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Highest-weight Vectors in Tensor Products of Verma Modules for $U_q(\mathfrak{sl}_2)$

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ABSTRACT: We obtain an explicit basis for the subspace spanned by highest-weight vectors in a tensor product of two highest-weight modules for the quantized universal enveloping algebra of $\mathfrak{sl}_2$. The structure constants provide a generalization of the Clebsch-Gordan coefficients. As a byproduct, we give an alternative proof for the decomposition of these tensor products as direct sums of indecomposable modules and supply generators for all highest weight summands.

Key Words: Clebsch-Gordan coefficients, Verma modules, highest-weight vectors, tensor product decomposition.

Contents

1. Introduction

The quantized universal enveloping algebras $U_q(\mathfrak{g})$, introduced independently by V. Drinfeld and M. Jimbo in the mid-1980's, form a class of Hopf algebras which can be considered as one-parameter deformations of the universal enveloping algebra of the underlying symmetrizable Kac-Moody algebra $\mathfrak{g}$. The algebra $U_q(\mathfrak{sl}_2)$ first appeared in 1981 in a paper by P.P. Kulish and N. Reshetikhin on the study of integrable XYZ models with highest spin, while its Hopf algebra structure was discovered later by E.K. Sklyanin.

The BGG category $\mathcal{O}(\mathfrak{g})$ introduced in the early 1970's by J. Bernstein, I. Gelfand, and S. Gelfand has garnered attention since it could be used to study several important problems, and it has become one of the most studied categories

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of representations of Kac-Moody algebras. It is interesting to note that, for $\mathfrak{g}$ finite-dimensional, the category $\mathcal{O}(\mathfrak{g})$ contains that of finite-dimensional $\mathfrak{g}$-modules. We refer the reader to the book [4] and references therein for more details on the significance of category $\mathcal{O}(\mathfrak{g})$. The definition of category $\mathcal{O}$ is easily adapted to the setting of quantum groups. It turns out that $\mathcal{O}(U_q(\mathfrak{g}))$ is closed under tensor products and is equipped with a contragredient duality.

The study of tensor products of representations of (classical and quantum) Kac-Moody algebras is particularly relevant as it leads to deep connections with combinatorics and has important applications to mathematical physics (see [1,5,8,9] and references therein). For instance, the study of highest-weight vectors in tensor products of two finite-dimensional irreducible modules for $U_q(\mathfrak{sl}_2)$ can be used to compute the celebrated Clebsch-Gordan coefficients which are numbers that arise in angular momentum coupling in quantum mechanics (see for instance [9, Sections 3.4 and 3.5]). On the other hand, tensor products of Verma modules for $U_q(\mathfrak{sl}_2)$ play a crucial role in [6] for constructing the dual quantum group $SL_q(2)$ as semi-infinite cohomology of the Virasoro algebra with values in a tensor product of two braided vertex operator algebras with complementary central charges. This motivates the main goal of the present paper: to explicitly describe a basis for the subspace of highest-weight vectors in a tensor product of two Verma modules for $U_q(\mathfrak{sl}_2)$. Since a basis of highest-weight vectors in a tensor product of two finite-dimensional irreducible modules can be read off from our main result, the structure constants we compute may be regarded as generalizations of the original Clebsch-Gordan coefficients. A basis of highest-weight vectors in a tensor product of a Verma module with one such irreducible module can also be read off from the given basis of highest-weight vectors for the tensor product of Verma modules.

It is known that the tensor product of any two objects in category $\mathcal{O}$ decomposes as a (typically infinite) direct sum of indecomposable modules with each summand having finite multiplicity. In the case that $\mathfrak{g} = \mathfrak{sl}_2$ this decomposition was described in [2]. As a direct consequence of our main result (Theorem 2.1), we give an alternative proof for the tensor product decomposition in the case of highest weight modules and supply for each summand in the decomposition an explicit generator of its maximal highest-weight submodule.

The paper is organized as follows. Section 1 collects the basic preliminaries on $\mathcal{O}(U_q(\mathfrak{sl}_2))$, introduces structure constants $d^p_{i,k}$, and states the main result on highest-weight vectors. The proofs are given in Section 2 as well as the applications of the main result to the indecomposable summands of the decomposition.

