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# Some Properties of a Class of Analytic Functions*

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ABSTRACT: Making use of convolution, we introduce and investigate a certain class of functions which is analytic in the open unit disk. We obtain interesting properties of starlikeness and convexity for this function class. Special cases and some useful consequences of our main results are also mentioned.

Key Words: Analytic functions; Starlike functions; Convex functions; Hadamard product (or Convolution); Subordination.

## Contents



## 1. Introduction and preliminaries

Let the functions

$$\phi_1(z) = \sum_{k=0}^{\infty} a_k z^k \quad \text{and} \quad \phi_2(z) = \sum_{k=0}^{\infty} b_k z^k$$

be analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\}$. The Hadamard product (or convolution) $(\phi_1 * \phi_2)(z)$ of $\phi_1(z)$ and $\phi_2(z)$ is defined by

$$(\phi_1 * \phi_2)(z) = \sum_{k=0}^{\infty} a_k b_k z^k = (\phi_2 * \phi_1)(z).$$

Let $\mathcal{A}$ be the class of normalized functions $\phi(z)$ of the form

$$\phi(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U}$. The subclasses of the class $\mathcal{A}$ denoted by $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ are , respectively, the subclasses of starlike functions of order $\alpha(0 \le \alpha < 1)$ in $\mathbb{U}$, and the convex functions of order $\alpha(0 \le \alpha < 1)$ in $\mathbb{U}$. In

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particular, the classes $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{K}(0) = \mathcal{K}$ are well known classes of starlike and convex functions in $\mathbb{U}$, respectively.

For functions $\phi(z) \in \mathcal{A}$ and $\varphi(z) \in \mathcal{A}$. We say that the function $\phi(z)$ is *subordinate* to $\varphi(z)$ in $\mathbb{U}$, and we write $\phi(z) \prec \varphi(z)$, if there exists an analytic function $w(z)$ in $\mathbb{U}$ such that

$$|w(z)| \leq |z| \quad \text{and} \quad \phi(z) = \varphi(w(z)) \quad (z \in \mathbb{U}).$$

Furthermore, if the function $\varphi(z)$ is univalent in $\mathbb{U}$, then

$$\phi(z) \prec \varphi(z) \Leftrightarrow \phi(0) = \varphi(0) \quad \text{and} \quad \phi(\mathbb{U}) \subset \varphi(\mathbb{U}).$$

Let the functions

$$f(z) = 1 + \sum_{k=1}^{\infty} a_k z^k \quad \text{and} \quad g(z) = 1 + \sum_{k=1}^{\infty} b_k z^k \tag{1.2}$$

be analytic in the open unit disk $\mathbb{U}$. We introduce and investigate a class of functions $\Phi^{\lambda}_{F,G}(z)$ defined by

$$\Phi(z) = \Phi^{\lambda}_{F,G}(z) = \frac{(F(z) * f(z)) - 1}{(G(z) * g(z))^{\lambda}} \; (\lambda > 0; f_1 a_1 = 1), \tag{1.3}$$

which is analytic in the open unit disc $\mathbb{U}$, where the functions $F$ and $G$ are of the form (1.2) with the coefficients, $a_k$ and $b_k$, respectively, replaced by $f_k \geq 0$ and $g_k \geq 0$.

By applying elementary calculations, we observe that

$$\Phi(0) = \Phi'(0) - 1 = 0,$$

which asserts that the class $\Phi(z) \in \mathcal{A}$.

It is observed that when

$$F(z) =_2 F_1(a', b'; c'; z) = \sum_{k=1}^{\infty} \frac{(a')_k (b')_k}{(c')_k (1)_k} z^k \quad (a' > 0, b' > 0, c' > 0; z \in \mathbb{U})$$

and

$$G(z) =_2 F_1(a'', b''; c''; z) = \sum_{k=1}^{\infty} \frac{(a'')_k (b'')_k}{(c'')_k (1)_k} z^k \quad (a'' > 0, b'' > 0, c'' > 0; z \in \mathbb{U}),$$

then

$$\Phi^{\lambda}_{F,G}(z) = \mathcal{H}^{\lambda}_{a',b',c';a'',b'',c''}(z) = \frac{({}_2F_1(a', b'; c'; z) * f(z)) - 1}{({}_2F_1(a'', b''; c''; z) * g(z))^{\lambda}}, \tag{1.4}$$

with $(\lambda > 0; a'b'a_1 = c'; z \in \mathbb{U})$, where ${}_2F_1$ is the well-known Gaussian hypergeometric function (see [11]).

