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Game Options Approach in Bankruptcy Triggering Asset Value

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ABSTRACT: In this paper, we develop a new numerical method, game theory and option pricing to compute a bankruptcy triggering asset value. we will draw our attention to determining a the numerical asset value, or price of a share, at which a bankruptcy is triggered. This paper develops and analyze a cubic spline collocation method for approximating solutions of the problem. This method converges quadratically. In addition, this article also provides with a real-life case study of the investment bank, and the optimal bankruptcy strategy in this particular case. As we will observe, the bankruptcy trigger computed in this example could have served as a good guide for predicting fall of this investment bank.

Key Words: Bankruptcy, Trigger, Game theory, Option pricing, Spline collocation method.

Contents

1. Introduction

The combination of modern financial economics and microeconomics can produce very interesting insights into the real-life on financial markets. The game theory serves here to define strategic interactions between the players (see [14]). The option pricing is used to translate uncertain future payoffs to the present value with a variety of use of its boundary conditions (see [12,1,5,2,3,15]).

Bankruptcy constitutes an important event in a firm's life and impacts many parties associated with the defaulting firm. The value of the firm's shares is reduced because of the direct and indirect costs of financial distress

The main determinant of success of a bankruptcy prediction model is the set of variables it utilizes to distinguish firms that are about to default from future solvent firms. While the list of variables that have been employed in various bankruptcy prediction models is quite lengthy, it typically excludes a measure of firms' investment opportunities. It turns out, however, that the mix of investment opportunities

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and assets in place is an important determinant of the likelihood of default. The logic is straightforward. On one hand, the more valuable a firm's growth options, the longer the shareholders are willing to wait and the higher the losses they are willing to sustain (and the more cash they are willing to inject into a struggling firm) before defaulting on their debt obligations and surrendering the firm to the debt holders. If a firm is endowed with a growth option then in the event of default its shareholders not only surrender the right to the profit flow generated by the assets in place but they also abandon the right to exercise the investment option in the future. Therefore, default is costlier for shareholders of a firm with considerable investment opportunities, and they would be willing to wait longer before making the decision to default on their debt obligations.

The lender provided funding to the borrower in order to finance investment in a project which had a finite, known maturity of T. Accordingly, the contract between lender and borrower specified a single payment from the borrower to the lender at time T, with bankruptcy occurring when the borrower was unable to meet his contractual payment obligation. Although such an analysis is valuable for an understanding of project financing, it is much less accurate to understand firm financing. Indeed, in practice, most firms are not liquidated as their debt matures. Rather, they issue new debt and keep operating as long as they can meet their contractual payment obligations. Bankruptcy usually takes place whenever firms default on promised payments on their debt. At any point in time, equity holders can decide whether they want the firm to make the promised payments or default and trigger bankruptcy. Thus, bankruptcy is a decision made endogenously by equity holders.

Consider a lender (say, a financial intermediary) and a borrower (say, a firm) that reach the following agreement: in exchange for a loan of F_0 at initial time, the borrower is to pay an instantaneous interest of $\phi D(t)dt$ to the lender, where $D(t) = D_0 e^{r^* t}$ denotes the face value of debt at time t and ϕ is the instantaneous interest rate to be effectively paid on debt. Debt is perpetual. Asset sales are prohibited (see [11,10,9]). Hence, any net cash outflows associated with interest payments must be financed by selling additional equity (see [11,9]). This setting generalizes the model in Leland [13] insofar as it distinguishes between effective interest payments $\phi D(t)dt$ and the increase in the face value of debt, which occurs at the rate r^*. Throughout, it is assumed that $r^* < r$, where r denotes the risk-free interest rate. The case $r^* < 0$ corresponds to a sinking fund provision.

Assume that the firm is liquidated if (and only if) the borrower defaults on his interest payments to the lender. If bankruptcy occurs, a fraction $0 < \alpha < 1$ of the firm's asset value is lost, leaving debt holders with $(1 - \alpha)S_B$, where S_B denotes the asset value at which bankruptcy occurs (see [7,8]). The structure of the game. In the first phase, the financing decision is made. The amount of debt, D_0, as well as the interest rates r^* and ϕ are determined, and the firm receives the fair value of the loan consistent with these parameters, F_0. Once its financing is completed, the firm chooses its investment program. We shall be concerned with two issues: underinvestment and the firm's choice of the risk of its assets, σ. Finally, equity holders choose their bankruptcy strategy S_B. If the firm goes bankrupt, its assets are liquidated and payoffs are realized, with debt holders receiving $(1-\alpha)S_B$ and equity holders nothing.

