80$
$t = 0$	92.3880	70.7107	38.2683	0.0000
$t = 80$	80.1320	61.3304	33.1918	0.0000
$t = 160$	69.7616	53.3932	28.8962	0.0000
$t = 240$	61.2769	46.8993	25.3817	0.0000
$t = 320$	54.6777	41.8485	22.6483	0.0000

Table 1: Example 1:Numerical values of temperacture distribution using TPDTM at different time intervals

Table 2 shows the absolute error at some particular points along the length of the rod for $\alpha = 0.75$ at time $t = 80$ seconds (Erwin Kreyszig, 2008). This proves the convergence of the series solution of equation 4.1 and Figure 1 depicts the comparison of the approximate absolute error for different sequence of partial sums.

x	$Eu_3(x,t)$	$Eu_4(x,t)$	$Eu_5(x,t)$
0	0	0	0 e-04
8	0.0530	0.0013	0.2682 e-04
16	0.1009	0.0025	0.5101 e-04
24	0.1389	0.0035	0.7021 e-04
32	0.1633	0.0041	0.8254 e-04
40	0.1717	0.0043	0.8678 e-04
48	0.1633	0.0041	0.8254 e-04
56	0.1389	0.0035	0.7021 e-04
64	0.1009	0.0025	0.5101 e-04
72	0.0530	0.0013	0.2682 e-04
80	0	0	0 e-04

Table 2: Absolute error calculation for the temperature distribution $u(x,t)$

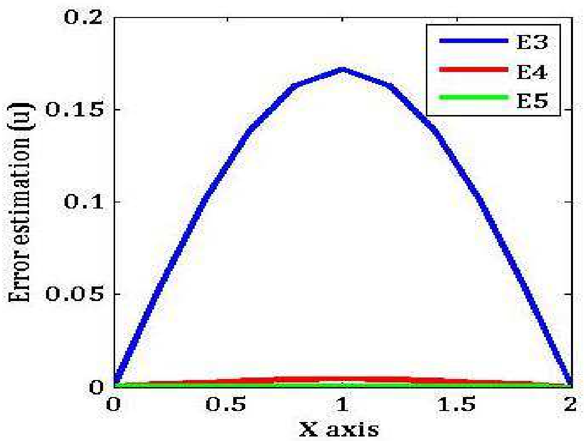

Figure 1: The comparison of absolute error (TPDTM)

4.2. Heat conduction in the presence of source term with constant thermal conductivity

Consider the problem in section [4.1] with the inclusion of source term $g(x,t) = M(x^2 + 2xt)$ as below, where M is a constant

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \kappa \frac{\partial^2 u}{\partial x^2} + M(x^2 + 2xt),$$

$$u(x,0) = 100 sin \frac{\pi x}{l}, 0 < x < l. \quad (4.8)$$

Using TPDTM, the solution of equation in terms of Mittag Leffler function is obtained as

$$u(x,t) = 100 sin\frac{\pi x}{l} E_\alpha\left(-\kappa\left(\frac{\pi}{l}\right)^2 t^\alpha\right)$$
$$+M\left(x^2\frac{t^\alpha}{\Gamma(\alpha+1)} + 2x\frac{t^{\alpha+2}}{\Gamma(\alpha+3)} + 2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right). \quad (4.9)$$

The influence of the source term is also analyzed in the temperature distribution and depicted graphically. It is observed that as $M \to 0$ the effect of the source term becomes small and the result exactly coincides with the solution obtained in the previous section in equation 4.7.

4.3. Heat conduction with variable thermal conductivity

Consider a fractional time linear heat conduction problem as in equation 4.1 with variable thermal conductivity as $k = k(1+\beta(u-u_0))$, where β is any constant and u_0 is the atmospheric temperature.