2. Preliminaries and Main Result

2.1. The Quantized Enveloping Algebra $U_q(\mathfrak{sl}_2)$

Fix a ground field $\Bbbk$ with char $\Bbbk \neq 2$ and let $q \in \Bbbk \setminus \{0\}$ which is not a root of unity. Then, for $n \in \mathbb{Z}$, the q-integer

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$$

is nonzero unless $n = 0$ and $[-n] = -[n]$. For $n, m \in \mathbb{Z}_{\geq 0}$ where $n \geq m$, the q-factorials and q-binomial coefficients, respectively, are defined as follows:

$$[n]! = [n][n-1]\dots[1],\ [0]! = 1, \quad \text{and} \quad \begin{bmatrix} n \\ m \end{bmatrix} = \frac{[n]!}{[m]![n-m]!}.$$

The universal enveloping algebra $U_q(\mathfrak{sl}_2)$ of $\mathfrak{sl}_2$ is the unital associative algebra with generators e, f, t, t^{-1} and relations:

$$tt^{-1} = t^{-1}t = 1, \quad tet^{-1} = q^2 e, \quad tft^{-1} = q^{-2} f, \quad ef - fe = \frac{t - t^{-1}}{q - q^{-1}}.$$

Abbreviate $U_q = U_q(\mathfrak{sl}_2)$ and denote by U_q^+, U_q^-, and U_q^0 the subalgebras of U_q generated by e, f, and $t^{\pm 1}$, respectively. The set $\{f^l t^p e^r : l, r \in \mathbb{Z}_{\geq 0}, p \in \mathbb{Z}\}$ is a basis for U_q and, therefore, the multiplication establishes an isomorphism of vector spaces

$$U_q \cong U_q^- \otimes U_q^0 \otimes U_q^+. \tag{2.1}$$

The divided powers of e and f are defined as follows: $e^{(i)} = \dfrac{e^i}{[i]!}$ and $f^{(i)} = \dfrac{f^i}{[i]!}$ for $i \in \mathbb{Z}_{\geq 0}$.

It is well known that U_q is a Hopf algebra. We will use the comultiplication $\Delta : U_q \to U_q \otimes U_q$ given by:

$$\Delta(t^{\pm 1}) = t^{\pm 1} \otimes t^{\pm 1}, \quad \Delta(e) = e \otimes 1 + t \otimes e, \quad \Delta(f) = f \otimes t^{-1} + 1 \otimes f.$$

Thus, for two U_q-modules V and W, $V \otimes W$ is given a module structure via Δ. We also consider the involutive antiautomorphism σ of U_q determined by

$$\sigma(e) = f, \quad \sigma(f) = e, \quad \text{and} \quad \sigma(t) = t. \tag{2.2}$$

2.2. Category $\mathcal{O}$

A U_q-module M is a weight module if it decomposes into a direct sum of eigenspaces of t:

$$M = \bigoplus_{\lambda \in \Bbbk} M_\lambda, \text{ where } M_\lambda = \{v \in M : tv = \lambda v\}.$$

Whenever $M_\lambda \neq 0$ we say λ a weight of M and refer to M_λ as the associated λ-weight space. A nonzero vector $v \in M_\lambda$ is called a weight vector of weight λ. For $v \in M_\lambda$, clearly

$$eM_\lambda \subseteq M_{\lambda q^2} \quad \text{and} \quad fM_\lambda \subseteq M_{\lambda q^{-2}}.$$

If v is a weight vector and $ev = 0$, then v is said to be a highest-weight vector. We set

$$M^e = \{v \in M : ev = 0 \text{ and } v \in M_\lambda \text{ for some } \lambda\}.$$

A weight module M is a highest-weight module of highest weight λ if it is generated by a highest-weight vector v of weight λ. Then, (2.1) implies $M = U_q^- v$.

If all weights of M are of the form q^r, for $r \in \mathbb{Z}$, M is said to be a weight module of type 1. The category $\mathcal{O}$ is the category of U_q-weight modules M of type 1 such that $\dim(M_{q^r}) < \infty$ for all $r \in \mathbb{Z}$, and such that there exists an integer n depending on M with $M_{q^r} = 0$ for all $r \geq n$. Since we will only work with modules in $\mathcal{O}$, we simplify notation and write M_r instead of M_{q^r}. The category $\mathcal{O}$ is abelian and every module in $\mathcal{O}$ is the direct sum of indecomposable ones also belonging to $\mathcal{O}$. We shall now recall the classification of these indecomposable objects.