The class of functions $\mathcal{H}^{\lambda}_{a',b',c';a'',b'',c''}(z)$ was investigated by Khosravianarab et al. [5].

On the other hand, when

$$F(z) = 1 + z \quad \text{and} \quad G(z) = 1 + \sum_{k=1}^{\infty} z^k,$$

then

$$\Phi^{\lambda}_{F,G}(z) = F^{\lambda}(z) = \frac{z}{(1 + \sum_{k=1}^{\infty} b_k z^k)^{\lambda}} \quad (\lambda > 0; z \in \mathbb{U}). \tag{1.5}$$

The class of functions $F^{\lambda}(z)$ was studied by Raina and Bansal [6] and contains as special cases the classes due to Fukui et al. [2] and Reade at al. [7]. In this paper we investigate the geometric properties of starlikeness and convexity for the function class $\Phi(z)$ defined above by (1.3). We also consider some relevant particular cases of our main results by mentioning few known (and new) results.

## 2. Main results

The starlikeness property satisfied by the class of functions $\Phi(z)$ defined by (1.3) is contained in Theorem 1 below.

**Theorem 1** *Let* $\Phi(z)$ *be defined by* (1.3), *then*

$$\Phi(z) \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}),$$

*provided that*

$$\sum_{k=2}^{\infty} f_k |a_k| \left( k - 1 + A - Bk + 2\lambda \sum_{k=1}^{\infty} k g_k |b_k| \right) + \sum_{k=1}^{\infty} (k\lambda + |k\lambda - (A - B)|) g_k |b_k|$$
$$+ \left( \sum_{k=2}^{\infty} (2k - 1 + A) f_k |a_k| \right) \left( \sum_{k=1}^{\infty} g_k |b_k| \right) \leq A - B, \tag{2.1}$$

*where* $-1 \leq B < A \leq 1, B \leq 0$.

*Proof.* In order to prove that $\Phi(z) \prec \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1, B \leq 0)$ it is sufficient to show that

$$\frac{z\Phi'(z)}{\Phi(z)} = \frac{1 + Aw(z)}{1 + Bw(z)},$$

where $w(z)$ is analytic in $\mathbb{U}$ and $|w(z)| < 1$, that is

$$\left| \frac{1 - \frac{z\Phi'(z)}{\Phi(z)}}{A - B\frac{z\Phi'(z)}{\Phi(z)}} \right| < 1 \quad (z \in \mathbb{U}). \tag{2.2}$$

Differentiating (1.3) with respect to $z$, we get

$$\frac{z\Phi'(z)}{\Phi(z)} = \frac{\sum_{k=1}^{\infty} k f_k a_k z^k}{\sum_{k=1}^{\infty} f_k a_k z^k} - \lambda \frac{\sum_{k=1}^{\infty} k g_k b_k z^k}{1 + \sum_{k=1}^{\infty} g_k b_k z^k}. \tag{2.3}$$

Using (2.2) and (2.3), we get

$$\left|\frac{1-\frac{z\Phi'(z)}{\Phi(z)}}{A-B\frac{z\Phi'(z)}{\Phi(z)}}\right| = \left|\frac{C(z)}{D(z)}\right|,$$

where

$$C(z) = \left(\sum_{k=2}^{\infty}(1-k)f_k a_k z^k\right)\left(1+\sum_{k=1}^{\infty} g_k b_k z^k\right) + \lambda\left(\sum_{k=1}^{\infty} f_k a_k z^k\right)\left(\sum_{k=1}^{\infty} k g_k b_k z^k\right)$$

and

$$\begin{aligned} D(z) &= \left(\sum_{k=1}^{\infty}(A-Bk)f_k a_k z^k\right)\left(1+\sum_{k=1}^{\infty} g_k b_k z^k\right) \\ &\quad + B\lambda\left(\sum_{k=1}^{\infty} f_k a_k z^k\right)\left(\sum_{k=1}^{\infty} k g_k b_k z^k\right). \end{aligned}$$

