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Cauchy Representation of Fractional Fourier Transform for Boehmians

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ABSTRACT: Results relating to fractional Fourier transform and their properties in the Lizorkin space are employed in this paper to investigate the Cauchy representation of fractional Fourier transform for integrable Boehmians. An inversion formula for the fractional Fourier transform is addressed. The conclusion remark of the paper spells the initiation for the present investigation.

Key Words: Cauchy integral, Fourier transform, Fractional Fourier transform, Distribution spaces, Boehmains.

Contents

1. Introduction

The Fourier transform of a function $u \in L_1(\mathbb{R})$, is defined by

$$\hat{u}(\omega) = (\mathcal{F}u)(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(t) e^{-i\omega t} dt, \qquad \omega \in \mathbb{R}. \tag{1.1}$$

If $\hat{u} \in L_1(\mathbb{R})$, then the inverse Fourier transform is given by

$$u(t) = (\mathcal{F}^{-1}\hat{u})(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{u}(\omega) e^{i\omega t} d\omega, \qquad t \in \mathbb{R}. \tag{1.2}$$

Fractional Fourier transform is a rotation operation on the time-frequency distribution and it can transform a function into the domain between time and frequency. One dimensional fractional Fourier transform with parameter α, of a function $u(x)$, is given by [16]:

$$\hat{u}_\alpha(\xi) = (\mathcal{F}_\alpha u)(\xi) = \int_{\mathbb{R}} K_\alpha(t,\xi) u(t) dt, \tag{1.3}$$

kernel of which is

$$K_\alpha(t,\xi) = \begin{cases} C_\alpha e^{\frac{i(t^2+\xi^2)\cot\alpha}{2} - i\,t\,\xi\csc\alpha} & \text{if } \alpha \neq n\pi \\ \frac{1}{\sqrt{2\pi}} e^{-it\xi} & \text{if } \alpha = \frac{\pi}{2} \end{cases}$$

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where n is an integer, and

$$C_\alpha = (2\pi i \sin\alpha)^{\frac{-1}{2}} e^{\frac{i\alpha}{2}} = \sqrt{\frac{1 - i\cot\alpha}{2\pi}}.$$

The corresponding inversion formula is given by

$$\varphi(t) = \frac{1}{2\pi}\int_{\mathbb{R}} \overline{K_\alpha(t,\xi)}\,\hat{u}_\alpha(\xi)dw, \tag{1.4}$$

where

$$\begin{aligned}\overline{K_\alpha(t,\xi)} &= \frac{(2\pi i\sin\alpha)^{\frac{1}{2}}}{\sin\alpha} e^{\frac{-i\alpha}{2}} e^{-\frac{i(t^2+\xi^2)\cot\alpha}{2}} + it\xi\csc\alpha \\ &= C'_\alpha e^{-\frac{i(t^2+\xi^2)\cot\alpha}{2}} + it\xi\csc\alpha, \\ C'_\alpha &= \frac{(2\pi i\sin\alpha)^{\frac{1}{2}}}{\sin\alpha} e^{\frac{-i\alpha}{2}} = \sqrt{2\pi(1+i\cot\alpha)}.\end{aligned}$$

On the other hand, Luchko et al. [8] introduced a new fractional Fourier transform $\mathcal{F}_\alpha$ of order α $(0 < \alpha \le 1)$ defined by the relation

$$\mathcal{F}_\alpha[u](\omega) = \int_{\mathbb{R}} u(t)e_\alpha(\omega,t)dt, \tag{1.5}$$

where

$$e_\alpha(\omega,t) = e^{-i|\omega|^{1/\alpha}t}, \ \ \omega \le 0\,; \quad e_\alpha(\omega,t) = e^{i|\omega|^{1/\alpha}t}, \ \ \omega \ge 0. \tag{1.6}$$

If $\alpha = 1$, then $e_\alpha(\omega,t) = e^{i\omega t}$, and $\mathcal{F}_1[u](\omega)$ coincides with the classical Fourier transform (1.1).

