Bol. Soc. Paran. Mat. (3s.) **v. 38** 3 (2020): 177–191.
©SPM –ISSN-2175-1188 ON LINE ISSN-00378712 IN PRESS
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v38i3.34498

Error Analysis of the Numerical Solution of the Benjamin-Bona-Mahony-Burgers Equation

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ABSTRACT: In this paper, the B-spline collocation scheme is implemented to find numerical solution of the nonlinear Benjamin-Bona-Mahony-Burgers equation. The method is based on collocation of quintic B-spline. We show that the method is unconditionally stable. Also the convergence of the method is proved. The method is applied on some test examples, and the numerical results have been compared with the analytical solutions. The L_∞ and L_2 in the solutions show the efficiency of the method computationally.

Key Words: Benjamin-Bona-Mahony-Burgers, B-spline, Collocation method, Stability, Convergence analysis, Finite difference.

Contents

1. Introduction

In [1], a generalized Benjamin-Bona-Mahony-Burgers equation has been consider as follows equation

$$u_t - u_{xxt} - \alpha u_{xx} + \beta u_x + \big(g(u)\big)_x = 0, \tag{1.1}$$

where α is a positive constant, $\beta \in R$ and $g(u)$ is a $\mathcal{C}^2$-smooth nonlinear function. Benjamin, Bona and Mahony proposed equation (1.1) as an alternative regularized long-wave equation with the same parameters. In this paper we consider $g(u) = \frac{u^2}{2}$ and we find the Benjamin- Bona- Mahony- Burgers (BBMB) equation as

$$u_t - u_{xxt} - \alpha u_{xx} + \beta u_x + u u_x = 0, \qquad x \in [a, b], \quad t \in [0, T], \tag{1.2}$$

2010 *Mathematics Subject Classification:* 65M70, 41A15, 65M15.
Submitted December 07, 2016. Published October 30, 2017

with the initial condition and boundary conditions

$$u(x,0) = f(x), \quad x \in [a,b], \tag{1.3}$$

$$u(a,t) = g_0(t), \ u(b,t) = g_1(t), \tag{1.4}$$

$$u_x(a,t) = u_x(b,t) = 0, \tag{1.5}$$

$$u_{xx}(a,t) = u_{xx}(b,t) = 0. \tag{1.6}$$

For $\alpha = 0$, (1.2) is called the Benjamin-Bona-Mahony (BBM) equation. BBMB equations play a dominant role in many branches of science and engineering [2]. In recent years, many different methods have been used to estimate the solution of the BBMB equation, for example, we note, Galerkin methods [3], The jacobi elliptic function solutions [4], Approximate wave solutions [5] and see [6,7]. Also a class of Benjamin-Bona-Mahony-initial value problems are studied in [8].
The layout of this article is as follows. In Section 2, we present a finite difference approximation to discretize the (1.2) in time variable and we applied quintic B-spline collocation method to solve the problem. In Section 3, the stability analysis of the method is given. In Section 4 we derive convergence of the B-spline collocation method. In Section 5, some examples have been conducted in order to validate the theoretical results. A summary is given at the end of the paper in Section 6.

2. Solution of BBMB equations via Quintic B-spline

The region $a \le x \le b$ partitioned into a mesh of uniform length $h = \frac{b-a}{N}$, by the knots $x_i = a + ih$ where $i = 0, 1, 2, ...,$N and $a = x_0 < x_1 < \ldots < x_{N-1} < x_N = b$. We use the following finite difference approximation to discretize the time variable

$$\frac{\delta t}{\Delta t(1+\frac{1}{2}\delta t)}(u^n - u^n_{xx}) - \alpha u^n_{xx} + \beta u^n_x + u^n u^n_x = 0, \tag{2.1}$$

where Δt is the time setp, $u^n(x) := u(x, n\Delta t)$ and $\delta t u^n := u^{n+1} - u^n$. This finite difference scheme is used in [9]. Rearranging the term and simplifying we get

$$u^{n+1} + \frac{\beta\Delta t}{2}u^{n+1}_x + (-1 - \frac{\alpha\Delta t}{2})u^{n+1}_{xx} + \frac{\Delta t}{2}(uu_x)^{n+1} = \Phi^n, \tag{2.2}$$

where

$$\Phi^n(x) := u^n(x) - \frac{\beta\Delta t}{2}u^n_x(x) + \big(-1 + \frac{\alpha\Delta t}{2}\big)u^n_{xx}(x) - \frac{\Delta t}{2}\big(uu_x\big)^n(x). \tag{2.3}$$

To linearized the non-linear term $(uu_x)^{n+1}$ in (2.2) we can use the Taylor expansions. We can get

$$\begin{aligned}
(uu_x)^{n+1} &= (uu_x)^n + \Delta t(uu_x)^n_t + \Delta t^2(uu_x)^n_{tt} + \mathcal{O}(\Delta t^3)\\
&= (uu_x)^n + \Delta t\Big(\frac{u^{n+1}-u^n}{\Delta t}u^n_x + \frac{u^{n+1}_x - u^n_x}{\Delta t}u^n\Big)\\
&+ \Delta t^2\Big(\frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2}u^n_x + 2\frac{u^{n+1}-u^n}{\Delta t}\frac{u^{n+1}_x - u^n_x}{\Delta t}\\
&+ \frac{u^{n+1}_x - 2u^n_x + u^{n-1}_x}{\Delta t^2}u^n\Big) + \mathcal{O}(\Delta t^2),
\end{aligned}$$

thus we have

$$(uu_x)^{n+1} = 3(uu_x)^n - u^{n-1}u_x^n - u^n u_x^{n-1} + \mathcal{O}(\Delta t^2). \tag{2.4}$$