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{k}{c\rho}(1+\beta(u-u_0))\left(\frac{\partial^2 u}{\partial x^2}\right), \ \kappa = \frac{k}{c\rho},$$

$$u(x,0) = 100 sin\frac{\pi x}{l}, 0 < x < l. \quad (4.10)$$

Let $A = \kappa(1-\beta u_0)$, $B = \kappa\beta$ to reduce complexity. Then, equation 4.10 can be expressed as,

$$\frac{\partial^\alpha u}{\partial t^\alpha} = A\left(\frac{\partial^2 u}{\partial x^2}\right) + Bu\left(\frac{\partial^2 u}{\partial x^2}\right), \quad (4.11)$$

Using the proposed hybrid method, we obtain

$$u(x,t) = 100 sin\frac{\pi x}{l}$$
$$+T^{-1}\left[v^{2\alpha}T\left[A\frac{\partial^2 u(x,m)}{\partial x^2} + B\sum_{m=0}^{h} u(x,m)\frac{\partial^2 u(x,h-m)}{\partial x^2}\right]\right], \quad (4.12)$$

$$u(x,t) = 100 sin\frac{\pi x}{l} - \left[100A\left(\frac{\pi}{l}\right)^2 sin\frac{\pi x}{l} + (100^2)B(\frac{\pi}{l})^2 sin^2\frac{\pi x}{l}\right]\frac{t^\alpha}{\Gamma(\alpha+1)}$$
$$+\left[100A^2\left(\frac{\pi}{l}\right)^4 sin\frac{\pi x}{l} - 2(100^2)AB\left(\frac{\pi}{l}\right)^4\left(cos^2\frac{\pi x}{l} - sin^2\frac{\pi x}{l}\right)\right]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$
$$+\left[2(100^2)AB\left(\frac{\pi}{l}\right)^4 sin^2\frac{\pi x}{l} + 3(100^3)B^2\left(\frac{\pi}{l}\right)^4 sin^3\frac{\pi x}{l}\right]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$
$$-\left[2(100^3)B^2\left(\frac{\pi}{l}\right)^4 sin\frac{\pi x}{l} cos^2\frac{\pi x}{l}\right]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + ... (4.13)$$

$\beta = 0.05$	$x = 0$	$x = 10$	$x = 20$	$x = 30$	$x = 40$
$t = 0$	0	38.2683	70.7107	92.3880	100.0000
$t = 80$	0	32.3805	59.2340	79.1969	86.8862
$t = 160$	0	25.9790	53.1489	80.5128	92.7072
$t = 240$	0	18.7747	48.9746	88.5426	107.6562
$t = 320$	0	10.8481	46.0961	101.5759	129.4879
$\beta = 0.10$	$x = 0$	$x = 10$	$x = 20$	$x = 30$	$x = 40$
$t = 0$	0	38.2683	70.7107	92.3880	100.0000
$t = 80$	3.4331	26.3083	52.8718	83.5906	98.0237
$t = 160$	9.7103	10.8874	45.8391	112.8686	148.0476
$t = 240$	17.8389	-7.4786	43.0951	162.0644	226.1661
$t = 320$	27.4647	-28.3341	43.3909	226.6077	326.1061

$\beta = 0.05$	$x = 50$	$x = 60$	$x = 70$	$x = 80$
$t = 0$	92.3880	70.7107	38.2683	0.0000
$t = 80$	79.1969	59.2340	32.3805	0.0000
$t = 160$	80.5128	53.1489	25.9790	0.0000
$t = 240$	88.5426	48.9746	18.7747	0.0000
$t = 320$	101.5759	46.0961	10.8481	0.0000
$\beta = 0.10$	$x = 50$	$x = 60$	$x = 70$	$x = 80$
$t = 0$	92.3880	70.7107	38.2683	0.0000
$t = 80$	83.5906	52.8718	26.3083	3.4331
$t = 160$	112.8686	45.8391	10.8874	9.7103
$t = 240$	162.0644	43.0951	-7.4786	17.8389
$t = 320$	226.6077	43.3909	-28.3341	27.4647

Table 3: Example 3:Numerical values of temperature distribution at different time intervals ($\alpha = 0.75$)

The approximate closed form solution to the time fractional nonlinear heat conduction equation is obtained by considering the first two terms of the series, which give more accurate result for the nonlinearity. Using TPDTM, the temperature distributions are predicted at different points along the rod for different values of nonlinearity (variable thermal conductivity with respect to β) $\beta = 0.05,\ 0.10$ when $\alpha = 0.75$ and is presented in Table 3.