The structure of the paper is as follows. Section 2 values the firm and its different securities using option pricing theory and analyzes the last stage of the game, namely, equity holders' bankruptcy decision and demonstrates that equity holders' optimal bankruptcy choice is suboptimal both from the perspective of social welfare and from the standpoint of the lender. Section 3 is devoted to the spline collocation method for Bankruptcy triggering asset value, using a cubic spline collocation method. Next, the error bound of the spline solution is analyzed. In order to validate the theoretical results presented in this paper, we present numerical test in Section 4 to validate our methodology. Finally, a conclusion is given in Section 5.

2. Bankruptcy triggering asset value

After we have specified the game, we need to value each player's payoffs using option pricing theory, treating all the players' decision variables as parameters.We can now compute the value of lender's claim F.
The value of the borrower's assets, X, is assumed to follow the usual geometric Brownian motion

$$dX_t = \mu X_t dt + \sigma X_t dB_t$$

For the time being, assume that asset substitution is not possible, so that the parameters μ and σ are known to the lender. Since asset sales are prohibited, any net cash outflows associated with interest payments must be financed by selling additional equity. The value of the lender's claim satisfies the partial differential equation

$$\left\{\begin{array}{lcll} \mathcal{L}F(X) & = & -\phi D(t), & X \in (X_B, +\infty), \\ F(X_B) & = & (1-\alpha)X_B, & \\ F(\infty) & = & \dfrac{\phi D(t)}{r-r^*}, & \end{array}\right. \tag{2.1}$$

where X_B denotes the asset value at which bankruptcy occurs. Note that F depends on t only through the face value of debt $D(t)$, and $\mathcal{L} : C(\overline{\Omega}) \cap C^{2,1}(\Omega) \to C(\Omega)$, with $\Omega = (X_B, +\infty)$, is defined by

$$\mathcal{L} := 0.5\sigma^2 X^2 \frac{\partial^2}{\partial X^2} + rX\frac{\partial}{\partial X} + r^* D(t) X \frac{\partial}{\partial D} - rI, \tag{2.2}$$

with $\sigma(X) \geq \widetilde{\alpha} > 0$, $r(X) \geq 0$, $r^*(X) \geq 0$ on $\overline{\Omega}$, and σ, D, r, r^*, ϕ are sufficiently smooth functions.
Making the change in variables $V = \dfrac{X_t}{D(t)}$ and defining $G(V) = \dfrac{F(X)}{D(t)}$, one can show that G satisfies the ordinary differential equation

$$\begin{cases} \mathcal{L}G(V) &= 0, & V \in (V_B, +\infty), \\ G(V_B) &= (1-\alpha)V_B, \\ G(\infty) &= \dfrac{\phi}{r-r^*}, \end{cases} \tag{2.3}$$

where

$$V_B = X_B/D(t),$$
$$\mathcal{L} = \frac{1}{2}\sigma^2 V^2 \frac{\partial^2}{\partial V^2} + (r-r^*)V\frac{\partial}{\partial V} - (r-r^*)I,$$
$$r = r(V),\ r^* = r^*(V),\ \sigma^2 = \sigma^2(V).$$

Here we assume that the problem satisfies sufficient regularity and compatibility conditions which guarantee the problem has unique solution $F \in \mathcal{C}([V_B, +\infty]) \cap \mathcal{C}^{2,1}(]V_B, +\infty[)$(see [4]):

$$\left|\frac{\partial^i G(V)}{\partial V^i}\right| \leq k \text{ on } [V_B, +\infty]; \quad \text{ for } i = 1, ..., 4 \tag{2.4}$$

where k is a positive constant.