So (2.2) can be rewritten as

$$u^{n+1} + \frac{\beta\Delta t}{2}u_x^{n+1} + \big(-1 - \frac{\alpha\Delta t}{2}\big)u_{xx}^{n+1} = \Lambda^n, \tag{2.5}$$

where

$$\begin{aligned}\Lambda^n(x) :=& u^n(x) + \frac{\Delta t}{2}\big(u^n(x)u_x^{n-1}(x) + u^{n-1}(x)u_x^n(x)\big) - 2\Delta t\big(uu_x\big)^n(x)\\ &- \frac{\beta\Delta t}{2}u_x^n(x) + \big(-1 + \frac{\alpha\Delta t}{2}\big)u_{xx}^n(x).\end{aligned} \tag{2.6}$$

We define the quintic B-spline basis functions at knots, by the following relationships [10,11]

$$B_i(x) = \frac{1}{h^5}\begin{cases}(x - x_{i-3})^5, & x \in [x_{i-3}, x_{i-2}),\\ (x - x_{i-3})^5 - 6(x - x_{i-2})^5, & x \in [x_{i-2}, x_{i-1}),\\ (x - x_{i-3})^5 - 6(x - x_{i-2})^5 + 15(x - x_{i-1})^5, & x \in [x_{i-1}, x_i),\\ (x_{i+3} - x)^5 - 6(x_{i+2} - x)^5 + 15(x_{i+1} - x)^5, & x \in [x_i, x_{i+1}),\\ (x_{i+3} - x)^5 - 6(x_{i+2} - x)^5, & x \in [x_{i+1}, x_{i+2}),\\ (x_{i+3} - x)^5, & x \in [x_{i+2}, x_{i+3}).\end{cases} \tag{2.7}$$

To continue we define the approximation for u(x,t) as

$$U(x,t) = \sum_{i=-2}^{n+2} c_i(t)B_i(x), \tag{2.8}$$

where $B_i(x)$ are the quintic B-spline basis functions, and $c_i(t)$ are time-dependent quantities. We can determine $c_i(t)$ from boundary conditions and collocation form of the differential equations. We calculate U, $U^{'}$ and $U^{''}$ at node points as

$$u(x_i,t_n) \approx U_i^n := U(x_i,t_n) = c_{i+2}^n + 26c_{i+1}^n + 66c_i^n + 26c_{i-1}^n + c_{i-2}^n, \tag{2.9}$$

$$u_x(x_i,t_n) \approx (U_i')^n := U_x(x_i,t_n) = \frac{5}{h}(c_{i+2}^n + 10c_{i+1}^n - 10c_{i-1}^n - c_{i-2}^n), \tag{2.10}$$

$$u_{xx}(x_i,t_n) \approx (U_i'')^n := U_{xx}(x_i,t_n) = \frac{20}{h^2}(c_{i+2}^n + 2c_{i+1}^n - 6c_i^n + 2c_{i-1}^n + c_{i-2}^n), \tag{2.11}$$

where $c_i^n := c_i(t_n)$. Substituting the approximate solution U for u and using (2.5) and (2.9)-(2.11) at the knots we get

$$\dot{a}c_{i+2}^{n+1} + \dot{b}c_{i+1}^{n+1} + \dot{c}c_i^{n+1} + \dot{d}c_{i-1}^{n+1} + \dot{e}c_{i-2}^{n+1} = \Psi_i^n, \ i = 0, 1, \ldots, N, \tag{2.12}$$

where

$$\begin{aligned}\Psi_i^n \ :=\ & U_i^n + \frac{\Delta t}{2}\big(U_i^n(U_i')^{n-1} + U_i^{n-1}(U_i')^n\big) - 2\Delta t(U_iU_i')^n\\ &-\frac{\beta\Delta t}{2}(U_i')^n + (-1 + \frac{\alpha\Delta t}{2})(U_i'')^n,\end{aligned}$$

and
$\dot{a} = 1 + \frac{5\beta\Delta t}{2h} + \frac{20}{h^2}(-1 - \frac{\alpha\Delta t}{2})$, $\dot{b} = 26 + \frac{25\beta\Delta t}{h} + \frac{40}{h^2}(-1 - \frac{\alpha\Delta t}{2})$,
$\dot{c} = 66 - \frac{120}{h^2}(-1 - \frac{\alpha\Delta t}{2})$, $\dot{d} = 26 - \frac{25\beta\Delta t}{h} + \frac{40}{h^2}(-1 - \frac{\alpha\Delta t}{2})$,
$\dot{e} = 1 - \frac{5\beta\Delta t}{2h} + \frac{20}{h^2}(-1 - \frac{\alpha\Delta t}{2})$.

