Note on p_1-Lindelöf spaces which are not contra second countable spaces in bitopology

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Abstract. We prove that a contra second countable bitopological space is a p_1-Lindelöf space, but the converse is not true in general. We provide suitable example with the help of concepts of nest and interlocking from LOTS. The relation between pairwise regular spaces and p_1-normal spaces is studied. At the end, we propose some open questions which may enrich various concepts related to Lindelöfness in a bitopological space and other areas of mathematical ideas.

However, this article is continuation of Acharjee and Tripathy [25] since Theorem 3.6 of theorem 3.7 of [25] can be connected easily with results of this article.

AMS Classification No (2010): 54E55, 06B30, 54E05, 97P20.

Keywords : p_1-Lindelöfness, contra second countable space, nest, interlocking.

1. Introduction.

Kelly [12] introduced bitopological spaces via quasi-pseudo metric and investigated their various properties. Bitopological spaces have drawn the attention of many topologists ([2], [5], [6], [7], [11], [13], [14], [15], [16], [17]), engineers, researchers in medical sciences [21], economists ([3], [18]), computer scientists [32] and many others including fuzzy set topologists ([27],[28],[29]), rough set topologists [31].

It can be found that countability, Lindelöfness are old concepts in general topology and suitable relations along with applications can be found almost in every area of science and social science, but they are different for the case of bitopological space since bitopological space has more than two types of Lindelöf spaces viz. pairwise Lindelöf space, p-Lindelöf space, p_1-Lindelöf space etc.

Literature survey on bitoppological space suggests possible scopes of establishing relations between various types of Lindelöfness viz. p_1-Lindelöf space, p-Lindelöf space and pairwise Lindelöf space under suitable conditions.

In this paper, we give answer to the following question which was raised from literature reviews related to bitopological space.
"What type of a countable space in a bitopological space is a p_1-Lindelöf space?"

A cover $\mathcal{U}$ of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*open cover* [11] if $\mathcal{U} \subseteq \tau_1 \cup \tau_2$. If in addition $\mathcal{U}$ contains at least one non-empty member of τ_1 and at least one nonempty member of τ_2, then it is called *pairwise open cover* [8]. Pairwise compactness was defined by Fletcher et al.[8]. Notions of p-compact space, p_1-compact space, p-Lindelöf space and p_1-Lindelöf space can be found in the paper of Kilićkman, Salleh [13] and Birsan [2]. According to Reilly [20], (X, τ_1, τ_2) is *pairwise Lindelöf space* if each pairwise open cover has a countable subcover. Cooke and Reilly [5] investigated the relation between semi-compactness and pairwise compactness in bitopological spaces.

Kilićkman and Salleh ([4], [13], [14], [15], [16],[17], [22]) also studied various properties of pairwise Lindelöfness. Cocompactness, cotopology, (i, j)-Baire space etc. were all studied by Dvalishvili [6].

Frolik [9] introduced weakly Lindelöf spaces. Willard and Dissanayake [24] introduced almost Lindelöf spaces and their bitopological generalizations were studied by Kilićkman and Salleh [14].

From the prospective of applications, one may find uses of various concepts of bitopological spaces in science and social science. In 2010, Salama [21] worked on lower and upper approximations of rough sets by using a class of generalized closed sets of bitopological space to detect rheumatic fever. Recently, Acharjee and Tripathy [1] used the concept of bitopological space to reveal poverty patterns and equilibria between strategies of consumers and governments. For some other applications, one may refer to the extensive works of Bosi [3].

In this article, we consider p_1-Lindelöf space due to Birsan [2] and we consider notions of Good and Papadopoulos [10] to give counter example 3.1. One may find recent application of results of LOTS in Minkowski Space of relativity [33]. A systematic study on various types of Lindelöf spaces w.r.t. generalized continuous functions of a bitopological space can be found in [22].

One may refer [25], [26] for some results in the direction of this article.