4.4. System of fractional nonlinear PDE

Consider a system of fractional nonlinear coupled differential equations.

$$\frac{\partial^\alpha U}{\partial t^\alpha} + W\frac{\partial U}{\partial x} + U = 1, U(x,0) = e^x, \tag{4.14}$$

$$\frac{\partial^\alpha W}{\partial t^\alpha} + U\frac{\partial W}{\partial x} - W = -1, W(x,0) = e^{-x},\ 0 < \alpha \leq 1. \tag{4.15}$$

By applying Tarig transform and PDTM, we obtain

$$U(x,t) = e^x - T^{-1}\left[v^{2\alpha}T\left[\sum_{m=0}^{h} W(x,m)\frac{\partial U(x,h-m)}{\partial x} + U(x,m) - 1\right]\right], \tag{4.16}$$

$$W(x,t) = e^{-x} - T^{-1}\left[v^{2\alpha}T\left[\sum_{m=0}^{h} U(x,m)\frac{\partial W(x,h-m)}{\partial x} - W(x,m) + 1\right]\right], \tag{4.17}$$

where, $U(x,0) = e^{x}, W(x,0) = e^{-x}$ and

$$U(x,m+1) = -T^{-1}\left[v^{2\alpha}T\left[\sum_{m=0}^{h} W(x,m)\frac{\partial U(x,h-m)}{\partial x} + U(x,m) - 1\right]\right], \tag{4.18}$$

$$W(x,m+1) = -T^{-1}\left[v^{2\alpha}T\left[\sum_{m=0}^{h} U(x,m)\frac{\partial W(x,h-m)}{\partial x} - W(x,m) + 1\right]\right]. \tag{4.19}$$

Solving these equations we get,

$$U(x,1) = -e^{x}\frac{t^{\alpha}}{\Gamma(\alpha+1)},$$
$$U(x,2) = e^{x}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$
$$U(x,3) = -e^{x}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, etc\ .$$

Similarly,

$$W(x,1) = e^{-x}\frac{t^{\alpha}}{\Gamma(\alpha+1)},$$
$$W(x,2) = e^{-x}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$
$$W(x,3) = e^{-x}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, etc\ .$$

The closed form solution of the coupled system of nonlinear fractional differential equations using Mittag Leffler can be expressed as,

$$U(x,t) = e^{x}E_{\alpha}(-t^{\alpha}),\ W(x,t) = e^{-x}E_{\alpha}(t^{\alpha}). \tag{4.20}$$

If $\alpha = 1$, the results obtained in equation 4.20 are reduced to their exact solution $U(x,t) = e^{x-t}$, $W(x,t) = e^{-x+t}$ respectively. Table 4 shows the temperature distribution for a coupled system and Table 5 shows the absolute error when $\alpha = 0.75$ at time $t = 0.2$. This proves the convergence of the series solution of the coupled system of fractional differential equations 4.14 and 4.15. Also, Figure 2 depicts the comparison of the approximate absolute error for different sequence of partial sums.

$\alpha=\beta=0.5$	$x=0$	$x=0.25$	$x=0.5$	$x=0.75$	$x=1$
$t=0$, U	1.0000	1.2840	1.6487	2.1170	2.7183
V	1.0000	0.7788	0.6065	0.4724	0.3679
$t=0.1000$, U	0.7432	0.9543	1.2253	1.5733	2.0202
V	1.4568	1.1346	0.8836	0.6882	0.5359
$t=0.2000$, U	0.6954	0.8929	1.1465	1.4721	1.8902
V	1.7046	1.3276	1.0339	0.8052	0.6271
$t=0.3000$, U	0.6820	0.8757	1.1244	1.4437	1.8538
V	1.9180	1.4938	1.1633	0.9060	0.7056
$t=0.4000$, U	0.6864	0.8813	1.1316	1.4530	1.8657
V	2.1136	1.6461	1.2820	0.9984	0.7776
$\alpha=\beta=0.75$	$x=0$	$x=0.25$	$x=0.5$	$x=0.75$	$x=1$
$t=0$, U	1.0000	1.2840	1.6487	2.1170	2.7183
V	1.0000	0.7788	0.6065	0.4724	0.3679
$t=0.1000$, U	0.8303	1.0661	1.3689	1.7577	2.2570
V	1.2173	0.9480	0.7383	0.5750	0.4478
$t=0.2000$, U	0.7419	0.9526	1.2231	1.5706	2.0166
V	1.3927	1.0846	0.8447	0.6579	0.5123
$t=0.3000$, U	0.6825	0.8764	1.1253	1.4450	1.8554
V	1.5647	1.2186	0.9490	0.7391	0.5756
$t=0.4000$, U	0.6430	0.8257	1.0602	1.3613	1.7480
V	1.7376	1.3532	1.0539	0.8208	0.6392