For $\lambda \in \mathbb{k}$, a Verma module is a universal highest weight module of U_q with highest weight λ. Thus, such a module is isomorphic to the quotient of U_q by the left ideal generated by $t - \lambda$ and e. In particular, (2.1) implies that $M(\lambda)$ is isomorphic to U_q^- as a vector space. Letting v be the image of 1 in the aforementioned quotient and setting $v_i = f^{(i)} v$ for $i \in \mathbb{Z}_{\geq 0}$ and $v_{-1} = 0$, we have

$$tv_i = \lambda q^{-2i} v_i, \qquad fv_i = [i+1] v_{i+1}, \qquad ev_i = \frac{\lambda q^{-(i-1)} - \lambda^{-1} q^{i-1}}{q - q^{-1}} v_{i-1}, \tag{2.3}$$

In particular, $M(\lambda) \in \mathcal{O}$ iff $\lambda = q^r$ for some $r \in \mathbb{Z}$ and, in that case, (2.3) can be rewritten as

$$tv_i = q^{r-2i} v_i, \qquad fv_i = [i+1] v_{i+1}, \qquad ev_i = [r - i + 1] v_{i-1}. \tag{2.4}$$

Since we will only work with objects in $\mathcal{O}$, we abbreviate the notation and write $M(r)$ instead of $M(q^r)$.

The Verma module $M(r)$ is irreducible if and only if $r \notin \mathbb{Z}_{\geq 0}$. Otherwise, it has a unique proper nonzero submodule which is generated by v_{r+1}. It easily follows from (2.4) that this submodule is isomorphic to $M(-r-2)$. We denote by $V(r)$ the irreducible quotient of $M(r)$. In particular, $V(r) \cong M(r)/M(-r-2)$ has dimension $r + 1$. Any simple object in $\mathcal{O}$ is isomorphic to either $V(r), r \geq 0$, or $M(r), r < 0$.

Given any U_q-module $M \in \mathcal{O}$, consider the subspace of the dual space $M^* = Hom_{\mathbb{k}}(M, \mathbb{k})$ consisting of elements $m^* : M \to \mathbb{k}$ such that $m^*(M_r) = 0$ for all but finitely many $r \in \mathbb{Z}$. The morphism σ defined by (2.2) induces an action of U_q on this subspace defined as follows:

$$(um^*)(m) = m^*(\sigma(u)m), \quad \text{for} \quad u \in U_q, m^* \in M^\sigma, \ m \in M.$$

One easily checks that this module, which we denote by M^σ, is an object in $\mathcal{O}$. We refer to it as the contragredient dual of M. It can be easily seen that

$$V(r)^\sigma \cong V(r) \quad \text{and} \quad M(s)^\sigma \cong M(s) \quad \text{if} \quad r \geq 0, s < 0.$$

However, $M(r)^\sigma, r \geq 0$, belongs to a new isomorphism class of indecomposable objects and we have a nonsplit short exact sequence

$$0 \to V(r) \to M(r)^\sigma \to M(-r-2) \to 0.$$

There is one more class of indecomposable objects in $\mathcal{O}$ (it was originally considered in [3]). For $r \in \mathbb{Z}_{\geq 0}$, the module $T(r)$ is the quotient of U_q by the left ideal generated by

$$t - q^{-r-2}, \quad e^{r+2}, \quad \text{and} \quad (\Omega - c)^2, \quad \text{where} \quad \Omega = fe + \frac{qt + q^{-1}t^{-1}}{(q-q^{-1})^2}$$

is the quantum Casimir element and $c = \dfrac{q^{r+1} + q^{-r-1}}{(q-q^{-1})^2}$.

The image of e^{r+1} in $T(r)$ generates a submodule isomorphic to $M(r)$ and we have a nonsplit short exact sequence

$$0 \to M(r) \to T(r) \to M(-r-2) \to 0.$$

Moreover, $T(r)^\sigma \cong T(r)$ (see [7]) and, hence, we also have an exact sequence

$$0 \to M(-r-2) \to T(r) \to M(r)^\sigma \to 0.$$

For $M \in \mathcal{O}$, denote by $H(M)$ the submodule of M generated by M^e. The following result is obvious.