The system (2.12) consists of $N+1$ linear equations in $N+5$ unknowns $\{c_{-2}^{n+1}, \dots, c_{N+2}^{n+1}\}$. To obtain a unique solution for $\{c_{-2}^{n+1}, \dots, c_{N+2}^{n+1}\}$, we must use the boundary conditions. From (1.4)-(1.5) and (2.8), we can write

$$\begin{aligned}
c_{-1}^{n+1} &= \frac{g_0}{16} - \frac{1}{8}c_2^{n+1} - \frac{9}{4}c_1^{n+1} - \frac{33}{8}c_0^{n+1},\\
c_{-2}^{n+1} &= \frac{-5g_0}{8} + \frac{9}{4}c_2^{n+1} + \frac{65}{2}c_1^{n+1} + \frac{165}{4}c_0^{n+1},\\
c_{N+1}^{n+1} &= \frac{g_1}{16} - \frac{1}{8}c_{N-2}^{n+1} - \frac{9}{4}c_{N-1}^{n+1} - \frac{33}{8}c_N^{n+1},\\
c_{N+2}^{n+1} &= \frac{-5g_1}{8} + \frac{9}{4}c_{N-2}^{n+1} + \frac{65}{2}c_{N-1}^{n+1} + \frac{165}{4}c_N^{n+1}.
\end{aligned}$$

Then we write the last system in the matrix form

$$AC = D, \tag{2.13}$$

where

$$A = (A_1 \ A_2) \tag{2.14}$$

where,

$$A_1 = \begin{pmatrix}
\dot{a} - \frac{33\dot{d}}{8} + \frac{165\dot{e}}{4} & \dot{b} - \frac{9\dot{d}}{4} + \frac{65\dot{e}}{2} & \dot{a} - \frac{\dot{d}}{8} + \frac{9\dot{e}}{4}\\
\dot{d} - \frac{33\dot{e}}{8} & \dot{c} - \frac{9\dot{e}}{4} & \dot{b} - \frac{\dot{e}}{8}\\
\dot{e} & \dot{d} & \dot{c}\\
 & \ddots & \ddots\\
0 & \dots & \dot{e}\\
0 & \dots & 0\\
0 & \dots & 0
\end{pmatrix}$$

and

$$A_2 = \begin{pmatrix}
0 & 0 & \dots & 0\\
\dot{a} & 0 & \dots & 0\\
\dot{b} & \dot{a} & \dots & 0\\
\ddots & \ddots & & \vdots\\
\dot{d} & \dot{c} & \dot{b} & \dot{a}\\
\dot{e} & \dot{d} - \frac{\dot{a}}{8} & \dot{c} - \frac{9\dot{a}}{4} & \dot{b} - \frac{33\dot{a}}{8}\\
0 & \dot{e} - \frac{\dot{b}}{8} + \frac{9\dot{a}}{4} & \dot{d} - \frac{9\dot{b}}{4} + \frac{65\dot{a}}{2} & \dot{c} - \frac{33\dot{b}}{8} + \frac{165\dot{a}}{4}
\end{pmatrix}.$$

$$C = \left(c_0^{n+1}, c_1^{n+1}, \dots, c_{N-1}^{n+1}, c_N^{n+1}\right)^T, \tag{2.15}$$

$$\begin{aligned} D =&\Big(\Psi_0^n + (-\frac{\dot{d}}{16} + \frac{5\dot{e}}{8})g_0, \Psi_1^n - \frac{\dot{e}}{16}g_0, \Psi_2^n, \dots, \\ &\Psi_{N-3}^n, \Psi_{N-1}^n - \frac{\dot{a}}{16}g_1, \Psi_N^n + (\frac{5\dot{a}}{8} - \frac{\dot{b}}{16})g_1\Big)^T. \end{aligned} \tag{2.16}$$

The above system of equations given in (2.13) has been solved using the computer algebra system Mathematica-9.

To start any calculate, we must know $U^1(x)$. We assume that

$$U^1(x) = \sum_{i=-2}^{N+2} c_i^1 B_i(x).$$

By using (2.2) and (2.9)-(2.11) we can write

$$\begin{aligned} &\dot{a}c_{i+2}^1 + \dot{b}c_{i+1}^1 + \dot{c}c_i^1 + dc_{i-1}^1 + \dot{e}c_{i-2}^1 + \frac{5\Delta t}{2h}\Big(c_{i+2}^1 + 26c_{i+1}^1 + 66c_i^1 + 26c_{i-1}^1 \\ &+ c_{i-2}^1\Big)\Big(c_{i+2}^1 + 10c_{i+1}^1 - 10c_{i-1}^1 - c_{i-2}^1\Big) = \Phi^0(x_i),\ i = 0, \dots, N. \end{aligned} \tag{2.17}$$

The nonlinear system (2.17) consists of $N+1$ equations in $N+5$ unknowns $C^1 = \{c_{-2}^1, \dots, c_{N+2}^1\}$. To obtain a unique solution for C^1, similar to the above discussion, we use the boundary conditions. From (1.4)-(1.5), we can write

$$c_2^1 + 26c_1^1 + 66c_0^1 + 26c_{-1}^1 + c_{-2}^1 = g_0(\Delta t), \tag{2.18}$$

$$c_2^1 + 10c_1^1 - 10c_{-1}^1 - c_{-2}^1 = 0, \tag{2.19}$$

$$c_{N+2}^1 + 26c_{N+1}^1 + 66c_N^1 + 26c_{N-1}^1 + c_{N-2}^1 = g_1(\Delta t), \tag{2.20}$$

$$c_{N+2}^1 + 10c_{N+1}^1 - 10c_{N-1}^1 - c_{N-2}^1 = 0. \tag{2.21}$$

Then we obtain the nonlinear system consists of $N+5$ equations in $N+5$ unknowns. This system is solved by the computer algebra system Mathematica-9.