Let S be the space of rapidly decreasing test functions namely, the space of infinitely differentiable functions: $v(x)$ on $\mathbb{R}$ satisfying the relation

$$\gamma_{m,k}(v) = \sup_{x\in\mathbb{R}}(1+|x|)^m \left|v^{(k)}(x)\right| < \infty,$$

for any $m \in \mathbb{N}_0$ and $k \in \mathbb{N}_0$ $(\mathbb{N}_0 = \mathbb{N}\cup\{0\})$. It follows from here that if $v(x) \in S$, then

$$\left|v^{(k)}(x)\right| \le \frac{A}{|x|^m} \qquad (m,k \in \mathbb{N}_0,\ m > k;\ |x| \to \infty). \tag{1.7}$$

Denoted by $V(\mathbb{R})$, is the set of functions $v \in S$, satisfying

$$\frac{d^n v}{dx^n}|_{x=0} = 0 \qquad (n = 0,1,2,...).$$

The Lizorkin space [7] $\Phi(\mathbb{R})$ is introduced as the Fourier pre-image of the space $V(\mathbb{R})$ in the space S,

$$\Phi(\mathbb{R}) = \{\varphi \in S(\mathbb{R}) : \mathcal{F}\varphi \in V(\mathbb{R})\}\,. \tag{1.8}$$

According to the definition of the Lizorkin space, any function $\varphi \in \Phi(\mathbb{R})$ satisfies the orthogonality condition

$$\int_{-\infty}^{\infty} x^n \varphi(x)dx = 0, \qquad n = 0, 1, 2, ...$$

The space $\Phi(\mathbb{R})$ is also invariant with respect to the Fourier transform (1.1) and its inverse (1.2), and simultaneously, these transforms are inverse of each other

$$\mathcal{F}^{-1}\mathcal{F}u = u \qquad (u \in \Phi(\mathbb{R})). \tag{1.9}$$

Definition 1.1. *Let u be a function belonging to $\Phi(\mathbb{R})$. Then the fractional Fourier transform of order α, $0 < \alpha \leq 1$, is defined by*

$$\mathcal{F}_\alpha[u](\omega) = \int_{\mathbb{R}} u(t)e^{i\omega^{1/\alpha}t}dt, \tag{1.10}$$

whereas the corresponding inverse fractional Fourier transform of order α is given by

$$u(t) = \frac{1}{2\pi\alpha}\int_{\mathbb{R}} e^{-i\omega^{1/\alpha}t}(\mathcal{F}_\alpha[u](\omega))\omega^{\frac{1-\alpha}{\alpha}}d\omega. \tag{1.11}$$

Now, considering L_1 as the space of complex valued Lebesgue integrable functions on real line $\mathbb{R}$, the norm of the function is defined by

$$||f|| = \int_{\mathbb{R}} |\ f(x)|\ dx$$

and the convolution product

$$(f * g)(x) = \int_{\mathbb{R}} f(u)g(x-u)du, \qquad f, g \in L_1$$

is an element of L_1, and

$$\|f * g\| \leq \|f\|\|g\|.$$

Lemma 1.2. *Let u be a function belonging to $\Phi(\mathbb{R})$. Then the fractional Fourier transform of order $\alpha, 0 < \alpha \leq 1$, is*

$$\mathcal{F}_\alpha u(\omega^\alpha) = \sqrt{2\pi}\hat{u}(\omega). \tag{1.12}$$

Lemma 1.3. *Let f and g are functions belonging to $\Phi(\mathbb{R})$. Then*

$$\mathcal{F}_\alpha[f * g] = \mathcal{F}_\alpha[f].\mathcal{F}_\alpha[g] \tag{1.13}$$

The proof of the Lemma may be referred to [14].

2. Cauchy representation and the fractional Fourier transform

Let us enumerate the basic properties and some definitions of the Cauchy representation of the fractional Fourier transform by following [3], and [6].

Suppose $f \in L_2$, $g(\omega) = \mathfrak{F}_\alpha(f, \omega)$ and $\hat{g}_\alpha(z)$ is the Cauchy representation of $g_\alpha(z)$, where $z = x + iy$. Then

$$\hat{g}_\alpha(z) = \begin{cases} \displaystyle\int_0^\infty f(t) e_\alpha(t, z) dt, & y > 0 \\ \displaystyle -\int_{-\infty}^0 f(t) e_\alpha(t, z) dt, & y < 0 \end{cases} \tag{2.1}$$

where $e_\alpha(t, z)$ defined by (1.6).