Proposition 2.1. *Let $M \in \mathcal{O}$ be an indecomposable object. Then, $H(M)$ is the unique maximal highest-weight submodule of M. Moreover, if $H(M) \neq M$, then $M/H(M)$ is isomorphic to $M(-r-2)$ for some $r \geq 0$ and, if $z \in M_{-r-2} \setminus H(M)$, then z generates M and satisfies*

$$e^{r+2} z = 0, \qquad tz = q^{-r-2} z, \qquad (\Omega - c)^2 z = 0.$$

$\diamond$

2.3. Highest-weight Vectors in Tensor Products

Since $\mathcal{O}$ is closed under tensor products, the tensor product of any two objects from $\mathcal{O}$ is isomorphic to a direct sum of the indecomposable modules described in the previous subsection. Of course, it suffices to describe such decomposition when both tensor factors are indecomposable. In that case, the multiplicities of the summands were computed in [2, Section 3.5] and, in particular, it follows that

$$V \otimes W \cong W \otimes V \quad \text{for all} \quad V, W \in \mathcal{O}.$$

However, it is also useful to describe the highest-weight vectors inside such tensor products. Thus, we now fix indecomposable modules V and W and describe M^e for

$$M = V \otimes W$$

in terms of certain basis of V and W. Let v and w be highest-weight vectors of $H(V)$ and $H(W)$ (see Proposition 2.1), respectively, and set

$$v_i = f^{(i)} v, \qquad w_i = f^{(i)} w \qquad \text{for} \qquad i \geq 0.$$

We also set $v_i = w_i = 0$ if $i < 0$ and let n and m be such that

$$tv = q^n v \qquad \text{and} \qquad tw = q^m w.$$

The set $\{v_i : i \geq 0\} \setminus \{0\}$ is a basis for $H(V)$, and similarly for $H(W)$. Note that

$$v_i = 0 \quad \text{for some} \quad i > 0 \quad \Leftrightarrow \quad i \geq n+1 > 0 \quad \text{and} \quad H(V) \cong V(n), \tag{2.5}$$

and similarly for W.

Clearly $M_r \neq 0$ only if $r = m + n - 2p$ for some $p \in \mathbb{Z}_{\geq 0}$. For each fixed p and $i, k \in \mathbb{Z}_{\geq 0}$ such that $k \leq i$ and $n \notin \{k, ..., i-1\}$, set

$$d^p_{i,k} = q^{i(n-i+1)} \prod_{\ell=k+1}^{i} \frac{[p-m-\ell]}{[n+1-\ell]}.$$

We record some immediate properties of $d^p_{i.k}$ in the following lemma.

Lemma 2.1.

1. *Let either $k > n$ or $i \leq n$. The scalar $d^p_{i,k} = 0$ if and only if $k < p - m \leq i$.*
2. *If $p \leq m$ and $i \leq n$, then $d^p_{i,0} = (-1)^i q^{i(n-i+1)} \dfrac{\begin{bmatrix} m-p+i \\ i \end{bmatrix}}{\begin{bmatrix} n \\ i \end{bmatrix}}$.*
3. *If $p > m \geq 0$ and $i \leq n, p - m - 1$, then $d^p_{i,0} = q^{i(n-i+1)} \dfrac{\begin{bmatrix} p-m-1 \\ i \end{bmatrix}}{\begin{bmatrix} n \\ i \end{bmatrix}}$.*
4. *If $p \leq n + m + 1$ and $i \geq n + 1 \geq 1$, then $d^p_{i,n+1} = q^{i(n-i+1)} \begin{bmatrix} m-p+i \\ i-n-1 \end{bmatrix}$.*
5. *If $p = n + m + 1$, then $d^p_{i,k} = q^{i(n-i+1)}$ whenever $k > n$ or $i \leq n$.* $\diamond$

For $p \geq k \geq 0$, define the elements $u_{p,k}$ of M_{m+n-2p} as follows:

$$u_{p,k} = \sum_{i=k}^{p-s} d^p_{i,k} v_i \otimes w_{p-i} \tag{2.6}$$

where

$$s = \begin{cases} 0 & \text{if} \;\; k > n \text{ or } p \leq n; \\ m+1 & \text{if} \;\; k = 0 \leq n, m < p \leq n+m+1. \end{cases}$$

Notice that the structure constants $d^p_{i,k}$ are well-defined for each of the above cases.