3. Stability analysis

In this section, we discuss the stability of the quintic B-spline approximation (2.1) using the Von Numann method [12,13]. According to the Von-Neumann method, we have

$$c_i^n = \xi^n exp(\lambda khi), \qquad \lambda^2 = -1, \tag{3.1}$$

where k is the mode number and h is the element size. To apply this method, we have linearized the nonlinear term uu_x by consider u as a constant ϖ in equation

(2.1). We obtain the equation:

$$\breve{a}_1 c_{i+2}^{n+1} + \breve{b}_1 c_{i+1}^{n+1} + \breve{c}_1 c_i^{n+1} + \breve{d}_1 c_{i-1}^{n+1} + \breve{e}_1 c_{i-2}^{n+1} = \breve{a}_2 c_{i+2}^{n} + \breve{b}_2 c_{i+1}^{n} + \breve{c}_2 c_i^{n} + \breve{d}_2 c_{i-1}^{n} + \breve{e}_2 c_{i-2}^{n}, \tag{3.2}$$

where
$\breve{a}_1 = 1 + \frac{5\Delta t}{2h}(\beta+\varpi) + \frac{20}{h^2}(-1-\frac{\alpha\Delta t}{2}),\ \breve{a}_2 = 1 - \frac{5\Delta t}{2h}(\beta+\varpi) + \frac{20}{h^2}(-1+\frac{\alpha\Delta t}{2}),$
$\breve{b}_1 = 26 + \frac{25\Delta t}{h}(\beta+\varpi) + \frac{40}{h^2}(-1-\frac{\alpha\Delta t}{2}),\ \breve{b}_2 = 26 - \frac{25\Delta t}{h}(\beta+\varpi) + \frac{40}{h^2}(-1+\frac{\alpha\Delta t}{2}),$
$\breve{c}_1 = 66 - \frac{120}{h^2}(-1-\frac{\alpha\Delta t}{2}),\ \breve{c}_2 = 66 - \frac{120}{h^2}(-1+\frac{\alpha\Delta t}{2}),$
$\breve{d}_1 = 26 - \frac{25\Delta t}{h}(\beta+\varpi) + \frac{40}{h^2}(-1-\frac{\alpha\Delta t}{2}),\ \breve{d}_2 = 26 + \frac{25\Delta t}{h}(\beta+\varpi) + \frac{40}{h^2}(-1+\frac{\alpha\Delta t}{2}),$
$\breve{e}_1 = 1 - \frac{5\Delta t}{2h}(\beta+\varpi) + \frac{20}{h^2}(-1-\frac{\alpha\Delta t}{2}),\ \breve{e}_2 = 1 + \frac{5\Delta t}{2h}(\beta+\varpi) + \frac{20}{h^2}(-1+\frac{\alpha\Delta t}{2}).$

With substituting $c_i^n = \xi^n exp(\lambda khi)$ into linearized form (3.2) and simplifying , we obtain

$$\xi = \frac{X_1 + iY}{X_2 - iY}, \tag{3.3}$$

where

$$\begin{aligned} X_1 =& \Big(2 - \frac{40}{h^2}\Big)\cos(2\phi) + \Big(52 - \frac{80}{h^2}\Big)\cos(\phi) + 66 \\ &+ \frac{120}{h^2} - \Big(\frac{20\alpha\Delta t}{h^2}\cos(2\phi) + \frac{40\alpha\Delta t}{h^2}\cos(2\phi) - \frac{60\alpha\Delta t}{h^2}\Big), \end{aligned}$$

$$\begin{aligned} X_2 =& \Big(2 - \frac{40}{h^2}\Big)\cos(2\phi) + \Big(52 - \frac{80}{h^2}\Big)\cos(\phi) + 66 \\ &+ \frac{120}{h^2} + \Big(\frac{20\alpha\Delta t}{h^2}\cos(2\phi) + \frac{40\alpha\Delta t}{h^2}\cos(2\phi) - \frac{60\alpha\Delta t}{h^2}\Big), \end{aligned}$$

$$Y = \Big(\frac{5\Delta t}{h}(\beta+\varpi)\Big)\sin(2\phi) + \Big(\frac{50\Delta t}{h}(\beta+\varpi)\Big)\sin(\phi).$$

From (3.3), we can write

$$|\xi|^2 = \xi\bar{\xi} = \frac{X_1^2 + Y^2}{X_2^2 + Y^2}. \tag{3.4}$$

We note that $X_1 \le X_2$, so $|\xi| \le 1$. This implies $\mid \xi \mid \le 1$. Therefore the linearized numerical scheme for BBMB equation is unconditionally stable.