Lemma 2.1. *(see [13]): Problem (2.3) generally has no analytic solution, but if r, u, I and σ are constants, then the exact solution of problem (2.3) is :*

$$F(X) = \frac{\phi D(t)}{r-r^*}\left(1 - \left(\frac{X}{X_B}\right)^{-\gamma^*}\right) + (1-\alpha)X_B\left(\frac{X}{X_B}\right)^{-\gamma^*}. \tag{2.5}$$

with $\gamma^* = 2\dfrac{r-r^*}{\sigma^2}$.

3. Cubic spline collocation method

In this section we construct a cubic spline spline which approximates the solution G of problem (2.1), in the interval $\Omega_V = (V_B, V_F) \subset \mathbb{R}$.

We denote by $\| . \|$ the Euclidean norm on $\mathbb{R}^{n+1}$, $\| . \|_\infty$ the uniform norm.
Let $\Theta = \{V_B = V_{-3} = V_{-2} = V_{-1} = V_0 < V_1 < \cdots < V_{n-1} < V_n = V_{n+1} = V_{n+2} = V_{n+3} = V_F\}$ be a subdivision of the interval Ω_X. Without loss of generality, we put $V_j = jh$, where $0 \leq j \leq n$ and $h = \dfrac{V_B + V_F}{n}$. Denote by $\mathbb{S}_4(\Omega_V, \Theta) = \mathbb{P}_3^2(\Omega_V, \Theta)$ the space of piecewise polynomials of degree 3 over the subdivision Θ and of class C^2 everywhere on Ω_V. Let B_i, $i = -3, \cdots, n-1$, be the B-splines of degree 3 associated with Θ. These B-splines are positives and form a basis of the space $S_4(\Omega_V, \Theta)$.

Proposition 3.1. *Let G be the solution of problem (2.1). Then, there exists a unique cubic spline interpolant $S_c \in \mathbb{S}_4(\Omega_X, \Theta)$ of G which satisfies:*

$$S_c(\tau_j) = G(\tau_j), \quad j = 0, \cdots, n+2,$$

where $\tau_0 = V_0$, $\tau_i = \dfrac{V_{j-1} + V_j}{2}$, $j = 1, \cdots, n$, $\tau_{n+1} = V_{n-1}$ *and* $\tau_{n+2} = V_n$.

Proof: Using the Schoenberg-Whitney theorem (see [6]), it is easy to see that there exits a unique cubic spline which interpolates G at the points τ_i, $i = 0, \cdots, n+2$. $\square$

If we put $S_c = \sum_{i=-3}^{n-1} c_i B_i$, then by using the boundary conditions of problem (2.1) we obtain $c_{-3} = S_c(0) = G(V_B) = (1-\alpha)V_B \equiv g_B$ and $c_{n-1} = S_c(V_F) = \frac{\phi}{r-r^*} \equiv g_F$. Hence

$$S_c = S + g_B B_{-3} + g_F B_{n-1} \text{ with } S = \sum_{i=-2}^{n-2} c_i B_i.$$

Furthermore, since the interpolation with splines of degree d gives uniform norm errors of order $O(h^{d+1})$ for the interpolant, and of order $O(h^{d+1-r})$ for the rth derivative of the interpolant (see [2], for instance), then for any $G \in C^4(\Omega_V)$ we have

$$\mathcal{L}S(\tau_j) = \psi(\tau_j) + O(h^2),\ j = 1, ..., n+1, \tag{3.1}$$

where

$$\psi(\tau_j) \quad = \quad -\mathcal{L}(g_B B_{-3} + g_F . B_{n-1})(\tau_j), \quad j = 1, \ldots, n+1.$$

The cubic spline collocation method, that we present in this paper, constructs numerically a cubic spline $\widetilde{S_c} = \sum_{i=-3}^{n-1} \widetilde{c}_i B_i$ which satisfies the equation of problem (2.1) at the points τ_j, $j = 0, \ldots, n+2$. It is easy to see that $\widetilde{c}_{-3} = g_B$ and $\widetilde{c}_{n-1} = g_F$, and the coefficients $\widetilde{c}_i$, $i = -2, \ldots, n-1$ satisfy the following:

$$\mathcal{L}\widetilde{S}(\tau_j) = \psi(\tau_j),\ j = 1, ..., n+1. \tag{3.2}$$

Taking $C = [c_{-2}, \ldots, c_{n-2}]^T$ and $\widetilde{C} = [\widetilde{c}_{-2}, \ldots, \widetilde{c}_{n-2}]^T$, and using equations (3.1) and (3.2), we get:

$$(PA_h^{(2)} + QA_h^{(1)} + LA_h^{(0)})C \quad = \quad \Psi + E, \tag{3.3}$$

$$(PA_h^{(2)} + QA_h^{(1)} + LA_h^{(0)})\widetilde{C} \quad = \quad \Psi, \tag{3.4}$$

with:
$\Psi = [\psi_1, \ldots, \psi_{n+1}]^T$, and $\psi_j = \psi(\tau_j)$,

$E = [O(h^2), \ldots, O(h^2)]^T \in \mathbb{R}^{n+1}$,

$P = (diag\ 0.5\sigma^2(\tau_j)\tau_j^2)_{1 \le j \le n+1}$,

$Q = (diag(r(\tau_j) - r^*(\tau_j))\tau_j)_{1 \le j \le n+1}$,

$L = (diag(-(r(\tau_j) - r^*(\tau_j)))_{1 \le j \le n+1}$,

$A_h^{(k)} = (B_{-3+l}^{(k)}(\tau_j))_{1 \le j, l \le n+1}, \quad k = 0, 1, 2.$

It is well known that $A_h^{(k)} = \dfrac{1}{h^k} A_k$ for $k = 0, 1, 2$ where matrices A_0, A_1 and A_2 are independent of h, with the matrix A_2 is invertible (See [5], for instance).

We deduce that (3.3) and (3.4) can be written also in the following form

$$PA_2 \left(I + A_2^{-1} P^{-1} (hQA_1 + h^2 LA_0)\right) C = h^2 \Psi + h^2 E \tag{3.5}$$

$$PA_2 \left(I + A_2^{-1} P^{-1} (hQA_1 + h^2 LA_0)\right) \widetilde{C} = h^2 \Psi, \tag{3.6}$$

In order to determine the bound of $\| C - \widetilde{C} \|$, we need the following remark.

Remark 3.1. For a small real h such that

$$\|A_2^{-1} P^{-1}\|_\infty \left(h\|Q\|_\infty \|A_1\|_\infty + h^2 \|L\|_\infty \|A_0\|_\infty\right) < 1,$$

the matrix $\left(I + A_2^{-1} P^{-1} (hQA_1 + h^2 LA_0)\right)^{-1}$ exists, and

$$\| \left(I + A_2^{-1} P^{-1} (hQA_1 + h^2 LA_0)\right)^{-1} \|_\infty < \frac{1}{1 - (h\|Q\|_\infty \|A_1\|_\infty + h^2 \|L\|_\infty \|A_0\|_\infty)}.$$

Hence, in this case, there exists a unique cubic spline that approximates the exact solution u of problem (2.3).

Proposition 3.2. *If we choose the real h such that*

$$\|A_2^{-1} P^{-1}\|_\infty \left(h\|Q\|_\infty \|A_1\|_\infty + h^2 \|L\|_\infty \|A_0\|_\infty\right) \leq \frac{1}{2}, \tag{3.7}$$

then there exists a constant cte that depends only on the functions p, q, l and f such that

$$\|C - \widetilde{C}\| \leq cte.h^2. \tag{3.8}$$

Proof: from relations (3.5) and (3.6) we have

$$C - \widetilde{C} = h^2 (PA_2)^{-1} \left(I + (PA_2)^{-1} (hQA_1 + h^2 LA_0)\right)^{-1} E.$$

Since $E = O(h^2)$, then there exists a constant K_1 such that $\|E\| \leq K_1 h^2$. This implies that

$$\|C - \widetilde{C}\| \leq K_1 \frac{h^2 \|(A_2 P)^{-1}\|_\infty}{1 - h^2 \|(A_2 P)^{-1}\|_\infty (h^{-1} \|Q\|_\infty \|A_1\|_\infty + \|L\|_\infty \|A_0\|_\infty)} h^2$$