Let $k \in \{0, n+1\}$. Then $d^p_{k,k} = 1$. Also, whenever $p - k > m \geq 0$,

$$u_{p,k} = \sum_{i=k}^{p-m-1} d^p_{i,k} v_i \otimes w_{p-i} \tag{2.7}$$

by part (1) of Lemma 2.1.

We are now ready to state our main result.

Theorem 2.1. *Suppose V and W are highest-weight modules with highest weights q^n and q^m, respectively, for $n, m \in \mathbb{Z}$. Given $p \in \mathbb{Z}_{\geq 0}$, the nonzero vectors among the vectors*

$$\begin{cases} u_{p,0}, & \text{if } \ n < 0 \text{ or } 0 \leq p \leq n; \\ u_{p,0}, u_{p,n+1} & \text{if } \ 0 \leq n, m < p \leq n+m+1; \\ u_{p,n+1}, & \text{otherwise,} \end{cases}$$

form a basis $\mathcal{B}_p$ of M^e_{n+m-2p}. If, in particular, V and W are Verma modules, then all of the above vectors are nonzero and, thus, comprise $\mathcal{B}_p$. ◇

Note that, if V and W are finite-dimensional, then $u_{p,k} \neq 0$ iff $k = 0$ and $0 \leq p \leq \min\{m, n\}$. Since M is finite-dimensional, the submodule generated by $u_{p,0}, 0 \leq p \leq \min\{m, n\}$, is isomorphic to $V(m+n-2p)$. In particular, this recovers the well-known Clebsch-Gordan Theorem:

$$M \cong \bigoplus_{p=0}^{\min\{m,n\}} V(n+m-2p). \tag{2.8}$$

As we shall see in Section 3.3, once Theorem 2.1 is proved for the case that V and W are Verma modules, the other cases, including this one, can be easily deduced. Moreover, in Section 3.4 we use Theorem 2.1 to obtain the decomposition of M as a direct sum of indecomposable modules.

3. Proofs

3.1. Recursive Relation

The following lemma will be used in the proof of Theorem 2.1.

Lemma 3.1. *Let $\mu, p \in \mathbb{Z}_{\geq 0}$ such that $\mu \leq p$ and $I = \{\mu+1, \ldots, p\}$. Let $a_i, b_i \in \Bbbk$ $(i \in I)$ be such that at most one a_i and at most one b_i are zero. Set*

$$S = \{(c_\mu, \ldots, c_p) \in \Bbbk^{p-\mu+1} : c_i a_i - c_{i-1} b_i = 0, \forall i \in I\} \quad \text{and} \quad d_{\alpha,\beta} = \prod_{\ell=\beta+1}^{\alpha} \frac{b_\ell}{a_\ell}$$

for $\alpha > \beta$, $\alpha, \beta \in I \cup \{\mu\}$. Let $k \in I$ be such that $a_k = 0$ and $a_k \neq 0$ for $i \neq k$. If there is no such k, set $k = \mu$. We have:

1. *If there exists $j \in I, j \leq k$ such that $b_j = 0$, then the set $\{u, u'\}$ is a basis for S where $u = (0, ..., 0, 1, d_{k+1,k}, ..., d_{p,k})$ and $u' = (1, d_{\mu+1,\mu}, ..., d_{j-1,\mu}, 0, ..., 0)$.*
2. *Otherwise, u (as above) spans S.* ◇

Proof. By solving the recursive relation that defines S, we obtain for the above two cases:

1. If $\mu + 1 \leq i < j$, then $c_i = c_\mu d_{i,\mu}$; $c_j = ... = c_{k-1} = 0$; and $c_i = c_k d_{i,k}$ for $i > k$. Thus, $S = \{c_\mu(1, d_{\mu+1,\mu}, ..., d_{j-1,\mu}, 0, ..., 0) + c_k(0, ..., 0, 1, d_{k+1,k}, ..., d_{p,k}) : c_\mu, c_k \in \Bbbk\}$.