4. Convergence analysis

In this section we study the convergence of the quintic B-spline collocation method has been given in Section 2.

Theorem 4.1. *Suppose that $f(x) \in \mathcal{C}^5[a,b]$. Then for the unique quintic spline $S(x)$ associated with f, we have*

$$\| f^{(j)} - S^{(j)} \|_\infty \leq K_j \omega_5(h) h^{4-j} \ , \ j = 0, 1, 2, 3, \tag{4.1}$$

where $\omega_5(h)$ denotes the modulus of continuity of $f^{(5)}$ and the coefficients λ_j are independent of f and h.

Proof: For the proof see [14]. □

Remark 4.2. *By using Theorem 4.1 and definition of the modulus of continuity, we can say that if $|f^{(5)}(x)| \leq L$, we can write (4.1) as*

$$\| f^{(j)} - S^{(j)} \|_\infty \leq \lambda_j L h^{4-j} \ , \ j = 0, 1, 2, 3. \tag{4.2}$$

Lemma 4.3. *For the B-splines $\{B_{-2}, \cdots, B_{N+2}\}$ we have the following inequality:*

$$\Big| \sum_{i=-2}^{N+2} B_i(x) \Big| \leq 186, \quad (a \leq x \leq b). \tag{4.3}$$

Proof: From the real analysis we have

$$\Big| \sum_{i=-2}^{N+2} B_i(x) \Big| \leq \sum_{i=-2}^{N+2} \Big| B_i(x) \Big|,$$

if $x = x_i$, $i = 1, \ldots, N$, then, we have

$$\Big| \sum_{i=-2}^{N+2} B_i(x) \Big| = 120 \leq 186,$$

and if $x_{i-1} \leq x \leq x_i$, then, we can write

$$\begin{aligned} \Big| \sum_{i=-2}^{N+2} B_i(x) \Big| \leq & \mid B_{i-3}(x) \mid + \mid B_{i-2}(x) \mid + \mid B_{i-1}(x) \mid \\ & + \mid B_i(x) \mid + \mid B_{i+1}(x) \mid + \mid B_{i+2}(x) \mid \\ & \leq 1 + 26 + 66 + 66 + 26 + 1 \leq 186. \end{aligned}$$

□

Theorem 4.4. *Suppose that $u(x,t)$ be the exact solution of (1.2) and $u(x,t) \in \mathcal{C}^5[a,b]$ also $|\frac{\partial^5 u(x,t)}{\partial x^5}| \leq L$ and $U(x,t)$ be the numerical approximation by our methods, then we can write*

$$\| u(x,t) - U(x,t) \|_\infty = \mathcal{O}(h^2 + \Delta t^2). \tag{4.4}$$

Proof: At the $(n+1)$th time step, we assume that S^* be the unique spline interpolate to the exact solution u of (1.2)-(1.6) given by

$$S^*(x) = \sum_{i=-2}^{N+2} c^* B_i(x). \tag{4.5}$$

We note that matrix A is strictly diagonally dominant matrix. Let η_i, $(1 \leq i \leq N+1)$ be the summation of the ith row of the matrix A. From the theory of matrices we know that

$$\sum_{i=1}^{N+1} a_{ki}^{-1} \eta_i = 1, \tag{4.6}$$

where a_{ki}^{-1} are the elements of A^{-1}. As a result we can write

$$\|A^{-1}\|_\infty = \sum_{i=1}^{N+1} |a_{ki}^{-1}| \leq \frac{1}{\min_{1\leq i\leq N+1} \eta_i} \leq \frac{1}{G}, \tag{4.7}$$

where G is is constant. We substituting $S^*(x)$ in (2.5), we get

$$AC^* = D^*. \tag{4.8}$$

Subtracting (2.13), (4.8) and taking the infinity norm, we can write

$$\|C^* - C\|_\infty \leq \|A^{-1}\|_\infty \|D^* - D\|_\infty. \tag{4.9}$$

By using (4.2), we get the result as

$$\begin{aligned} |\Psi_i^* - \Psi_i| &\leq |\ S^*(x_i) - U(x_i)\ | + |\ \frac{\beta\Delta t}{2}\big(S^{*\prime}(x_i) - U'(x_i)\big)\ | \\ &\quad + |\ (-1 - \frac{\alpha\Delta t^2}{2})\big(S^{*\prime\prime}(x_i) - U''(x_i)\big)\ | \\ &\leq \lambda_0 L h^4 + |\ \frac{\beta\Delta t}{2}|\lambda_1 L h^3 + |\ -1 - \frac{\alpha\Delta t^2}{2}\ |\ \lambda_2 L h^2. \end{aligned} \tag{4.10}$$