$\alpha=\beta=0.5$	$x=1.25$	$x=1.5$	$x=1.75$	$x=2.00$
$t=0$, U	3.4903	4.4817	5.7546	7.3891
V	0.2865	0.2231	0.1738	0.1353
$t=0.1000$, U	2.5939	3.3307	4.2767	5.4914
V	0.4174	0.3251	0.2532	0.1972
$t=0.2000$, U	2.4271	3.1164	4.0016	5.1382
V	0.4884	0.3804	0.2962	0.2307
$t=0.3000$, U	2.3803	3.0563	3.9244	5.0391
V	0.5495	0.4280	0.3333	0.2596
$t=0.4000$, U	2.3956	3.0760	3.9497	5.0715
V	0.6056	0.4716	0.3673	0.2861
$\alpha=\beta=0.75$	$x=1.25$	$x=1.5$	$x=1.75$	$x=2.00$
$t=0$, U	3.4903	4.4817	5.7546	7.3891
V	0.2865	0.2231	0.1738	0.1353
$t=0.1000$, U	2.8980	3.7211	4.7780	6.1351
V	0.3488	0.2716	0.2115	0.1647
$t=0.2000$, U	2.5894	3.3249	4.2692	5.4818
V	0.3990	0.3108	0.2420	0.1885
$t=0.3000$, U	2.3823	3.0590	3.9278	5.0434
V	0.4483	0.3491	0.2719	0.2118
$t=0.4000$, U	2.2444	2.8819	3.7004	4.7514
V	0.4978	0.3877	0.3019	0.2352

Table 4: Example 4:Numerical values of temperature distribution at different time intervals

x	$EU_3(x,t)$	$EU_4(x,t)$	$EU_5(x,t)$		$EW_3(x,t)$	$EW_4(x,t)$	$EW_5(x,t)$
0	0.0105	0.0013	0.0001		0.0105	0.0013	0.1442 e-03
0.2	0.0128	0.0016	0.0002		0.0086	0.0011	0.1181 e-03
0.4	0.0157	0.0020	0.0002		0.0070	0.0009	0.0967 e-03
0.6	0.0191	0.0024	0.0003		0.0058	0.0007	0.0792 e-03
0.8	0.0234	0.0030	0.0003		0.0047	0.0006	0.0648 e-03
1.0	0.0285	0.0036	0.0004		0.0039	0.0005	0.0531 e-03
1.2	0.0348	0.0044	0.0005		0.0032	0.0004	0.0434 e-03
1.4	0.0426	0.0054	0.0006		0.0026	0.0003	0.0356 e-03
1.6	0.0520	0.0066	0.0007		0.0021	0.0003	0.0291 e-03
1.8	0.0635	0.0081	0.0009		0.0017	0.0002	0.0238 e-03
2.0	0.0775	0.0099	0.0011		0.0014	0.0002	0.0195 e-03

Table 5: Absolute error calculation for the functions $U(x,t)$ and $W(x,t)$

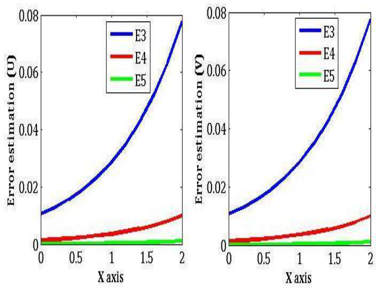

Figure 2: The comparison of absolute error (TPDTM)

4.5. Fractional boundary value problem

In this section, the TPDTM is implemented to solve a fractional boundary value problem as shown below. Let us consider a fractional ODE with boundary conditions as

$$\frac{d^\beta y}{dt^\beta} = y + t^2,\ y(0) = 0,\ y(1) = 1;\ 1 < \beta \leq 2. \tag{4.21}$$

Applying TPDTM on both sides imply

$$\frac{T\left[y(t)\right]}{v^{2\beta}} - v^{2((1-\beta)-1)}y(0) - v^{2((2-\beta)-1)}y'(0) = T\left[t^2 + y\right]. \tag{4.22}$$