It follows that

$$\begin{aligned} |C(z)| &\leq \sum_{k=2}^{\infty}(k-1)f_k|a_k| + \lambda\sum_{k=1}^{\infty} k g_k|b_k| \\ &\quad + \left(\sum_{k=2}^{\infty}(k-1)f_k|a_k|\right)\left(\sum_{k=1}^{\infty} g_k|b_k|\right) \\ &\quad + \lambda\left(\sum_{k=2}^{\infty} f_k|a_k|\right)\left(\sum_{k=1}^{\infty} k g_k|b_k|\right) \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} |D(z)| &\geq A-B-\sum_{k=2}^{\infty}(A-Bk)f_k|a_k| \\ &\quad - \sum_{k=1}^{\infty}|k\lambda-(A-B)|g_k|b_k| \\ &\quad - \left(\sum_{k=2}^{\infty}(A-Bk)f_k|a_k|\right)\left(\sum_{k=1}^{\infty} g_k|b_k|\right) \\ &\quad - \lambda\left(\sum_{k=2}^{\infty} f_k|a_k|\right)\left(\sum_{k=1}^{\infty} k g_k|b_k|\right). \end{aligned} \tag{2.5}$$

Making use of the inequalities (2.1), (2.4) and (2.5), the assertion (2.2) is established which proves Theorem 1.

**Definition.** A function $\Phi(z) \in \mathcal{A}$ given by (1.3) is said to be in the class

$$\mathcal{S}^*(A, B) \;\; (-1 \leq B < A \leq 1, B \leq 0; \; z \in \mathbb{U})$$

if and only if it satisfies

$$\frac{z\Phi'(z)}{\Phi(z)} \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U})$$

and the coefficient inequality (2.1).

**Remark 1.** Let

$$\begin{aligned} \Psi(A, B) \quad = \quad & \frac{1}{A - B}\left(\sum_{k=2}^{\infty} f_k|a_k| \left(k - 1 + A - Bk + 2\lambda \sum_{k=1}^{\infty} kg_k|b_k|\right)\right. \\ & + \sum_{k=1}^{\infty}(k\lambda + |k\lambda - (A - B)|)g_k|b_k| \\ & \left. + \left(\sum_{k=2}^{\infty}(2k - 1 + A)f_k|a_k|\right)\left(\sum_{k=1}^{\infty} g_k|b_k|\right)\right), \end{aligned}$$

then it is easy to verify that

$$\frac{\partial\Psi}{\partial A} \leq 0 \quad \text{and} \quad \frac{\partial\Psi}{\partial B} \geq 0,$$

where $f_k \geq 0, g_k \geq 0, -1 \leq B < A \leq 1, B \leq 0$. We have thus the following inclusion relations:

If $-1 \leq B_0 \leq B < A \leq A_0 \leq 1$ and $B \leq 0$, then

$$\mathcal{S}^*(A, B) \subseteq \mathcal{S}^*(A_0, B_0) \subseteq \mathcal{S}^*(-1, 1) = \mathcal{S}^*.$$

To establish the next result, we state here a known lemma which is due to Ruscheweyh [8].

**Lemma 1.** *Let $\phi(z) \in \mathcal{K}, \varphi(z) \in \mathcal{S}^*$ and $\Psi(z)$ be analytic in $\mathbb{U}$. Then*

$$\frac{\phi * \Psi\varphi}{\phi * \varphi}(\mathbb{U}) \subset \overline{\text{co}}\Psi(\mathbb{U}),$$

*where $\overline{\text{co}}\Psi(\mathbb{U})$ denotes the convex hull of $\Psi(\mathbb{U})$.*

By applying Lemma 1, we derive following convolution conditions for $\mathcal{S}^*(A, B)$.