2. Preliminary definitions.

Definition 2.1.([13], Definition 6) A bitopological space (X, τ_1, τ_2) is said to be *p-Lindelöf*, if the topological spaces (X, τ_1) and (X, τ_2) are both Lindelöf.

Definition 2.2. ([13], Definition 7) In a bitopological space (X, τ_1, τ_2), τ_1 *is said to* *be Lindelöf with respect to* τ_2 if, every τ_1-open cover of X can be reduced to a countable τ_2-open cover.

(X, τ_1, τ_2) is a p_1-*Lindelöf space* if τ_1 is Lindelöf with respect to τ_2 and τ_2 is Lindelöf with respect to τ_1.

Theorem 2.1. ([13], Theorem 6) If (X, τ_1, τ_2) is a second countable space, then (X, τ_1, τ_2) is p-Lindelöf.

The following notions will be used in the counterexample of Remark 3.1.

Definition 2.3.([10], Definition 1) Let X be a set. We say that a collection of subsets $\mathcal{S}$ of X:

1. T_0-*separates* X, if and only if for all $x, y \in X$, such that $x \neq y$, there exists $S \in \mathcal{S}$ such that $x \in S$,$y \notin S$ or $y \in S$,$x \notin S$.

2. T_1-*separates* X, if and only if for all $x, y \in X$, such that $x \neq y$, there exist $S, T \in \mathcal{S}$, such that $x \in S$, $y \notin S$ and $y \in T$,$x \notin T$.

One can easily see that a space is T_0 (resp. T_1) if and only if its topology is generated by a T_0- (resp. T_1-) separating subbase, but the statement of Definition 2.3 is not valid for the T_2 separation axiom, if one defines a T_2-separating subbase in an analogous way.

Definition 2.4.([10], Definition 2) Let X be a set and let $\mathcal{L} \subseteq \mathcal{P}(X)$. The order $\lhd_{\mathcal{L}}$ on X is defined by declaring $x \lhd_{\mathcal{L}} y$, if and only if there exists some $L \in \mathcal{L}$, such that $x \in L$ and $y \notin L$.

Theorem 2.2.([10], Theorem 8) Let X be a set and let $\mathcal{L}$ be a T_0-separating nest on X. The following are equivalent:

(i) $\mathcal{L}$ is interlocking;

(ii) for each $L \in \mathcal{L}$, if L has a $\lhd_{\mathcal{L}}$-maximal element, then $X - L$ has a $\lhd_{\mathcal{L}}$-minimal element;

(iii) for all $L \in \mathcal{L}$, either L has no $\lhd_{\mathcal{L}}$-maximal element or $X - L$ has a $\lhd_{\mathcal{L}}$-minimal element.

Theorem 2.3. ([10], Theorem 10) Let (X, τ) be a topological space. Then:

(i) If $\mathcal{L}$ and $\mathcal{R}$ are two nests of open sets whose union is T_1-separating, then every $\lhd_{\mathcal{L}}$-order open set is open in X.

(ii) X is a GO space if and only if there are two nests, $\mathcal{L}$ and $\mathcal{R}$, of open sets whose union is T_1-separating and forms a subbase for τ.

(iii) X is a LOTS if and only if there are two interlocking nests $\mathcal{L}$ and $\mathcal{R}$, of open sets whose union is T_1-separating and forms a subbase for τ.

3. Main results.

In this section we shall show that a contra second countable space is a p_1-Lindelöf space but the converse is not true.

Definition 3.1 Let (X, τ_1, τ_2) be a bitopological space, then:

(i) (X, τ_1, τ_2) is said to be an (i, j)-*second countable bitopological space*, if (X, τ_i) is second countable with respect to τ_j where $i, j \in \{1, 2\}$.

(ii) (X, τ_1, τ_2) is said to be a *contra second countable bitopological space*, if it is both $(1, 2)$-second countable and $(2, 1)$-second countable bitopological space.

Theorem 3.1. Let (X, τ_1, τ_2) be a contra second countable bitopological space, then it is p_1-Lindelöf.