By assuming the unknown value as $y'(0) = b$, we obtain the accurate series solution to the boundary value problem as

$$y(t) = bt + 2\frac{t^{\beta+2}}{\Gamma(\beta+3)} + b\frac{t^{\beta+1}}{\Gamma(\beta+2)} + 2\frac{t^{2\beta+2}}{\Gamma(2\beta+3)} + b\frac{t^{2\beta+1}}{\Gamma(2\beta+2)} + 2\frac{t^{3\beta+2}}{\Gamma(3\beta+3)}. \tag{4.23}$$

Using the boundary condition $y(1) = 1$, we obtain the value $b = 0.777$. Only few terms in the series are enough to produce the accurate solution to the boundary value problem and when $\beta = 2$ the result is identical with the exact solution $y(t) = 1.387e^{x} + 0.613e^{-x} - (x^2 + 2)$ shown in Figure 8.

5. Results and Discussion

The temperature distribution is predicted along a rod for a linear fractional time heat conduction problem (Example 1) by using the Tarig Projected differential transform method with the parameters $\kappa = 1.158$ and length of the rod $l = 80\ cm$. The Table 1 represent the temperature distribution along a rod during the heat conduction process for different values of fractional order $\alpha = 0.25, 0.50, 0.75$. The Figure 3 (a), (b) and Figure 4 (a) depict the instability phenomenon during the heat conduction and provides the prelocal information about the heat transfer process. The Figure 4 (b) presents the temperature distribution for alpha=1, where the comparison shows well agreement between the result obtained by TPDTM and the exact integer order solution. The influence of the source term is also analyzed (Example 2) for a small value of M and the results are graphically shown in Figure 5. The TPDTM is applied to solve nonlinear fractional time heat conduction problem with variable thermal conductivity (Example 3) and the temperature distributions are presented for the case beta=0.05 and 0.10 when alpha=0.75. The Figure 6 (a), (b) provides the nonlinear behavior of temperature distribution of the system for fractional time and temperature dependent thermal conductivity.

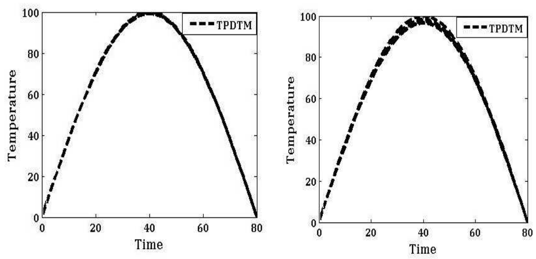

Figure 3: Example 1:Temperature distribution at: (a) α=0.25, (b) α=0.50

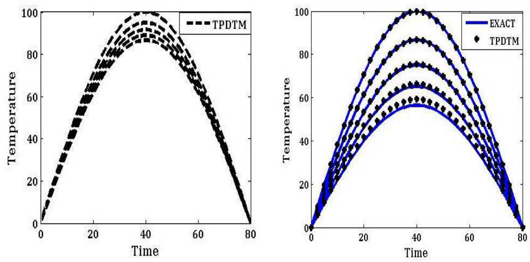

Figure 4: Example 1:Temperature distribution at: (a) α=0.75, (b) Comparison result: α=1

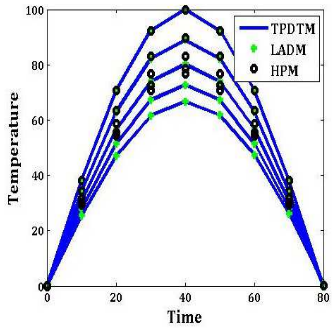

Figure 5: Example 2:Temperature distribution with source term at α=0.95

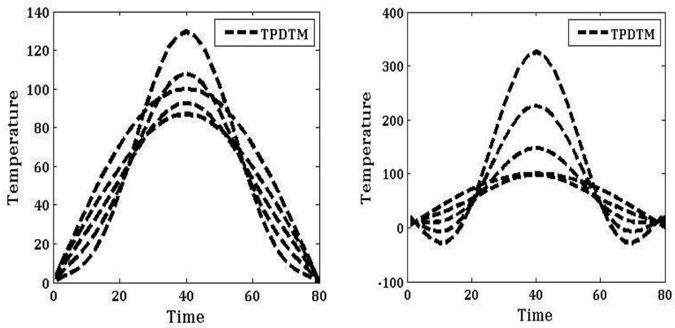

Figure 6: Example 3:Temperature distribution response to fractional time and variable thermal conductivity:(a) α=0.75, β=0.05, (b) α=0.75, β=0.10