**Theorem 2.** *Let $\Phi(z) \in \mathcal{S}^*(A, B)(-1 \leq B < A \leq 1, B \leq 0)$ and $\varphi(z) \in \mathcal{K}$. Then $(\Phi * \varphi)(z) \in \mathcal{S}^*(A, B)$.*

*Proof.* Let $\Phi(z) \in \mathcal{S}^*(A, B)(-1 \leq B < A \leq 1, B \leq 0)$, then

$$W(z) = \frac{z\Phi'(z)}{\Phi(z)} \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}). \tag{2.6}$$

From Remark 1, we have $\Phi(z) \in \mathcal{S}^*$. If we let $\Psi = \Phi * \varphi, \varphi \in \mathcal{K}$, then

$$\frac{z\Psi'(z)}{\Psi(z)} = \frac{z(\Phi * \varphi)'(z)}{(\Phi * \varphi)(z)} = \frac{\varphi(z)(z\Phi'(z))}{\varphi(z) * \Phi(z)} = \frac{\varphi(z) * (\Phi(z)W(z))}{\varphi(z) * \Phi(z)} \quad (z \in \mathbb{U}). \tag{2.7}$$

Since $\Phi(z) \in \mathcal{S}^*, \varphi(z) \in \mathcal{K}$ and $\frac{1+Az}{1+Bz}(-1 \leq B < A \leq 1)$ is convex univalent in $\mathbb{U}$, then upon using (2.6), (2.7) and Lemma 1, we deduce that

$$\frac{z\Psi'(z)}{\Psi(z)} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}),$$

that is $(\Phi * \varphi)(z) \in \mathcal{S}^*(A,B)$ and the proof is complete.

For a function $\phi(z) \in \mathcal{A}$, the generalized Bernardi-Libera operator is defined by (see[1,3])

$$J_\beta \phi(z) = \frac{\beta+1}{z^\beta} \int_0^z t^{\beta-1}\phi(t)dt \quad (\beta > -1). \tag{2.8}$$

We prove the following

**Corollary 1.** *Let $\Phi(z) \in \mathcal{S}^*(A,B)(-1 \leq B < A \leq 1, B \leq 0)$ and $J_\beta\Phi(z)$ be defined by* (2.8) *with $\beta > 0$. Then $J_\beta\Phi(z) \in \mathcal{S}^*(A,B)$.*

*Proof.* Let $\Phi(z) \in \mathcal{S}^*(A,B)(-1 \leq B < A \leq 1, B \leq 0)$ and $J_\beta\Phi(z)$ be defined by (2.8) with $\beta > 0$. Then

$$J_\beta\Phi(z) = \frac{\beta+1}{z^\beta} \int_0^z t^{\beta-1}\Phi(t)dt = (\Phi * \varphi)(z),$$

where

$$\varphi(z) = z + \sum_{k=2}^{\infty} \frac{\beta+1}{\beta+k} z^k \quad (\beta > 0; z \in \mathbb{U}).$$

From [9, Theorem 5] when $\beta > 0, \varphi(z)$ is convex univalent in $\mathbb{U}$. Hence, from Theorem 2, we have $J_\beta\Phi(z) \in \mathcal{S}^*(A,B)$.

**Corollary 2.** *Let $\Phi(z) \in \mathcal{S}^*(A,B)(-1 \leq B < A \leq 1, B \leq 0)$, then functions $G_\mu(z)$ is also in the class $\mathcal{S}^*(A,B)(|z| < r_0)$, where $G_\mu(z)$ is defined by*

$$G_\mu(z) = (1-\mu)\Phi(z) + \mu z\Phi'(z) \quad (0 < \mu \leq 1) \tag{2.9}$$

*and*

$$r_0 = \frac{1}{2\mu + \sqrt{4\mu^2 - 2\mu + 1}}, \tag{2.10}$$

*the radius $r_0$ is best possible.*

*Further, if $G_\mu(z) \in \mathcal{S}^*(A,B)(z \in \mathbb{U})$, then $\Phi(z) \in \mathcal{S}^*(A,B)$ for $z \in \mathbb{U}$.*