From (4.10), we get

$$\|\ D^* - D\ \|_\infty \leq M_1 h^2, \tag{4.11}$$

where $M_1 = \lambda_0 L h^2 + |\frac{\beta\Delta t}{2}|\lambda_1 L h + |1 + \frac{\alpha\Delta t}{2}|\lambda_2 L$. Thus by taking norm and using Lemma 4.3, (4.7), (4.9), (4.11) we obtain

$$\|\ S^*(x) - U(x)\ \|_\infty = \|\ \sum_{i=-2}^{N+2} (c_i^* - c_i) B_i(x)\ \|_\infty \leq \Big|\sum_{i=-2}^{N+2} B_i(x)\Big|\ \|\ C^* - C\ \|_\infty \leq 186 M_2 h^2, \tag{4.12}$$

where $M_2 = \frac{M_1}{G}$ is constant. Also from Theorem 4.1 we can write

$$|\ u(x) - S^*(x)\ | \leq \lambda_0 L h^4, \tag{4.13}$$

and therefore with helping (4.12) and (4.13), we get

$$\begin{aligned} \| u(x) - U(x) \|_\infty &\leq \| u(x) - S^*(x) \|_\infty + \| S^*(x) - U(x) \|_\infty \\ &\leq \lambda_0 L h^4 + 186 M_2 h^2 \\ &= \gamma h^2, \end{aligned} \tag{4.14}$$

where $\gamma = \lambda_0 L h^2 + 186 M_2$.

In the next step, suppose that $\varepsilon_i = u(x, t_i) - U(t_i)$ be the local truncation error for (2.1) at the ith level of time. By using the truncation error, we get

$$| \varepsilon_i | \leq \varrho_i \Delta t^3 \ , \ i \geq 1. \tag{4.15}$$

We assume that E_{n+1} be the global error in time discretizing process and $\varrho = max\{\varrho_1, ..., \varrho_n\}$. We can write the following global error estimate at $n+1$ level
$E_{n+1} = \sum_{i=1}^{n} \varepsilon_i, \ (\Delta t \leq T/n)$,

with the help of (4.15) we can write

$$| E_{n+1} | = | \sum_{i=1}^{n} \varepsilon_i | \leq n \varrho \Delta t^3 \leq n \varrho \frac{T}{n} \Delta t^2 = \rho \Delta t^2, \tag{4.16}$$

where $\rho = \varrho T$. Which completes the proof. □

5. Numerical examples

In this section to illustrate the performance of the B-spline collocation method in solving BBMB equation and the efficiency of the method, the following examples are considered. We defined L_2 and L_∞ as

$$L_2 = \sqrt{h \sum_{i=0}^{N} |U_i - u_i|}, \ L_\infty = \max_{i=0}^{N} |U_i - u_i|.$$

Note that we have computed the numerical results by Mathematica-9 programming.

Example 1. Consider the BBMB equation with $\alpha = 0$ and $\beta = 1$ in the interval $[-40, 60]$, with the analytical solution $u(x,t) = 3c\,\mathrm{sech}^2(k(x - vt - x_0))$ with $c = 0.1$, $v = 1 + c$, $x_0 = 0$, $k = \sqrt{\frac{c}{4v}}$. For comparison, we consider the our results with methods [15,16,17]. We assume that $\Delta t = 0.1$ and $N = 1000$. Table 1 exhibits the compared results. Also Table 2 and Table 3 give a comparison between numerical and analytical solutions for different partitions. From Table 2, we see that the L_2 and error decrease as Δt decreases or N increases. Also numerical results in Table 3, are in accordance with the order of convergence of our presented scheme. From

Figure 1 we can see that numerical solutions show the same behavior as analytical solution. Also Figure 2 shows that the solution obtained by our method is close to the analytical solution. Figure 3 shows absolute errors.

Table 1: Comparison of errors for Example 1 with $N = 1000, \Delta t = 0.1$ and $c = 0.1$.

Method	Error \ Time	4	8	12	16	20
Present method	$L_2 \times 10^3$	0.0203045	0.0382871	0.0524730	0.0630462	0.0709222
	$L_\infty \times 10^3$	0.0084461	0.0160410	0.0210382	0.0241158	0.0259986
Method in [15]	$L_2 \times 10^3$	0.12	0.23	0.34	0.45	0.55
	$L_\infty \times 10^3$	0.05	0.09	0.14	0.18	0.21
Method in [16]	$L_2 \times 10^3$	0.046	0.090	0.135	0.179	0.220
	$L_\infty \times 10^3$	0.017	0.036	0.054	0.071	0.086
Method in [17]	$L_2 \times 10^3$	39.82	79.46	118.8	157.7	196.1
	$L_\infty \times 10^3$	13.74	27.66	41.35	54.60	67.35

Table 2: L_2 errors for Example 1 at different times.

Partition	Error \ Time	1	5	10	15
$\Delta t = 0.5,\ N = 500$	$L_2 \times 10^3$	0.12098300	0.51809700	1.07851000	1.49870000
$\Delta t = 0.1,\ N = 500$	$L_2 \times 10^3$	0.00655321	0.02512740	0.04587060	0.06070200
$\Delta t = 0.01,\ N = 500$	$L_2 \times 10^3$	0.00519310	0.00579594	0.00628573	0.00643410
$\Delta t = 0.01,\ N = 100$	$L_2 \times 10^3$	0.00755186	0.01044510	0.01407850	0.01638590
$\Delta t = 0.01,\ N = 200$	$L_2 \times 10^3$	0.00594902	0.00637034	0.00692952	0.00715433
$\Delta t = 0.01,\ N = 300$	$L_2 \times 10^3$	0.00552811	0.00601056	0.00652286	0.00670635

Table 3: L_∞ errors for Example 1 at different partitions.