Using the inequality $\|A_2^{-1} P^{-1}\|_\infty \left(h\|Q\|_\infty \|A_1\|_\infty + h^2 \|L\|_\infty \|A_0\|_\infty\right) \leq \frac{1}{2}$ and $0 < h \leq 1$, we deduce that

$$\|C - \widetilde{C}\| \leq K_1 \frac{h^2 \|(A_2 P)^{-1}\|_\infty}{\|Q\|_\infty \|A_1\|_\infty + \|L\|_\infty \|A_0\|_\infty)} h^2$$

Finally, we deduce that
$$\|C - \widetilde{C}\| \leq cte.h^2.$$
□

Now, we are in position to prove the main theorem of our work.

Theorem 3.3. *The spline approximation $\widetilde{S}$ converges quadratically to the exact solution u of problem (2.1), i.e., $\|G - \widetilde{S}\|_\infty = O(h^2)$.*

Proof: It is well known that $\|G - S\|_\infty = O(h^4)$ (see [6]), so $\|G - S\|_\infty \leq Kh^4$, where K is a positive constant. On the other hand we have

$$S(X) - \widetilde{S}(X) = \sum_{i=-2}^{n-2} (c_i - \widetilde{c}_i) B_i(X).$$

Therefore, by using (3.8) and $\sum_{i=-2}^{n-2} B_i(X) \leq 1$, we get

$$|S(X) - \widetilde{S}(X)| \leq \|C - \widetilde{C}\| \sum_{i=-2}^{n-2} B_i(X) \leq \|C - \widetilde{C}\| \leq cte.h^2.$$

Since $\|X - \widetilde{S}\|_\infty \leq \|X - S\|_\infty + \|S - \widetilde{S}\|_\infty$, we deduce the stated result. □

4. Numerical example

In this section we verify the obtained theoretical results in the previous section. If the exact solution is known, then the maximum error E^{max} can be calculated as:
$$E^{max} = \max_{V \in [V_B, V_F]} \mid S^N(V) - G(V) \mid .$$
Otherwise it can be estimated by the following double mesh principle:
$$E_N^{max} = \max_{V \in [V_B, V_F]} \mid S^N(V) - S^{2N}(V) \mid,$$
where $S^N(V)$ is the numerical solution on the $N + 1$ grids in space, and $S^{2N}(V)$ is the numerical solution on the $2N + 1$ grids in space.

We present an example to better illustrate the use of the cubic spline collocation approach and the proposed pricing methodology in concrete situations.

We consider the same test-case presented in [13], in which the model parameters and the option's data are chosen as in Table 1.

Table 2 shows values of the maximum error (max_error) obtained in our numerical experiments on the interval $[0, 100]$.

Table 1: Parameters and data.

θ	σ	ϕ	$r-r^*$	X_B	X_F	D
1/3	0.2	0.05	0.05	0	100	90

Table 2: Numerical results for Example.

n	10^2	2.10^2	4.10^2
our max_error	8.8039×10^{-5}	1.1244×10^{-6}	3.9151×10^{-10}

Assume that the lender would be entitled to choose the bankruptcy-triggering asset value X_B. Then, he would either set it at zero (this would make his claim riskless), or choose X_B as high as possible, that is, force bankruptcy immediately, or as soon as the condition $X_B > 112.5$ is met.
It is worth contrasting these bankruptcy strategies with the socially optimal bankruptcy strategy. The socially optimal bankruptcy trigger is the one that maximizes the overall value of the firm. the bankruptcy trigger that maximizes overall firm value W is $X_B = 0$.

5. Conclusion

In this paper, we developed a new numerical method, game theory and option pricing to compute a bankruptcy triggering asset value. We have determined a the numerical asset value, or price of a share, at which a bankruptcy is triggered. We have developed and analyzed a cubic spline collocation method for approximating solutions of the problem. We have provided an error estimate of order $O(h^2)$ with respect to the maximum norm: $\| \, . \, \|_\infty$. The computational results show that the proposed numerical method is an efficient alternative method and we have provided reader with a real-life case study of the investment bank, and the optimal bankruptcy strategy in this particular case. As have observed, the bankruptcy trigger computed in this example could have served as a good guide for predicting fall of this investment bank.

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