*Proof.* For $0 < \mu \leq 1, \Phi(z) \in \mathcal{S}^*(A,B)(-1 \leq B < A \leq 1, B \leq 0)$, we can write (2.9) as

$$G_\mu(z) = (\upsilon * \Phi)(z),$$

where

$$\upsilon(z) = (1-\mu)\frac{z}{1-z} + \mu\frac{z}{(1-z)^2} = z + \sum_{k=2}^{\infty}(1+(k-1)\mu)z^k \in \mathcal{A}.$$

It is well known that the function $\upsilon(z)$ is convex for $|z| < r_0$, where $r_0$ is given by (2.10) and this radius is best possible. Applying Theorem 2, we infer that $G_\mu(z) \in \mathcal{S}^*(A,B)(|z| < r_0)$.

On the other hand, we have from (2.9) that

$$\Phi(z) = \omega(z) * G_\mu(z),$$

where

$$\omega(z) = \sum_{k=1}^{\infty}\frac{1}{1+(k-1)\mu}z^k \quad (z \in \mathbb{U}).$$

Since $\omega(z)$ is convex univalent for $0 < \mu \leq 1$ (see [9, Theorem 5]), hence we have $\Phi(z) \in \mathcal{S}^*(A,B)$ for $z \in \mathbb{U}$.

**Theorem 3.** *Let* $\Phi(z) \in \mathcal{S}^*(A,B)(-1 \leq B < A \leq 1, B \leq 0)$, *then*

$$(\Phi * h_\sigma)(z) \neq 0 \quad (z \in \mathbb{U}\backslash\{0\}; \sigma \in \mathbb{C}, |\sigma| = 1), \tag{2.11}$$

*where*

$$h_\sigma(z) = z + \frac{1+A\sigma-2(1+B\sigma)}{\sigma(A-B)}\cdot\frac{z^2}{1-z} - \frac{1+B\sigma}{\sigma(A-B)}\cdot\frac{z^3}{(1-z)^2}.$$

*Proof.* Let $\Phi(z) \in \mathcal{S}^*(A,B)(-1 \leq B < A \leq 1, B \leq 0)$, then

$$\frac{z\Phi'(z)}{\Phi(z)} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}),$$

which is equivalent to

$$\frac{z\Phi'(z)}{\Phi(z)} \neq \frac{1+A\sigma}{1+B\sigma} \quad (z \in \mathbb{U}; \sigma \in \mathbb{C}, |\sigma| = 1, 1+B\sigma \neq 0)$$

or to

$$(1+B\sigma)z\Phi'(z) - (1+A\sigma)\Phi(z) \neq 0 \quad (z \in \mathbb{U}\backslash\{0\}; \sigma \in \mathbb{C}, |\sigma| = 1, 1+B\sigma \neq 0). \tag{2.12}$$

We note that

$$\Phi(z) = \Phi(z) * \left(z + \frac{z^2}{1-z}\right) \tag{2.13}$$

and

$$z\Phi'(z) = \Phi(z) * \left(z + \sum_{k=2}^{\infty}kz^k\right) = \Phi(z) * \left(z + \frac{2z^2}{1-z} + \frac{z^3}{(1-z)^2}\right). \tag{2.14}$$

Making use of (2.12)-(2.14), we have

$$\Phi(z) * \left\{(1+B\sigma)\left(z+\frac{2z^2}{1-z}+\frac{z^3}{(1-z)^2}\right)-(1+A\sigma)\left(z+\frac{z^2}{1-z}\right)\right\} \neq 0$$

for $z \in \mathbb{U}\backslash\{0\}; \sigma \in \mathbb{C}$ and $|\sigma| = 1$. This given the desired result (2.11).

Next, we give some inequalities.