2. If j as above does not exist, then $0 = c_{k-1} = ... = c_\mu$ and $c_i = c_k d_{i,k}$ for $i > k$. Thus, $S = \{c_k(0, ..., 0, 1, d_{k+1,k}, ..., d_{p,k}) : c_k \in \Bbbk\}$.

□

3.2. Proof of Theorem 2.1 for a Tensor Product of Two Verma Modules

Assume that V and W are Verma modules with highest weights q^n and q^m, respectively, where $n, m \in \mathbb{Z}$, and observe that, in this case, all the vectors listed in Theorem 2.1 are in fact nonzero. To shorten notation, let $r = n+m-2p$. Then, a $\Bbbk$-basis of M_r is $\{v_i \otimes w_{p-i} : 0 \leq i \leq p\}$ with $\dim(M_r) = p + 1$. Therefore, the kernel of e from M_r into M_{r+2} is nontrivial. Let u_p denote a nonzero element of the kernel of e in M_r. Note that since V and W are $U_q^-(\mathfrak{sl}_2)$-torsion free weight modules, then M is also such. Thus, u_p generates a Verma submodule of M with highest weight q^r and, hence,

$$M(n + m - 2p) \hookrightarrow M(n) \otimes M(m)$$

for every $p \in \mathbb{Z}_{\geq 0}$.

Note that $u_p = \sum_{i=0}^{p} c_i v_i \otimes w_{p-i}$ for some $c_i \in \Bbbk$, not all 0, and we have the following:

$$\begin{aligned}
0 = eu_p &= \sum_{i=0}^{p} e(c_i v_i \otimes w_{p-i}) \\
&= \sum_{i=1}^{p} c_i[n - i + 1]v_{i-1} \otimes w_{p-i} + \sum_{i=0}^{p-1} c_i q^{n-2i} v_i \otimes [m - p + i + 1]w_{p-i-1} \\
&= \sum_{i=1}^{p} (c_i[n - i + 1] + c_{i-1}[m - p + i]q^{n-2(i-1)})v_{i-1} \otimes w_{p-i}.
\end{aligned}$$

It suffices to solve the following recursive relation:

$$c_i[n - i + 1] + c_{i-1}[m - p + i]q^{n-2(i-1)} = 0, \qquad i = 1, \ldots, p.$$

Let $a_i = [n - i + 1]$ and $b_i = -[m - p + i]q^{n-2(i-1)}$, and set $I = \{1, \ldots, p\}$. Then, $a_i = 0$ if and only if $i = n + 1 \in I$, while $b_i = 0$ if and only if $i = p - m \in I$. By applying Lemma 3.1 with $\mu = 0$, we obtain

$$u_p = \sum_{i=k}^{p} d_{i,k} v_i \otimes w_{p-i},$$

where

$$k = \begin{cases} n+1 & \text{if } \ p > n \geq 0; \\ 0 & \text{otherwise}, \end{cases} \tag{3.1}$$

and

$$d_{i,k} = \prod_{\ell=k+1}^{i} \frac{b_\ell}{a_\ell} = q^{(i-k)(n-i+1-k)} \prod_{\ell=k+1}^{i} \frac{[p-m-\ell]}{[n+1-\ell]} = q^{i(n-i+1)} \prod_{\ell=k+1}^{i} \frac{[p-m-\ell]}{[n+1-\ell]}.$$

Also, Lemma 3.1 implies that there is an additional linearly independent solution $u_p^{'}$ if and only if $a_k = 0$ and $b_j = 0$ for some $k, j \in I, k \geq j$. This condition is equivalent to $k = n+1, \ j = p-m, \ \text{ and } \ 0 \leq n, m < p \leq n+m+1$. In such a case,

$$u_p^{'} = \sum_{i=0}^{p-m-1} d_{i,0} v_i \otimes w_{p-i}$$

with $d_{i,0}$ given as above. The vectors u_p and u_p' are precisely of the same form as the vectors defined in (2.6) with u_p corresponding to $u_{p,k}$ for k as in (3.1) and $s = 0$, and u_p' corresponding to $u_{p,k}$ for $k = 0$ and $s = m+1$. This completes the proof of Theorem 2.1 in the case of two Verma modules.