Proposition 2.1. *Suppose $g \in L_2$, $f(t) = \mathfrak{F}_\alpha^{-1}(g, t)$, and $\hat{f}(z), z = x + iy$, be the Cauchy representation of f. Then*

$$\hat{f}_\alpha(z) = \begin{cases} \displaystyle(2\pi)^{-1} \int_{-\infty}^0 g(\omega) e_\alpha(\omega, z) d\omega, & y > 0 \\ \displaystyle -(2\pi)^{-1} \int_{-\infty}^0 g(\omega) e_\alpha(\omega, z) d\omega, & y < 0 \end{cases} \tag{2.2}$$

Definition 2.2. *A function f is called a locally integrable function if f is a continuous function and for some α,*

$$| f | = O(| t |)^\alpha). \tag{2.3}$$

Definition 2.3. *The generalized fractional Fourier transform and the generalized inverse fractional Fourier transform of a tempered function f are defined and denoted, respectively, by*

$$\hat{\mathfrak{F}}_\alpha(f, z) = \begin{cases} \displaystyle\int_0^\infty f(t) e_\alpha(t, z) dt, & y > 0 \\ \displaystyle -\int_{-\infty}^0 f(t) e_\alpha(t, z) dt, & y < 0 \end{cases} \tag{2.4}$$

and

$$\hat{\mathfrak{F}}_\alpha^{-1}(f, z) = \begin{cases} \displaystyle(2\pi)^{-1} \int_{-\infty}^0 f(t) e_\alpha(t, z) dt, & y > 0 \\ \displaystyle -(2\pi)^{-1} \int_0^\infty f(t) e_\alpha(t, z) dt, & y < 0 \end{cases} \tag{2.5}$$

where $z = x + iy$.

Proposition 2.4. *$\hat{\mathfrak{F}}_\alpha(f,z)$ and $\hat{\mathfrak{F}}_\alpha^{-1}(f,z)$, defined in (2.4) and (2.5) are analytic functions for $y \neq 0$.*

Following identities hold true between $\hat{\mathfrak{F}}_\alpha(f,z)$ and the fractional Fourier transform:

$$\hat{\mathfrak{F}}_\alpha(f, x+i\epsilon) = \mathfrak{F}_\alpha(f(t)H(t)e^{-\epsilon|t|^{1/\alpha}}, x), \epsilon > 0 \tag{2.6}$$

and

$$\hat{\mathfrak{F}}_\alpha(f, x-i\epsilon) = -\mathfrak{F}_\alpha(f(t)H(-t)e^{-\epsilon|t|^{1/\alpha}}, x), \epsilon > 0 \tag{2.7}$$

where $H(t)$ is a positive function, which is convenient to consider in the proof of inversion theorem and has a positive fractional Fourier transform whose integral is easily calculated.

Therefore, we have

$$\hat{\mathfrak{F}}_\alpha(f, x+i\epsilon) - \hat{\mathfrak{F}}_\alpha(f, x-i\epsilon) \ = \ \int_{-\infty}^{\infty} f(t)e_\alpha(t,x)(2\cosh(|t|^{1/\alpha}\epsilon))dt. \tag{2.8}$$

Proposition 2.5. *Let f is a locally integrable function, then the generalized fractional Fourier transform of f has the property*

$$\langle \mathfrak{F}_\alpha(f), \varphi\rangle \ = \ \lim_{\epsilon\to 0}\int_{-\infty}^{\infty}(\hat{\mathfrak{F}}_\alpha(f, x+i\epsilon) - \hat{\mathfrak{F}}_\alpha(f, x-i\epsilon))\varphi(x)\, dx, \tag{2.9}$$

for all $\varphi \in S$ and $\epsilon > 0$.