**Theorem 4.** *Let* $\Phi(z) \in \mathcal{S}^*(A, B)$, *then*

$$\Re\left\{\frac{\Phi(z)}{z}\right\}^{\gamma} > \begin{cases} (1-B)^{\frac{(A-B)\gamma}{B}}, & (B \neq 0; z \in \mathbb{U}), \\ e^{-A\gamma}, & (B = 0; z \in \mathbb{U}), \end{cases} \tag{2.15}$$

*where* $-1 \leq B < A \leq 1, B \leq 0, A - B \leq 1$ *and* $0 < \gamma \leq 1$.

*Proof.* Let $\Phi(z) \in \mathcal{S}^*(A, B)(-1 \leq B < A \leq 1, B \leq 0, A - B \leq 1)$, then

$$\frac{z\Phi'(z)}{\Phi(z)} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}).$$

Applying the result of Suffridge [10, Theorem 3], then

$$\int_0^z \left(\frac{\Phi'(t)}{\Phi(t)} - \frac{1}{t}\right) dt \prec \int_0^z \frac{\frac{1+At}{1+Bt} - 1}{t} dt = \int_0^z \frac{A-B}{1+Bt} dt,$$

that is

$$\log\frac{\Phi(z)}{z} \prec \int_0^z \frac{A-B}{1+Bt} dt.$$

Hence

$$\frac{\Phi(z)}{z} \prec (1+Bz)^{\frac{A-B}{B}} = \varphi_1(z) \quad (B \neq 0; z \in \mathbb{U}) \tag{2.16}$$

and

$$\frac{\Phi(z)}{z} \prec e^{Az} \quad (B = 0; z \in \mathbb{U}). \tag{2.17}$$

For $-1 \leq B < A \leq 1, A - B \leq 1$, the $\varphi_1(\mathbb{U})$ is symmetric with respect to the real axis and the function $\varphi_1(z)$ is convex univalent in $\mathbb{U}$ because

$$\Re\left\{1 + \frac{z\varphi_1''(z)}{\varphi_1'(z)}\right\} = \Re\frac{1+(A-B)z}{1+Bz} > 0 \quad (z \in \mathbb{U}).$$

Therefore, with the aid of the elementary inequality $\Re(w^\gamma) \geq (\Re w)^\gamma (0 < \gamma \leq 1)$ and $\Re w > 0$, it follows from (2.16) and (2.17) that

$$\Re\left\{\frac{\Phi(z)}{z}\right\}^{\gamma} \geq \left\{\Re\frac{\Phi(z)}{z}\right\}^{\gamma} > \begin{cases} (1-B)^{\frac{(A-B)\gamma}{B}}, & (B \neq 0; z \in \mathbb{U}), \\ e^{-A\gamma}, & (B = 0; z \in \mathbb{U}) \end{cases}$$

for $0 < \gamma \leq 1$. This proves (2.15) and the proof is complete.

From (2.16) and (2.17), we have

**Corollary 3.** *Let* $\Phi(z) \in \mathcal{S}^*(A, B)$, *then*

$$\left|\arg \frac{\Phi(z)}{z}\right| \leq \begin{cases} \frac{(A-B)}{B} \arcsin B, & (B < 0, -1 \leq B < A \leq 1, A - B \leq 1; z \in \mathbb{U}), \\ A, & (B = 0, 0 < A \leq 1; z \in \mathbb{U}). \end{cases}$$

**Remark 2.** When $A = 0, B = -1$ and $\gamma = 1$, then from Theorem 4 and Corollary 3, we have

$$\Phi(z) \in \mathcal{S}^*(\frac{1}{2}) \Rightarrow \Re \frac{\Phi(z)}{z} > \frac{1}{2} \text{ and } \left|\arg \frac{\Phi(z)}{z}\right| < \frac{\pi}{2} \quad (z \in \mathbb{U}).$$

The next result gives sufficient conditions such that the function $\Phi(z)$ defined by (1.3) belongs to $\mathcal{K}(\alpha)(0 \leq \alpha < 1)$.