Time	5		15	
Partition	Error	Order	Error	Order
$\Delta t = 0.5,\ N = 500$	2.20791×10^{-4}	——	5.57623×10^{-4}	——
$\Delta t = 0.1,\ N = 500$	1.05724×10^{-5}	1.88822	2.34437×10^{-5}	1.96906
$\Delta t = 0.05,\ N = 500$	2.87871×10^{-6}	1.87681	5.93276×10^{-6}	1.98242

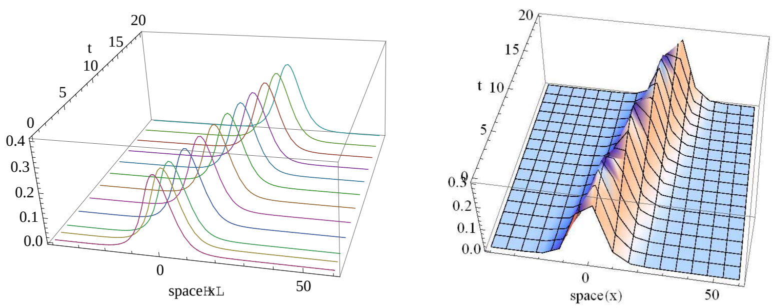

Figure 1: Analytical solution (right) and numerical solution (left) using $\Delta t = 0.1$ and $N = 100$ of Example 1.

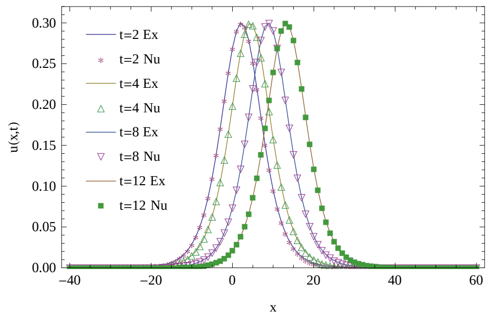

Figure 2: Numerical solution of Example 1 with $\Delta t = 0.1$ *and* $N = 100$.

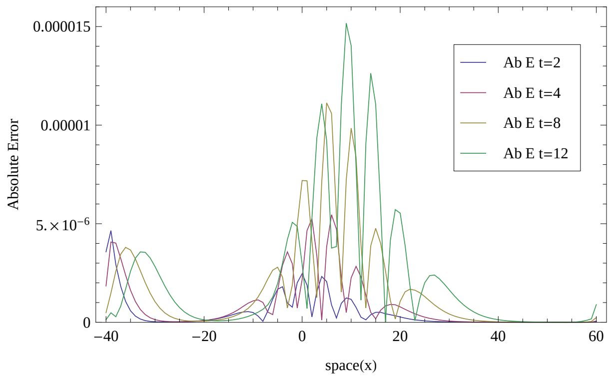

Figure 3: Absolute errors for Example 1 with $\Delta t = 0.1$ *and* $N = 100$.

Example 2. In this example we consider $\alpha = 0$ and $\beta = 1$ in the interval $[-10, 30]$, with the initial condition $u(x, 0) =\text{sech}^2(x/4)$. The analytical solution is $u(x, t) =\text{sech}^2(x/4 - t/3)$ [18]. Table 4 and Table 5 show L_2 error in different partitions. We can say that the numerical solution graph shows the same behavior as the analytical solution in the Figure 4. In addition Figure 5 shows absolute errors in different times. Also numerical results in Table 6 are in accordance with the order of convergence of our presented scheme.

Table 4: L_2 errors for Example 2 at different times.

Partition	Error \ Time	0.5	1	1.5	2
$\Delta t = 0.1,\ N = 900$	$L_2 \times 10^3$	0.1844620	0.39313900	0.6125150	0.8297940
$\Delta t = 0.01,\ N = 900$	$L_2 \times 10^3$	0.0985166	0.08748880	0.0814472	0.0788639

Table 5: L_2 errors for Example 2 at different times.

Partition	Error \ Time	0.5	1	1.5	2
$\Delta t = 0.01,\ N = 100$	$L_2 \times 10^3$	1.103180	1.071410	1.075280	1.103980
$\Delta t = 0.01,\ N = 200$	$L_2 \times 10^3$	0.490962	0.453793	0.437590	0.435878

Table 6: L_∞ errors for Example 2 at different partitions.

Time	3		20	
Partition	Error	Order	Error	Order
$\Delta t = 0.1,\ N = 10$	1.00500×10^{-1}	———	2.59682×10^{-1}	———
$\Delta t = 0.1,\ N = 40$	3.95893×10^{-3}	2.33297	1.97882×10^{-2}	1.85702
$\Delta t = 0.1,\ N = 100$	8.03935×10^{-4}	1.73985	3.27004×10^{-3}	1.96475