Proof. Let $\{\beta_1^m\}$ and $\{\beta_2^n\}$ be countable τ_1-base and countable τ_2-base respectively in (X, τ_1, τ_2), where m, n are positive integers.

Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be any τ_1-open cover of X. Then for every $x \in X$, there exists $U_x \in \mathcal{U}$ such that $x \in U_x$. Since (X, τ_1, τ_2) is a contra second countable bitopological space, it is both $(1, 2)$-second countable and $(2, 1)$-second countable. So, for each $x \in U_x$ and $U_x \in \mathcal{U}$, there exists $\beta_x \in \{\beta_2^n\}$ such that $x \in \beta_x \subseteq U_x$. Hence $X = \cup\{\beta_x : x \in X, x \in \beta_x \subseteq U_x\}$.

Since $\{\beta_x : x \in X, x \in \beta_x \subseteq U_x\} \subseteq \{\beta_2^n\}$, we have that $\{\beta_x : x \in X, x \in \beta_x \subseteq U_x\} = \{\beta_2^n : n \in N, x \in \beta_2^n\}$.

Thus, $X = \cup\{\beta_2^n : n \in N, x \in \beta_2^n\} = \cup\{U_n : n \in N, x \in \beta_2^n \subseteq U_n\}$ and so $\{U_n : n \in N, x \in \beta_2^n \subseteq U_n\}$ is a countable τ_2-open subcover of $\mathcal{U}$. Similarly, one can prove the other part.

Corollary 3.1. Every pairwise closed subset of a contra second countable bitopological space is p_1-Lindelöf.

Proof. The proof follows from Theorem 3.1. and Lemma 4 of [13].

Corollary 3.2. Every pairwise regular and contra second countable bitopological space is p_1-normal.

Proof. The proof follows from Theorem 3.1. and Theorem 8 of [13]. The following example shows that a p_1-Lindelöf space is not a contra second countable space. In this example, we use some order set theoretical notions, which are due to Good and Papadopoulos [10].

Example 3.1. We consider a bitopological space (X, τ_1, τ_2), such that τ_1 makes X a LOTS. Thus, there exists a subbase for τ_1 which consists of the union of two nests $\mathcal{L}$ and $\mathcal{R}$, such both $\mathcal{L}$ and $\mathcal{R}$ are interlocking and $\mathcal{L} \cup \mathcal{R}$ obviously T_1-separates X. Having stated this, we consider all elements of $\mathcal{L}$ that have a maximal element. Then, according to the property of interlocking, the complement L^c of L will have a minimal element. We isolate all these L^cs that have a minimal element to form a nest. We do the same for the corresponding R^cs for the nest $\mathcal{R}$. The union of the two nests consisting of L^cs and R^cs form a subbase for the topology τ_2, which violates the definition of contra-second countability of (X, τ_1, τ_2). It shows that a p_1-Lindelöf space is not a contra second countable space.

4. Open questions.

Thus, at the end of this article, we want to propose some open questions. One may find from literature survey of bitopological space, that following questions are still unanswered. The questions may be simple in nature but they may help to unify concepts of Pairwise Lindelöfness, p-Lindelöfness and p_1-Lindelöfness. They are as follows:

Q.1. On which conditions p_1-Lindelöf space $\Longrightarrow$ p-Lindelöf space and vice-versa?

Q.2. On which condition, p_1-Lindelöf space $\Longleftrightarrow$ p-Lindelöf space?

In Example 2 of [13], it can be found that a p-normal space is a p_1-normal space but the converse is not true in general. Thus, we have the following questions on the concept of normality in a bitopological space.

Q.3. On which condition, p_1-normal space $\Longrightarrow$$p$-normal space?

Q.4. What is the unified relation between p_1-Lindelöf space, p-Lindelöf space and pairwise Lindelöf space?

Conclusion.

In this paper, we provide answer and some results related to p_1-Lindelöf space and countability with the help of an example of LOTS specially nest and interlocking. This paper may find some impacts where bitopological space plays crucial roles. Since, rough set version of bitopological space is developed in [31], thus it is natural to find scopes of this paper in computer science in near future.

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