3.3. Tensor Product of a Verma Module with a Finite-Dimensional Module

We keep assuming that V and W are Verma modules with highest weights q^n and q^m, respectively. If $n \geq 0$,

$$V = V^+ \oplus V^-$$

where

$$V^+ = \bigoplus_{p=0}^{n} V_{n-2p} \qquad \text{and} \qquad V^- = \bigoplus_{p>n} V_{n-2p} \cong M(-n-2).$$

If $m \geq 0$, define $W^\pm$ similarly. Observe from (2.6) and (2.7) that, for $u_{p,k} \in \mathcal{B}_p$,

$$n \geq 0 \qquad \Rightarrow \qquad u_{p,0} \in V^+ \otimes W \quad \text{while} \quad u_{p,n+1} \in V^- \otimes W, \tag{3.2}$$

and

$$m \geq 0 \qquad \Rightarrow \qquad \begin{array}{ll} u_{p,k} \in V \otimes W^+ & \text{iff} \quad k \geq p-m \\ u_{p,k} \in V \otimes W^- & \text{iff} \quad k < p-m. \end{array} \tag{3.3}$$

Recall that, for $r \geq 0$, we have an exact sequence

$$0 \to M(-r-2) \to M(r) \to V(r) \to 0. \tag{3.4}$$

Assuming $n \geq 0$ and tensoring this sequence for $r = n$ from the right by W, we get the exact sequence

$$0 \to V^- \otimes W \to V \otimes W \to V(n) \otimes W \to 0.$$

Then, using (3.2), one easily deduces the following proposition which proves Theorem 2.1 for $M = V(n) \otimes M(m)$.

Proposition 3.1. *Let $M = V(n) \otimes M(m)$ for some $m, n \in \mathbb{Z}, n \geq 0$. Then, $M^e_{n+m-2p} \neq 0$ iff*

$$0 \leq p \leq n \quad \text{or} \quad 0 \leq n, m < p \leq n + m + 1$$

and, in that case, M^e_{n+m-2p} is spanned by $u_{p,0}$. ⋄

Similarly, assuming $m \geq 0$, tensoring (3.4) for $r = m$ from the left by $M(n)$, and using (3.3), leads to the following proposition which proves Theorem 2.1 for the remaining case where $M = M(n) \otimes V(m)$.

Proposition 3.2. *Let $M = M(n) \otimes V(m)$ for some $m, n \in \mathbb{Z}, m \geq 0$. Then, $M^e_{n+m-2p} \neq 0$ iff*

$$0 \leq p \leq m \quad \text{or} \quad 0 \leq n, m < p \leq n + m + 1$$

and, in that case, M^e_{n+m-2p} is spanned by $u_{p,k}$ where

$$k = \begin{cases} 0 & \text{if } \; n < 0 \text{ or } p \leq n; \\ n+1 & \text{if } \; 0 \leq n < p \leq n + m + 1. \end{cases}$$

⋄

The above propositions may also be proved directly, in a manner similar to the proof for tensor products of two Verma modules given in the previous section, by using Lemma 3.1 with $\mu = \max\{0, p - m\}$.

Let $n, m \geq 0$. Observe that, for each $u_{p,k}$ occurring in Proposition 3.2, $k \geq p-m$ and, moreover, $u_{p,k} \in V^+ \otimes V(m)$ iff $k = 0$ and $p \leq n$, while $u_{p,k} \in V^- \otimes V(m)$ otherwise. Similarly, for each $u_{p,k}$ occurring in Proposition 3.1, $k = 0$ and, clearly, $u_{p,0} \in V(n) \otimes W^+$ iff $p \leq n, m$, while $u_{p,0} \in V(n) \otimes W^-$ otherwise. Each of these two settings now leads, via exact sequences as before, to the expected conclusion that if $M = V(n) \otimes V(m)$, then $M^e_{n+m-2p} \neq 0$ iff $0 \leq p \leq n, m$ and, in that case, it is spanned by $u_{p,0}$, which recovers (2.8).

3.4. The Indecomposable Summands

Let M be as in Theorem 2.1 with at least one of the tensor factors not finite-dimensional. The decomposition of M as a direct sum of indecomposable modules was previously given in [2] and, for the case of the tensor product of a finite-dimensional module and a Verma module, also in [3]. We deduce the decomposition of M as a direct sum of indecomposable modules as an application of Theorem 2.1.