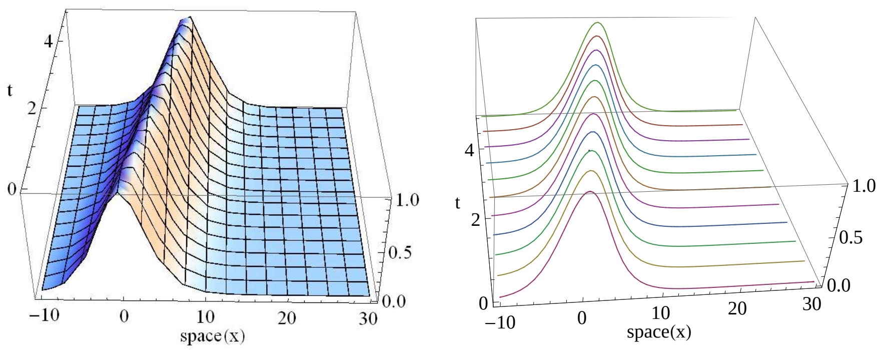

Figure 4: Analytical solution analytical solution (left) and numerical solution (right) using $\Delta t = 0.01$ and $N = 300$ of Example 2.

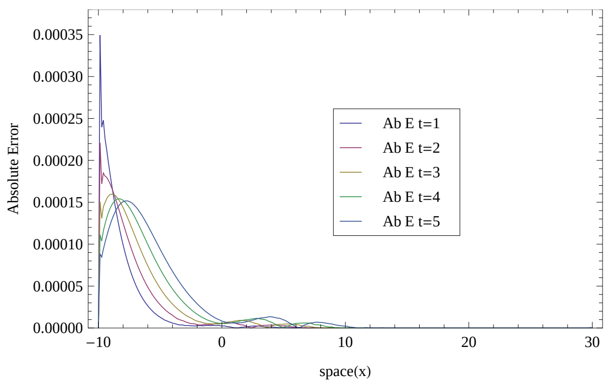

Figure 5: Absolute errors for Example 2 with $\Delta t = 0.01$ and $N = 300$.

Example 3. We consider here a numerical solution of the BBMB equation with $\alpha = 1$ and $\beta = 1$ in the interval $[-10, 10]$, with the initial condition $u(x, 0) = \exp(-x^2)$. The behavior of the approximated solution with $\Delta t = 0.01$ and $N = 300$ is presented in Figure 4. The graph shows the same behavior as in [19]. Also the numerical results are tabulated in Table 7 for $\Delta t = 0.01$ and $N = 300$.

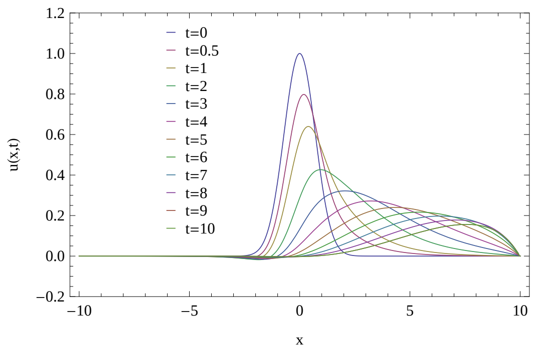

Figure 6: Numerical solution of the BBMB with $\alpha = 1$ and $\beta = 1$.

Table 7: Numerical results for Example 3 with $\Delta t = 0.01$ and $N = 300$.

x \ t	1	2	3	4	5
-5	-0.0010923500	-0.0010473800	-0.0007791350	-0.0005281950	-0.0003420730
0	0.5727480000	0.2869880000	0.1320010000	0.0555430000	0.0201706000
5	0.0396689000	0.1099050000	0.1805440000	0.2252390000	0.2345600000
x \ t	6	7	8	9	10
-5	-0.0002157390	-0.0001337230	-0.0000818514	-0.0000496067	-0.0000298137
0	0.0049344100	-0.0009074750	-0.0026172200	-0.0026737500	-0.0021879900
5	0.2149750000	0.1795830000	0.1400860000	0.1037240000	0.0736685000

Example 4. As a last study, we consider the non homogenous BBMB equation as follows

$$u_t - u_{xxt} - \alpha u_{xx} + \beta u_x + u u_x = G, \quad x \in [0, \pi], \ \ t \in [0, T], \tag{5.1}$$

where $G(x, t) = \exp(-t)[\cos(x) - \sin(x) + 12 \exp(-t) \sin(2x)]$. The exact solution for this problem is given as $u(x, t) = \exp(-t) \sin(x)$. The boundary and initial conditions can be found from exact solution. In Table 8, present method has been compared with method in [19]. In this table we consider $T = 10, N = N' + 1$ and $\Delta t = T/M$.

Table 8: Numerical results for Example 4 with different partitions in $t = 10$.

N'	M	Present method	Method in [19]
10	10	2.97032×10^{-4}	0.0218
20	20	1.14462×10^{-4}	0.0053
40	40	4.96031×10^{-5}	0.0013
80	80	2.32273×10^{-5}	3.3291×10^{-4}
160	160	1.12752×10^{-5}	8.3133×10^{-5}
320	320	5.5619×10^{-6}	2.0766×10^{-5}
640	640	2.76323×10^{-6}	5.1898×10^{-6}

6. Conclusion

In this work, the Quintic B-spline collocation method is used to solve the Benjamin-Bona-Mahony-Burgers(BBMB) equation. The stability analysis and convergence analysis of the method are shown. In addition, approximate numerical results given in the previous section. Also, obtained results showed that this approach can solve the problem effectively.

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