Proposition 2.6. *For a given locally integrable function f, the inversion formula for the generalized fractional Fourier transform $\hat{\mathfrak{F}}_\alpha(f,z)$ has the property*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}(\hat{\mathfrak{F}}_\alpha(f, x+i\epsilon) - \hat{\mathfrak{F}}_\alpha(f, x-i\epsilon))e_\alpha(t,x)\, dt = 2\cosh(|t|^{1/\alpha}\epsilon)f(t),\ \epsilon > 0 \tag{2.10}$$

and

$$\lim_{\epsilon\to 0}\mathfrak{F}_\alpha^{-1}(\hat{\mathfrak{F}}_\alpha(f, x+i\epsilon) - \hat{\mathfrak{F}}_\alpha(f, x-i\epsilon), t) \ = \ 2f(t),\ \epsilon > 0. \tag{2.11}$$

Proposition 2.7. *Let T is a functional for the m th derivative of the locally integrable function defined in L_1, then the generalized fractional Fourier transform of T is defined as $\hat{\mathfrak{F}}_\alpha(T,z) = (-i\omega^{1/\alpha})^m\hat{\mathfrak{F}}_\alpha(f,z)$.*

Let $T \in S'$ and $\hat{\mathfrak{F}}_\alpha(T,z)$ be a generalized fractional Fourier transform of T. Then $\hat{\mathfrak{F}}_\alpha(t,z)$ is an analytic (Cauchy) representation of $\hat{\mathfrak{F}}_\alpha(T)$ in the sense, that

$$\lim_{\epsilon\to 0}\langle\hat{\mathfrak{F}}_\alpha(f, x+i\epsilon) - \hat{\mathfrak{F}}_\alpha(f, x-i\epsilon), \varphi(x)\rangle \ = \ \langle\mathfrak{F}_\alpha(T), \varphi\rangle,\ \varphi \in S, \tag{2.12}$$

where S' is the space of linear functional on S or the space of tempered distributions.

Definition 2.8. *The generalized fraction Fourier transform $\mathcal{F}_\alpha(f,z)$ for a locally integrable function f, for multi-variables is defined by*

$$\int_0^\infty \cdots \int_0^\infty f(t)e_\alpha(t,z)dt,\ y_1>0,\cdots,y_n>0, \tag{2.13}$$

and

$$-\int_{-\infty}^0 \int_0^\infty \cdots \int_0^\infty f(t)e_\alpha(t,z)dt,\ y_1<0,\ y_2,y_3,\cdots,y_n>0, \tag{2.14}$$

where $z_j = x_j + iy_j$, $j = 1,2,3,....$

Proposition 2.9. *Suppose $f \in L_1$, $\mathcal{F}_\alpha(f) \in L_1$ and $F = \mathcal{F}_\alpha(f)$, then the Cauchy representation of $\hat{F}(z)$ by means of the Cauchy kernel is given by*

$$\hat{F}(z) \ = \ \hat{\mathcal{F}}_\alpha(f,z),\ y_1,y_2,\cdots,y_n \neq 0. \tag{2.15}$$

Definition 2.10. *For any tempered distribution space S, the space of linear functional S' on S, the space $\mathcal{D}$ of all infinitely differentiable functions on $\mathbb{R}^n$ with compact support (A set $K \subset X$, a topological space, is called compact if every open cover of K contains a finite subcover) and its dual $\mathcal{D}'$ if $S \in \mathcal{D}'$ and $T \in \mathscr{E}'$, then the convolution of the distributions is defined by*

$$S \star T \ = \ \mathcal{F}_\alpha{}^{-1}(\mathcal{F}_\alpha(S)\cdot\mathcal{F}_\alpha(T)) \ = \ 2\pi\mathcal{F}_\alpha(\mathcal{F}_\alpha{}^{-1}(S)\cdot\mathcal{F}_\alpha{}^{-1}(T)). \tag{2.16}$$

The space $\mathscr{E}(a,b)$ is the space of smooth functions on (a,b) and $\mathscr{E}'(a,b)$, or simply $\mathscr{E}'$, is the dual of the space $\mathscr{E}$.

As a consequence of equation (2.16), if $S \in S'$ and $T \in E'$, then $S \star T \in S'$. Further, the convolution of the generalized Fourier transform is given by

$$\hat{\mathcal{F}}_\alpha(f\star g,z) \ = \ \begin{cases} \hat{\mathcal{F}}_\alpha(f,z)\hat{\mathcal{F}}_\alpha(g,z) & ,y>0 \\ 0 & ,y<0 \end{cases} \tag{2.17}$$

Proposition 2.11. *Let $S,T \in S'$ have support in the half axis $\{t : t > 0\}$. Then,*

$$S\star T \ = \ \lim_{\epsilon\to 0}\mathcal{F}_\alpha^{-1}(\hat{\mathcal{F}_\alpha}(S,x+i\epsilon)\hat{\mathcal{F}}_\alpha(T,x+i\epsilon)), \tag{2.18}$$

where $S \star T \in S'$.