One easily checks that, since one of the tensor factors of M is $U_q^-(\mathfrak{sl}_2)$-torsion free, so is M. Hence, for $p \geq 0$, every element of $\mathcal{B}_p$ from Theorem 2.1 generates a Verma submodule of M.

For notational convenience, set $T(-1) = M(-1)$.

Corollary 3.1. *Suppose V and W are Verma modules with highest weights q^n and q^m, respectively, and set $l = n + m$, $l_0 = \max\{0, l + 2\}$, and $h = \min\{n, m\}$.*

(1) If $h < 0$, then

$$M \cong \left(\bigoplus_{0 \leq 2p \leq l+1} T(l - 2p) \right) \bigoplus \left(\bigoplus_{p \geq l_0} M(l - 2p) \right).$$

Each summand $M(l-2p)$ is generated by $u_{p,n+1}$ when $p > n \geq 0$ and by $u_{p,0}$ otherwise, while each $H(T(l - 2p)) \cong M(l - 2p)$ is generated by $u_{p,0}$.

(2) If $h \geq 0$, then

$$M \cong \left(\bigoplus_{0 \leq p \leq h} M(l - 2p) \right) \bigoplus \left(\bigoplus_{2h+2 \leq 2p \leq l+1} T(l - 2p) \right)$$
$$\bigoplus \left(\bigoplus_{p \geq l-h+1} M(l - 2p) \right).$$

The summand $M(l - 2p)$ is generated by $u_{p,n+1}$ for $p > l + 1$, while $u_{p,0}$ generates a copy of $M(l - 2p)$ for every other p. The module $H(T(l - 2p)))$ is generated by $u_{p,k}$ where $k = 0$ if $n \geq m$ and $k = n + 1$ if $n < m$. ◇

Proof. Since M belongs to category $\mathcal{O}$ and f acts freely on M, then M decomposes into a direct sum of Verma modules and T-modules. The maximal indecomposable highest-weight submodule of each summand is a Verma module and the direct sum of such submodules, $H(M)$, is obtained directly from Theorem 2.1 since $H(M)$ is the sum of all Verma submodules of M. For $p \geq 0$, $\#\mathcal{B}_p = 2$ for $0 \leq n, m < p \leq n + m + 1$ and $\#\mathcal{B}_p = 1$ otherwise. In particular, M contains exactly two copies of $M(n + m - 2p)$ for each p such that $0 \leq n, m < p \leq n + m + 1$ and one copy of $M(n + m - 2p)$ for every other $p \geq 0$.

A simple dimension comparison for weight spaces of M and $H(M)$ leads to T-module summands as they have to account for the dimension difference and, thus, also distinguishes Verma submodules that appear as submodules of T-module summands from Verma modules that remain as summands in the decomposition of M. □

Using a similar dimension comparison argument and the basis vectors from Proposition 3.2, we also have:

Corollary 3.2. *Let V be a Verma module and W a finite-dimensional module with highest weights q^n and q^m, respectively, and set $l = n + m$, $l_0 = \max\{0, l+2\}$, and $h = \min\{n, m\}$.*

(1) If $n < 0$, then

$$M \cong \left(\bigoplus_{0 \le 2p \le l+1} T(l-2p) \right) \bigoplus \left(\bigoplus_{l_0 \le p \le m} M(l-2p) \right).$$

Each $M(l-2p)$ and each $H(T(l-2p))$ are generated by $u_{p,0}$ for the corresponding p.

(2) If $n \ge 0$, then

$$M \cong \left(\bigoplus_{0 \le p \le h} M(l-2p) \right) \bigoplus \left(\bigoplus_{2n+2 \le 2p \le l+1} T(l-2p) \right).$$

Each $M(l-2p)$ is generated by $u_{p,0}$, while each $H(T(l-2p))$ is generated by $u_{p,n+1}$. ◇

Since the tensor product is symmetric, the decomposition of the tensor product of a finite-dimensional module V and a Verma module W with highest weights q^n and q^m, respectively, can be read off from the previous corollary by switching the roles of n and m. However, in this case, each $M(l-2p)$ and each $H(T(l-2p))$ occurring in the decomposition are generated by $u_{p,0}$ for the corresponding p.

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