**Theorem 5.** *Let*$\Phi(z)$ *be defined by* (1.3), *then* $\Phi(z) \in \mathcal{K}(\alpha)(0 \leq \alpha < 1)$, *provided that there exist numbers* $p, q$, *where* $\frac{1}{p} + \frac{1}{q} \leq 1$, *satisfying the following inequalities:*

$$\sum_{k=1}^{\infty} (pk(\lambda + 1) + 1 - \alpha) g_k |b_k| \leq 1 - \alpha \tag{2.18}$$

*and*

$$\begin{aligned} &\sum_{k=1}^{\infty} |k\lambda - 1|(qk + 1 - \alpha) g_k |b_k| \\ &\quad + \left(1 + \sum_{k=1}^{\infty} g_k |b_k|\right) \sum_{k=2}^{\infty} k(1 - \alpha + q(k-1)) f_k |a_k| \\ &\quad + q\lambda \sum_{k=1}^{\infty} k^2 g_k |b_k| \sum_{k=2}^{\infty} f_k |a_k| \\ &+ \sum_{k=1}^{\infty} k g_k |b_k| \sum_{k=2}^{\infty} (q|\lambda + k(1-\lambda)| + \lambda(1-\alpha)) f_k |a_k| \leq 1 - \alpha. \end{aligned} \tag{2.19}$$

*Proof.* Let the inequalities (2.18) and (2.19) be satisfied for the function $\Phi(z)$. We prove that

$$\Re\left\{1 + \frac{z\Phi''(z)}{\Phi'(z)}\right\} > \alpha \quad (z \in \mathbb{U}). \tag{2.20}$$

After some calculations, we get

$$1 + \frac{z\Phi''(z)}{\Phi'(z)} = 1 - \left((\lambda + 1)\frac{\sum_{k=1}^{\infty} k g_k b_k z^k}{1 + \sum_{k=1}^{\infty} g_k b_k z^k} + \frac{M(z)}{N(z)}\right),$$

where

$$M(z) = \lambda \sum_{k=1}^{\infty} k^2 g_k b_k z^{k-1} \sum_{k=1}^{\infty} f_k a_k z^k - \sum_{k=1}^{\infty} k(k-1) f_k a_k z^{k-1} \left(1 + \sum_{k=1}^{\infty} g_k b_k z^k\right)$$

$$-\sum_{k=1}^{\infty} kg_k b_k z^{k-1} \sum_{k=1}^{\infty} f_k a_k (\lambda + (k(1-\lambda))z^k$$

and

$$N(z) = z + \sum_{k=2}^{\infty} kf_k a_k z^k + \sum_{k=1}^{\infty} g_k b_k z^{k+1} + \sum_{k=1}^{\infty} g_k b_k z^k \sum_{k=2}^{\infty} kf_k a_k z^k$$
$$-\lambda \left( \sum_{k=1}^{\infty} kg_k b_k z^{k-1} + \sum_{k=1}^{\infty} g_k b_k z^{k-1} \sum_{k=2}^{\infty} f_k a_k z^k \right).$$

It readily follows that

$$\Re \left\{ 1 + \frac{z\Phi''(z)}{\Phi'(z)} \right\} = 1 - \Re \left( (\lambda+1) \frac{\sum_{k=1}^{\infty} kg_k b_k z^k}{1 + \sum_{k=1}^{\infty} g_k b_k z^k} + \frac{M(z)}{N(z)} \right)$$
$$\geq 1 - \left| \frac{(\lambda+1)\sum_{k=1}^{\infty} kg_k b_k z^k}{1 + \sum_{k=1}^{\infty} g_k b_k z^k} \right| - \left| \frac{M(z)}{N(z)} \right| \tag{2.21}$$

and in view of (2.18), we infer that

$$\left| \frac{(\lambda+1)\sum_{k=1}^{\infty} kg_k b_k z^k}{1 + \sum_{k=1}^{\infty} g_k b_k z^k} \right| \leq \frac{(\lambda+1)\sum_{k=1}^{\infty} kg_k |b_k|}{1 - \sum_{k=1}^{\infty} g_k |b_k|} \leq \frac{1-\alpha}{p} \quad (0 \leq \alpha < 1). \tag{2.22}$$