3. Cauchy Representation for Integrable Boehmians

The general construction of Boehmians is given in [9,10] which when applied to various function spaces, various Boehmian spaces result. The term Boehmian is used for all objects by an abstract algebraic construction, similar to that of the field of quotients. Let G be an additive commutative semigroup and $S \subseteq G$, is a sub-semigroup, which has a mapping $*$ from $G \times S$ to G such that

(i) if $\delta, \eta \in S$, then $(\delta * \eta) \in S$ and $\delta * \eta = \eta * \delta$

(ii) if $\alpha \in G, \delta, \eta \in S$, then $(\alpha * \delta) * \eta = \alpha * (\delta * \eta)$
(iii) if $\alpha, \beta \in G, \delta \in S$, then $(\alpha + \beta) * \delta = (\alpha * \delta) + (\beta * \delta)$.
The member of the class Δ of sequence from S are called the delta sequence, which satisfies the following:
(i) if $\alpha, \beta \in G, (\delta_n) \in \Delta$ and $(\alpha * \delta_n) = (\beta * \delta_n)$, for all n, then $\alpha = \beta$ in G.
(ii) if $(\delta_n), (\varphi_n) \in \Delta$, then $(\delta_n * \varphi_n) \in \Delta$.

We consider a quotient of the sequence f_n/φ_n, numerator of which belongs to G and the denominator is a delta sequence, and

$$f_m * \varphi_n = f_n * \varphi_m, \qquad \text{for all } m, n \in \mathbb{N}. \tag{3.1}$$

Two quotients of sequences f_n/φ_n and g_n/ψ_n are said to be equivalent if

$$f_n * \psi_n = g_n * \varphi_n, \qquad \text{for all } n \in \mathbb{N}. \tag{3.2}$$

The equivalence classes, thus obtained, are called Boehmians, the space of all of which bears the notation B and an element of which is written as $x = f_n/\varphi_n$. If we consider G to be the set of all locally integrable functions on $\mathbb{R}$, then the Boehmian space B is called the space of locally integrable Boehmian B_{L_1}, which has properties of addition, scalar by multiplication, convolution in a convolution algebra [11].

A sequence of Boehmians F_n is Δ-convergent to a Boehmian F if there exists a delta sequence (δ_n) such that $(F_n - F) * \delta_n \in L_1$, for every $n \in \mathbb{N}$ and $\|(F_n - F) * \delta_n\| \to 0$, as $n \to \infty$.

A sequence of Boehmians F_n is δ-convergent to a Boehmian F if there exists a delta sequence (δ_n) such that $F_n * \delta_k \in L_1$ and $F * \delta_k \in L_1$, for every $n, k \in \mathbb{N}$ and $\|(F_n - F) * \delta_k\| \to 0$, for each $k \in \mathbb{N}$. For convergence of Boehmians, see [9] and refer to [10] for the properties of integrable Boehmians. If

$$\Delta - \lim_{n\to\infty} F_n = F$$

and

$$\Delta - \lim_{n\to\infty} G_n = G$$

then

$$\Delta - \lim_{n\to\infty} F_n * G_n = F * G$$

where Δ is a class of sequence $(\delta_n)\,(n = 1, 2, \ldots)$. For (δ_n), the quotient $[\delta_n/\delta_n]$ corresponds to Dirac delta function δ, all the derivatives of δ are also integrable Boehmians.

Let $F = [f_n/\delta_n] \in B_{L_1}$, then for each $n \in \mathbb{N}$, we have, $f_1 * \delta_n = f_n * \delta_1$. Since $\int_{\mathbb{R}} \delta_n(x)dx = 1, \forall n \in \mathbb{N}$, it follows that

$$\int_{\mathbb{R}} f_1(x)dx = \int_{\mathbb{R}} (f_1 * \delta_n)(x)dx = \int_{\mathbb{R}} (f_n * \delta_1)(x)dx = \int_{\mathbb{R}} f_n(x)dx. \tag{3.3}$$

The integral of a Boehmian as follows from the property, that, if $F = [f_n/\delta_n] \in B_{L_1}$ then

$$\int_{\mathbb{R}} F(x)dx = \int_{\mathbb{R}} f_1(x)dx. \tag{3.4}$$

For a function from L_1, the definition, given by (3.4), happens to be the analogous to the definition of the Lebesgue integral. However, there are functions which are integrable in the sense of Boehmians, but not so as an ordinary function. For instance, a continuously differentiable function on L_1 is such that its derivative does not exist in L_1.

