



Convergence of a Series Leading to an Analogue of Ramanujan's Assertion on Squarefree Integers*

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ABSTRACT: Let d be a squarefree integer. We prove that

(i) $\sum_n \frac{\mu(n)}{n} d(n')$ converges to zero, where n' is the product of prime divisors of n with $\left(\frac{d}{n}\right) = +1$. We use the Prime Number Theorem.

(ii) $\prod_{\left(\frac{d}{p}\right)=+1} \left(1 - \frac{1}{p^s}\right)$ is not analytic at $s=1$, nor is $\prod_{\left(\frac{d}{p}\right)=-1} \left(1 - \frac{1}{p^s}\right)$.

(iii) The convergence (i) leads to a proof that asymptotically half the squarefree ideals have an even number of prime ideal factors (analogue of Ramanujan's assertion).

Key Words: Dirichlet series; Prime Number Theorem.

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1. Introduction

Ramanujan's Assertion was that asymptotically half the squarefree integers had an even number of prime factors ([2], pp 64-65). We prove the convergence of a certain series using the Prime Number Theorem. (Lemma B). This fills a gap in the proof of the analogue of Ramanujan's Assertion for Squarefree ideals in a Quadratic Field" ([6]).

2. Mathematics

Notation: If $n = (p_1 p_2 \cdots p_k)(q_1 q_2 \cdots q_l)$

$$\left(\frac{d}{p_i}\right) = +1 \quad \forall i, \quad \left(\frac{d}{q_j}\right) = -1 \text{ or } 0 \quad \forall j$$

write $n' = p_1 p_2 \cdots p_k$, $n'' = q_1 q_2 \cdots q_l$, $\mu(n)$ and $d(n)$ denote Moebius and divisor functions respectively.

Lemma B:

$$\sum_n \frac{\mu(n) d(n')}{n} \text{ converges (to 0)}$$

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Proof. Recall that for a multiplicative function $f(n)$ one has ([1], Theorem 11.7)

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_{p \text{ prime}} \left(1 + \frac{f(p)}{p^s} + \cdots + \frac{f(p^k)}{p^{ks}} + \cdots\right)$$

Choose $f(n) = \mu(n)d(n')$, f is indeed multiplicative

$$f(p) = \mu(p)d(p') = \begin{cases} -2 & \text{if } \left(\frac{d}{p}\right) = +1 \\ -1 & \text{if } \left(\frac{d}{p}\right) = -1 \text{ or } 0 \end{cases}$$

$$\therefore \sum_{n=1}^{\infty} \frac{\mu(n)d(n')}{n^s} = \prod_{p,+1} \left(1 - \frac{2}{p^s}\right) \prod_{q,-1,0} \left(1 - \frac{1}{q^s}\right), \quad (\operatorname{Re} s > 1)$$

with the obvious notation

Now ([5], p321)

$$\prod_{p,+1} \left(1 - \frac{2}{p^s}\right) = \prod_{p,+1} \left(1 - \frac{1}{p^s}\right)^2 \prod_{p,+1} \left(1 - \frac{1}{p^{2s}(1 - \frac{1}{p^s})^2}\right)$$

so that by combining terms

$$\sum_{n=1}^{\infty} \frac{\mu(n)d(n')}{n^s} = \left(\prod_{p,+1} \left(1 - \frac{1}{p^s}\right)\right) \left(\prod_p \left(1 - \frac{1}{p^s}\right)\right) \left(\sum_{n=1}^{\infty} \frac{a_n}{n^s}\right) \quad (\star)$$

where the last series converges absolutely for $\operatorname{Re} s > \frac{1}{2}$.

Indeed for $\operatorname{Re} s > 1$

$$\prod_p \left(1 - \frac{1}{p^s}\right) = \frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

We check that (\star) extends to $s = 1$ as a convergent Cauchy product of series.

Firstly, since $\sum_p \frac{1}{p} = \infty$

$$\prod_p \left(1 - \frac{1}{p}\right) = 0$$

and the Prime Number Theorem ([1], p 97) is equivalent to $\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0$. Thus at $s = 1$ the middle term above in (\star) is a series converging to zero

Claim: The first term in (\star) is also zero i.e.

$$\prod_{p,+1} \left(1 - \frac{1}{p}\right) = \sum_{n'} \frac{\mu(n')}{n'} = 0$$

where the sum is over all numbers n' which are products of primes of type +1. We prove this Claim by means of Lemmas 2.1 and 2.2 below. \square

Lemma 2.1.

$$\sum_{n' \leq x} \mu(n') = o(x)$$

Proof.

$$\begin{aligned} |Lim_{x \rightarrow \infty} \frac{1}{x} \sum_{n' \leq x} \mu(n')| &\leq Lim_{x \rightarrow \infty} \frac{1}{x} \left(\sum_{n' \leq x} |\mu(n')| \right) \\ &\leq Lim_{x \rightarrow \infty} \left\{ \frac{\left(\sum_{n' \leq x} |\mu(n')| \right) \sum_{n \leq x} |\mu(n)|}{\sum_{n \leq x} |\mu(n)| x} \right\} \\ &= Lim_{x \rightarrow \infty} \left\{ \frac{\sum_{n' \leq x} |\mu(n')|}{\sum_{n \leq x} |\mu(n)|} \right\} \cdot \frac{1}{\zeta(2)} \end{aligned}$$

Now we claim the ratio $\frac{\left(\sum_{n' \leq x} |\mu(n')| \right)}{\left(\sum_{n \leq x} |\mu(n)| \right)}$ tends to 0 as $x \rightarrow \infty$ because it can be rewritten as the ratio of the number of squarefree integers formed from the first t primes of symbol +1:

$\frac{|\{p_{i_1} p_{i_2} \dots p_{i_m} | i_j \leq t, (\frac{d}{p_{i_j}}) = +1\}|}{|\{p_{i_1} p_{i_2} \dots p_{i_m} | i_j \leq t\}|} t$ depending on x . But counting the "favourable cases" and using the above Theorem on density, this ratio is asymptotically $(\frac{3}{4})^t$ which tends to 0 as $t \rightarrow \infty$ or $x \rightarrow \infty$. This proves Lemma 2.1. \square

Lemma 2.2.

$$Lim_{x \rightarrow \infty} \sum_{n' \leq x} \frac{\mu(n')}{n'} = 0$$

Proof. Note that the argument of Hlawka et al ([3], p200) applies since $|\mu(n')| \leq 1$ (boundedness) and $\sum_{n' \leq x} \mu(n') = o(x)$. Thus with $f(n)$ defined by

$$f(n) = \begin{cases} \mu(n') & \text{if } n = n' \\ 0 & \text{if not} \end{cases}$$

we have

$$\begin{aligned} \sum_{n \leq x} \frac{f(n)}{n} &= \sum_{n' \leq x} \frac{\mu(n')}{n'} \\ &= \frac{1}{x} \sum_{n \leq x} \left(\sum_{k|n} f(k) \right) + o(1) \\ &= \frac{1}{x} \sum_{n \leq x, n \text{ squarefree}} \left(\sum_{k|n'} f(k) \right) + o(1) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{x} \sum_{n \leq x} \left(\sum_{k|n'} \mu(k) \right) + o(1) \\
&= \frac{1}{x} \sum_{n \leq x, n'=1} 1 \\
&= \frac{1}{x} |\{n|n \leq x, n = n'\}| + o(1)
\end{aligned}$$

Let $x \rightarrow \infty$ so that (since the fraction of n'' tends to 0)

$$\lim_{x \rightarrow \infty} \sum_{n' \leq x} \frac{\mu(n')}{n'} = 0$$

Now

$$\sum_{n=1}^{\infty} \frac{\mu(n)d(n')}{n} = \left[\sum_{n'} \frac{\mu(n')}{n'} \right] \left[\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \right] \left[\sum_{n=1}^{\infty} \frac{a_n}{n} \right]$$

Now one notes that the Cauchy product of the first two series on the RHS is zero by adapting the argument for $(\sum \frac{\mu(n)}{n})^2$ ([7], Lemma 3). Let $\sum_{n=1}^{\infty} \frac{b_n}{n}$ be this Cauchy product. Thus $\sum_{n=1}^{\infty} \frac{\mu(n)d(n')}{n} = (\sum \frac{b_n}{n})(\sum \frac{a_n}{n}) = (0)(A) = 0$ by Mertens'. \square

Corollary 2.3. *Writing (for $\operatorname{Re} s > 1$)*

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) = \prod_{p, +1} \left(1 - \frac{1}{p^s}\right) \prod_{p, -1, 0} \left(1 - \frac{1}{p^s}\right)$$

we cannot have both the products on the right to be analytic at $s = 1$.

Proof. $\frac{1}{\zeta(s)}$ has a simple zero at $s = 1$ ([1], Theorem 12.4) and by Lemma 2.1 and Remark 2.1 the two products have limits 0 as $s \rightarrow 1^+$. If both products extend analytically to $s = 1$ then the order of zero for the function is at least 2, contradicting known behaviour of $\zeta(s)$. \square

Remark 2.4. *The Dedekind Zeta function of $K = \mathbb{Q}(\sqrt{d})$ has a factorization ($\operatorname{Re} s > 1$)*

$$\zeta_K(s) = \zeta(s)L_d(s)$$

where ζ is the Riemann Zeta function and L_d is the L -series ([3]). On the other hand

$$\begin{aligned}
\zeta_K(s) &= \prod_{\left(\frac{d}{p}\right)=+1} \left(1 - \frac{1}{p^s}\right)^{-2} \prod_{\left(\frac{d}{p}\right)=-1} \left(1 - \frac{1}{p^{2s}}\right)^{-1} \prod_{p|d} \left(1 - \frac{1}{p^s}\right)^{-1} \\
&= \prod_{\left(\frac{d}{p}\right)=+1} \left(1 - \frac{1}{p^s}\right)^{-2} f(s)
\end{aligned}$$

with f analytic for $\operatorname{Re} s > \frac{1}{2}$. By the above ζ_K has a simple pole at $s = 1$ coming from the pole of ζ . Thus if there is a zero at $s = 1$ for the "analytic" function

$$\prod_{\left(\frac{d}{p}\right)=+1} \left(1 - \frac{1}{p^s}\right)$$

then ζ_K has a double pole at $s = 1$, contradicting the above. Hence $\prod_{\left(\frac{d}{p}\right)=+1} \left(1 - \frac{1}{p^s}\right)$ is not analytic at $s = 1$.

Now if $\prod_{\left(\frac{d}{p}\right)=-1,0} \left(1 - \frac{1}{p^s}\right)$ is analytic at $s = 1$ then in Cor 2.3 by cross multiplication, we have the function $\prod_{\left(\frac{d}{p}\right)=+1} \left(1 - \frac{1}{p^s}\right)$ to be meromorphic at $s = 1$. But this is not the case, by above, there is no pole at $s=1$ and it is not analytic at $s=1$.

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