The TPDTM has been applied to solve nonlinear coupled fractional partial differential equations (Example 4) that commonly arise in the heat transfer analysis. The temperature distributions are obtained and the results were presented in Figure 7 (a), (b) for alpha = 0.5 and 1 respectively. Finally, the easiest adaptability of TPDTM in solving boundary value problem is shown (Example 5) by incorporating the boundary condition and the results obtained well coincides with the exact solution with acceptable error given in Figure 8. The results shows the instability nature of heat transfer process for fractional order derivative. The proposed technique provides the solution in series form and even the first three terms of the series are enough to predict the accurate behaviour of the heat transfer process. Absolute error calculation provided in Table 2 and 5 reveal the convergence of the series solution obtained by the proposed method.

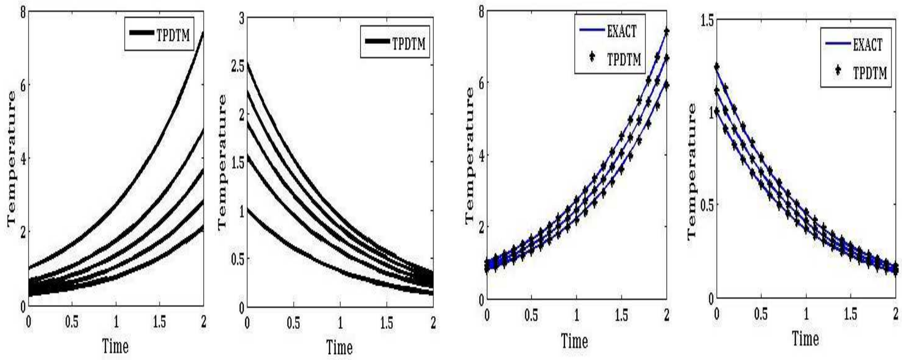

Figure 7: Example 4:Temperature distribution for a coupled system:(a) α=0.5, (b) α=1

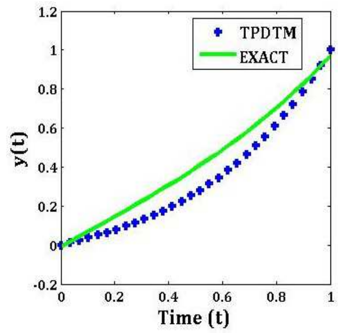

Figure 8: Example 5:Boundary value problem α=1

In order to show the effectiveness of TPDTM, for every numerical examples a comparative study has been carried out with the solution obtained by the LADM and HPM for a particular value of fractional order at some specified points and their computational time is calculated using CPU and the results are presented in Tables 6-9 including the graphical representation from Figures 9-12. An excellent agreement is observed between TPDTM, LADM and acceptable deviation with HPM. It is observed that, the CPU time is comparatively less in TPDTM than the other methods for both integer and fractional order derivatives in all the numerical examples. Even though there is a uniformity between the solutions obtained between TPDTM and LADM it is worth mentioning that, the proposed hybrid technique avoids the difficulty of manupulation of Adomian polynomials and evaluation of integrals in HPM. In the present method transformation technique is employed with projected differential transform method to overcome the difficulty and the results presented in this paper clearly reveals the computational efficiency and easiest adaptability of TPDTM.

	TPDTM			HPM		
$\alpha = 0.95$	$x = 10$	$x = 20$	Time	$x = 10$	$x = 20$	Time
$t = 0$	38.2683	70.7107	0.004333	38.2683	70.7107	0.004892
$t = 80$	34.0640	62.9423	0.000171	34.3833	63.5320	0.000189
$t = 160$	30.6397	56.6148	0.000166	31.7489	58.6643	0.000191
$t = 240$	27.7659	51.3048	0.000175	30.0843	55.5885	0.000256
$t = 320$	25.3837	46.9029	0.000172	29.3260	54.1874	0.000309

	LADM		
$\alpha = 0.95$	$x = 10$	$x = 20$	Time
$t = 0$	38.2683	70.7107	0.004333
$t = 80$	34.0625	62.939	0.000171
$t = 160$	30.6397	56.6148	0.000166
$t = 240$	27.7659	51.3048	0.000175
$t = 320$	25.3837	46.9029	0.000172