Lemma 3.1. *Let $f \in L_1$ and $z = x + iy$, from Properties 2.3, 2.4 and 2.5, we have*

$$f_n(t) = \mathcal{F}_\alpha{}^{-1}(\hat{\mathcal{F}}_\alpha(f,z)) = \mathcal{F}_\alpha{}^{-1}(\hat{\mathcal{F}}_\alpha F(z)) \tag{3.5}$$

$$= \frac{1}{2\pi\alpha}\int_{-\infty}^{\infty} \hat{\mathcal{F}}_\alpha(F(z))e_\alpha(t,z)z^{\frac{1-\alpha}{\alpha}}\, dt. \quad \text{by (1.11)} \tag{3.6}$$

Then the sequence (f_n) converges to f in L_1-norm.

Lemma 3.2. *If $[f_n/\delta_n] \in B_{L_1}$, then the sequence*

$$\hat{\mathcal{F}}_\alpha(f_n, z) = (\mathcal{F}_\alpha f_n(t)) = \int_{\mathbb{R}} f(t)H(t)e_\alpha(t,z)dt. \tag{3.7}$$

converges uniformly on each compact set in $\mathbb{R}$.

Proof: If (δ_n) is a delta sequence, then $\mathcal{F}_\alpha(\delta_n)$ converges uniformly on each compact set to a constant function 1. Therefore, for each compact set K, $\mathcal{F}_\alpha(\delta_k) > 0$ on K, and for almost all $k \in K$, we have

$$\mathcal{F}_\alpha(f_n) = \mathcal{F}_\alpha(f_n)\frac{\mathcal{F}_\alpha(\delta_k)}{\mathcal{F}_\alpha(\delta_k)} = \frac{\mathcal{F}_\alpha(f_n \star \delta_k)}{\mathcal{F}_\alpha(\delta_k)}$$

i.e.

$$= \frac{\mathcal{F}_\alpha(f_k \star \delta_n)}{\mathcal{F}_\alpha(\delta_k)} = \frac{\mathcal{F}_\alpha(f_k)}{\mathcal{F}_\alpha(\delta_k)}\mathcal{F}_\alpha(\delta_n)$$

on K.

In view of Lemma 3.1, the fractional Fourier transform of an integrable Boehmian $F = [f_n/\delta_n]$ can be defined as the limit of $\mathcal{F}_\alpha(f_n)$ in the space of continuous functions on $\mathbb{R}$. Hence, this proves that the fractional Fourier transform of an integrable Boehmian is a continuous function and, thereby, the lemma is proved. □

Theorem 3.3. *Let $F, G \in B_{L_1}$. Then,*
(i) $\hat{\mathcal{F}}_\alpha(\lambda(F,z)) = \lambda\hat{\mathcal{F}}_\alpha(F,z)$, (for any complex number)
(ii) $\hat{\mathcal{F}}_\alpha((F,z)+(G,z)) = \hat{\mathcal{F}}_\alpha(F,z) + \hat{\mathcal{F}}_\alpha(G,z)$
(iii)

$$\hat{\mathcal{F}}_\alpha(F \star G, z) = \begin{cases} \hat{\mathcal{F}}_\alpha(F,z)\hat{\mathcal{F}}_\alpha(G,z), & y > 0 \\ 0, & y < 0 \end{cases} \tag{2.16}$$

(iv) $\hat{\mathcal{F}}_\alpha[f(t-x)](\omega) = e^{i|\omega|^{1/\alpha}x}\hat{\mathcal{F}}_\alpha(f,z)$
(v) $\hat{\mathcal{F}}_\alpha[D^n f](\omega) = \left(-i\omega^{1/\alpha}\right)^n \hat{\mathcal{F}}_\alpha(f,z)$
(vi) If $\Delta - \lim_{n\to\infty} F_n = F$ *then* $\mathcal{F}_\alpha(F_n) \to \mathcal{F}_\alpha(F)$ *uniformly on each compact set.*