Table 6: Example 1: Computational efficiency between TPDTM, HPM and LADM at ($\alpha = 0.95$)

	TPDTM			HPM		
$C = 10^{-9}$	$x = 20$	$x = 30$	Time	$x = 20$	$x = 30$	Time
$t = 0$	70.7107	92.3880	0.005110	70.71	100.00	0.003146
$t = 80$	62.9423	82.2387	0.000103	63.5322	83.0088	0.000127
$t = 160$	56.6374	74.0048	0.000097	58.6648	76.6493	0.000129
$t = 240$	51.3794	67.1450	0.000098	55.5895	72.6314	0.000137
$t = 320$	47.0773	61.5432	0.000112	54.1891	70.8018	0.000126

	LADM		
$C = 10^{-9}$	$x = 20$	$x = 30$	Time
$t = 0$	70.7107	92.3880	0.005110
$t = 80$	62.9423	82.2387	0.000103
$t = 160$	56.6374	74.0048	0.000097
$t = 240$	51.3794	67.1450	0.000098
$t = 320$	47.0773	61.5432	0.000112

Table 7: Example 2: Computational efficiency between TPDTM, HPM and LADM at($\alpha = 0.95$) in seconds

	TPDTM			HPM		
$\beta = 0.05$	$x = 20$	$x = 40$	Time	$x = 20$	$x = 40$	Time
$t = 0$	70.71	100.00	0.000253	70.71	100.00	0.000245
$t = 80$	44.01	105.11	0.000206	44.01	105.11	0.000317
$t = 160$	35.36	263.28	0.000206	35.36	263.28	0.000320
$t = 240$	44.74	574.53	0.000207	44.74	574.53	0.000318
$t = 320$	72.22	1039	0.000207	72.22	1039	0.000313

	LADM		
$\beta = 0.05$	$x = 20$	$x = 40$	Time
$t = 0$	70.71	100.00	0.000253
$t = 80$	44.01	105.11	0.000206
$t = 160$	35.36	263.28	0.000206
$t = 240$	44.74	574.53	0.000207
$t = 320$	72.22	1039	0.000207

Table 8: Example 3: Computational efficiency between TPDTM, HPM and LADM ($\alpha = 1,\ \beta = 0.05$) in seconds

	TPDTM			HPM		
$\alpha = 0.75$	$x = 0.25$	$x = 0.5$	Time	$x = 0.25$	$x = 0.5$	Time
$t = 0,\ U$	1.2840	1.6487	0.000051	1.2840	1.6487	0.000032
V	0.7788	0.6065	0.000029	0.7788	0.6065	0.000034
$t = 0.1,\ U$	1.0661	1.3689	0.000040	1.0977	1.4094	0.000030
V	0.9480	0.7383	0.000026	0.8995	0.7004	0.000032
$t = 0.2,\ U$	0.9526	1.2231	0.000041	0.9503	1.2195	0.000030
V	1.0846	0.8447	0.000026	1.0113	0.7870	0.000032
$t = 0.3,\ U$	0.8764	1.1253	0.000042	0.8320	1.0658	0.000033
V	1.2186	0.9490	0.000026	1.1195	0.8699	0.000032
$t = 0.4,\ U$	0.8257	1.0602	0.000040	0.7414	0.9459	0.000033
V	1.3532	1.0539	0.000026	1.2242	0.9487	0.000032

	LADM		
$\alpha = 0.75$	$x = 0.25$	$x = 0.5$	Time
$t = 0,\ U$	1.2840	1.6487	0.000051
V	0.7788	0.6065	0.000029
$t = 0.1,\ U$	1.0661	1.3689	0.000040
V	0.9480	0.7383	0.000026
$t = 0.2,\ U$	0.9526	1.2231	0.000041
V	1.0846	0.8447	0.000026
$t = 0.3,\ U$	0.8764	1.1253	0.000042
V	1.2186	0.9490	0.000026
$t = 0.4,\ U$	0.8257	1.0602	0.000040
V	1.3532	1.0539	0.000026

Table 9: Example 4: Computational efficiency between TPDTM, HPM and LADM at($\alpha = 0.75$) in seconds