Proof: By virtue of the properties of fractional Fourier transform [8], indeed, the proofs of (i) and (ii) are obvious. By the definition of the convolution transform [5, p. 785] and by Lemma 1.2, the property (iii) can easily be proved. The proofs of properties (iv) and (v) are same as in [5, pp. 785-786], we obtain these results after some simplifications. Proof of the property (vi) is as follows :

We have

$$\delta - \lim_{n\to\infty} F_n - F \Rightarrow \ \mathcal{F}_\alpha(F_n) \to \mathcal{F}_\alpha(F)$$

uniformly on each compact set. Let (δ_n) be a delta sequence such that

$$F_n \star \delta_k, F \star \delta_k \in L_1, \forall n, k \in \mathbb{N}$$

and

$$||(F_n - F) \star \delta_k|| \to 0, \text{ as } n \to \infty, \text{ for all } k \in \mathbb{N},$$

where k is well defined. Then $\mathcal{F}_\alpha(\delta_k) > 0$ on K for some $k \in \mathbb{N}$. Since $\mathcal{F}_\alpha(\delta_k)$ is a continuous function, it is enough to prove that $\mathcal{F}_\alpha(F_n) \cdot \mathcal{F}_\alpha(\delta_k) \to \mathcal{F}_\alpha(F)\ \mathcal{F}_\alpha(\delta_k)$ uniformly on K. But we have,

$$\mathcal{F}_\alpha(F_n)\mathcal{F}_\alpha(\delta_k) - \mathcal{F}_\alpha(F) \cdot (\delta_k) = \mathcal{F}_\alpha((F_n - F) \star \delta_k)$$

and

$$||(F_n - F) \star \delta_k|| \to 0, \qquad \text{as } n \to \infty.$$

This, explicitly, proves the property (vi). The theorem is thus, completely proved. □

Theorem 3.4. *Let* $f \in B_{L_1}$ *and*

$$f_n(t) \ = \ \frac{1}{2\pi\alpha}\int_{-\infty}^{\infty} \hat{\mathcal{F}}_\alpha(F(z))e_\alpha(t,z)z^{\frac{1-\alpha}{\alpha}}\ dt. \tag{2.17}$$

Then

$$\delta - \lim_{n\to\infty} f_n \ = \ F,$$

and hence, also $\qquad \Delta - \lim_{n\to\infty} f_n \ = \ F.$

Proof: Let $F = [g_n/\delta_n]$, $k \in \mathbb{N}$. Then

$$(f_n \star \delta_k)(t) \ = \ \int_{\mathbb{R}} f_n(t-u)\delta_k(u)\, du$$

i.e.

$$= \ \frac{1}{2\pi\alpha}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-iz^{1/\alpha}(t-u)}\hat{\mathcal{F}}_\alpha(F(z))\delta_k(u)z^{\frac{1-\alpha}{\alpha}}\ dudz,$$

$$= \frac{1}{2\pi\alpha}\int_{\mathbb{R}} z^{\frac{1-\alpha}{\alpha}} e^{-iz^{1/\alpha}t}\hat{\mathfrak{F}}_\alpha(F(z))\hat{\mathfrak{F}}_\alpha\delta_k(z)\,dz$$

i.e.

$$(f_n\star\delta_k)(t) = \frac{1}{2\pi\alpha}\int_{\mathbb{R}} z^{\frac{1-\alpha}{\alpha}} e_\alpha(t,z)\hat{\mathfrak{F}}_\alpha(F\star\delta_k)\,dz.$$

By Theorem 3.3, $\|f_n\star\delta_k - F\star\delta_k\| \to 0$, as $n\to\infty$. Since k being an arbitrary positive integer, thus, $\delta - \lim_{n\to\infty} f_n = F$. This proves the theorem.

□

Conclusion

This paper can be considered a fair and reasonable generalization of the work given in the citation [6] in this paper. The concerned paper investigates the Cauchy representation of integrable and tempered Boehmians, which justifies the natural generalization of the tempered distributions (in the sense of Schwartz). As a result and for natural consequence, the present article becomes more of general nature by addressing the extension with the involvement of the fractional integral operator, the reference [14] of this paper highlights the concept.

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