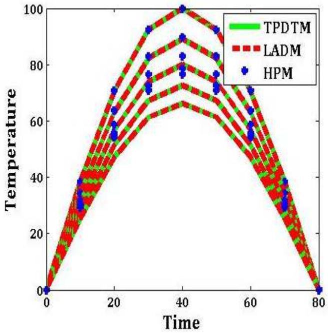

Figure 9: Example 1:Comparison result with HPM and LADM at $\alpha = 0.95$

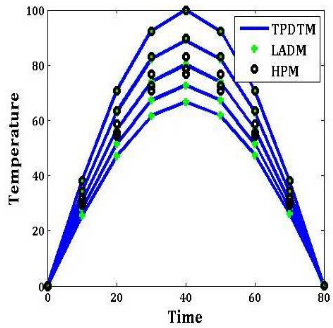

Figure 10: Example 2:Comparison result with HPM and LADM at $\alpha = 0.95$

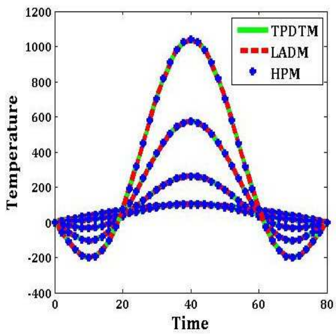

Figure 11: Example 3:Comparison result with HPM and LADM at $\alpha = 1, \beta = 0.05$

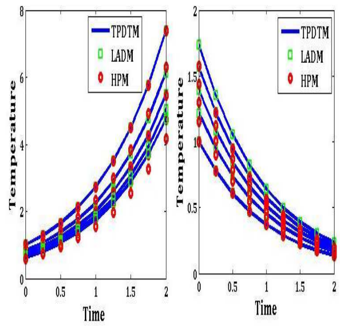

Figure 12: Example 4:Comparison result with HPM and LADM at $\alpha = 0.75$

6. Conclusion

In this paper, a hybrid technique 'Tarig projected differential transform method' is proposed to solve the nonlinear fractional partial differential equations. The TPDTM has been applied to study the heat conduction process in different types of linear and nonlinear fractional partial differential equations. The explicit series solutions are obtained for different category of fractional differential equations and studied their historical behavior. The convergences of the series solutions are well proved. The computational efficiency of the proposed combined technique is clearly illustrated. The observation shows the instability nature of the temperature distribution due to the nonlocal property of fractional derivative present in the equation. The instability nature cannot be analyzed with usual integer order ordinary differential equations or partial differential equations. The proposed hybrid technique is simple, faster and avoids complex computations of Adomian polynomials, integrals and discretization of variables.

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Nomenclature

u	Temperature function (K)
u_0	Atmospheric Temperature (K)
k	Thermal conductivity of the material (W/mK)
c_p	Specific heat of the fluid (J/kgK)
ρ	Density of the fluid (kg/m^3)
κ, A, B	Simplification constants
l	Length of the domain
α	Fractional constants $0 < \alpha \leq 1$
$M,\ \beta$	Real constants
δ	Kronecker Delta Function
U, W	Two dimensional variable functions
E	Mittag leffler function

[A].Appendix

The Tarig transform of standard functions and few basic properties are listed below.

S.No.	Time domain function	Frequency domain Function
1	$f(t) = t^n$	$n!v^{(2n+1)}$
2	$\exp(at)$	$\frac{v}{1-av^2}$
3	$\sin(at)$	$\frac{av^3}{1+a^2v^4}$
4	$\cos(at)$	$\frac{v}{1+a^2v^4}$
5	$\sinh(at)$	$\frac{av^3}{1-a^2v^4}$
6	$\cosh(at)$	$\frac{v}{1-a^2v^4}$
7	$\alpha f(t) + \beta g(t)$	$\alpha F(v) + \beta G(v)$
8	$T(f'(t))$	$\frac{1}{v^2}F(v) - \frac{1}{v}f(0)$
9	$T(f''(t))$	$\frac{1}{v^4}F(v) - \frac{1}{v^3}f(0) - \frac{1}{v^2}f'(0)$
9	$T(f^n(t))$	$\frac{1}{v^{2n}}F(v) - \sum_{i=1}^{n} v^{2(i-n)-1}f^{(i-1)}(0)$
10	$(f * g)(t)$	$vF(v)G(v)$

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