Also

$$|M(z)| \leq \sum_{k=1}^{\infty} |k(k\lambda - 1)| g_k |b_k| + \lambda \sum_{k=1}^{\infty} k^2 g_k |b_k| \sum_{k=2}^{\infty} f_k |a_k|$$
$$+ \sum_{k=2}^{\infty} k(k-1) f_k |a_k| \left( 1 + \sum_{k=1}^{\infty} g_k |b_k| \right)$$
$$+ \sum_{k=1}^{\infty} kg_k |b_k| \sum_{k=2}^{\infty} |\lambda + k(1-\lambda)| f_k |a_k|$$

and

$$|N(z)| \geq \left( 1 - \sum_{k=1}^{\infty} |k\lambda - 1| g_k |b_k| \right) - \sum_{k=2}^{\infty} kf_k |a_k| \left( 1 + \sum_{k=1}^{\infty} g_k |b_k| \right)$$
$$-\lambda \sum_{k=1}^{\infty} kg_k |b_k| \sum_{k=2}^{\infty} f_k |a_k|.$$

Making use of (2.19), we get

$$\left| \frac{M(z)}{N(z)} \right| \leq \frac{1-\alpha}{q}. \tag{2.23}$$

Applying (2.21) to (2.23), we conclude that the inequality (2.20) holds true and the proof is complete.

## 3. Some consequences of main results

In this concluding section, we consider some consequences of our main results proved in section 2.

**Remark 3.** For

$$F(z) ={}_2F_1(a',b';c';z)=\sum_{k=1}^{\infty}\frac{(a')_k(b')_k}{(c')_k(1)_k}z^k \quad (a'>0,b'>0,c'>0;z\in\mathbb{U})$$

and

$$G(z) ={}_2F_1(a'',b'';c'';z)=\sum_{k=1}^{\infty}\frac{(a'')_k(b'')_k}{(c'')_k(1)_k}z^k \quad (a''>0,b''>0,c''>0;z\in\mathbb{U}),$$

$A = 1-2\alpha(0\leq\alpha<1), B=-1$, Theorems 1 and 5 after some elementary simplification reduce to the results [5, Theorems 1 and 2].

If we set

$$F(z)=1+\sum_{k=1}^{n}z^k, \quad \text{and} \quad G(z)=1+\sum_{k=1}^{\infty}z^k,$$

then

$$\Phi_{F,G}^{\lambda}(z)=W_n^{\lambda}(z)=\frac{s_n(z)}{(1+\sum_{k=1}^{\infty}b_kz^k)^{\lambda}} \quad (\lambda>0;z\in\mathbb{U}), \tag{2.24}$$

where

$$s_1(z)=z, \quad s_n(z)=z+\sum_{k=2}^{n}a_kz^k(n=2,3,\cdots) \quad (z\in\mathbb{U}).$$

From Theorems 1 and 4, we have

**Corollary 4.** *The function*

$$W_n^{\lambda}(z)=\frac{s_n(z)}{(1+\sum_{k=1}^{\infty}b_kz^k)^{\lambda}}\in\mathcal{S}^*(A,B) \quad (-1\leq B<A\leq 1,B\leq 0,\lambda>0;z\in\mathbb{U})$$

*if and only if*

$$\sum_{k=2}^{n}|a_k|\left(k-1+A-Bk+2\lambda\sum_{k=1}^{\infty}k|b_k|\right)+\sum_{k=1}^{\infty}(k\lambda+|k\lambda-(A-B)|)|b_k|$$
$$+\sum_{k=1}^{\infty}|b_k|\sum_{k=2}^{n}(2k-1+A)|a_k|\leq A-B,$$

*and when* $-1\leq B<A\leq 1, B\leq 0, A-B\leq 1,\lambda>0, 0<\gamma\leq 1,$ *then*

$$\Re\left\{\frac{W_n^{\lambda}(z)}{z}\right\}^{\gamma}>\begin{cases}(1-B)^{\frac{(A-B)\gamma}{B}}, & (B\neq 0;z\in\mathbb{U}),\\ e^{-A\gamma}, & (B=0;z\in\mathbb{U}).\end{cases}$$

Also, from Theorem 5, we have

**Corollary 5.** *The function* $W_n^\lambda(z)$ *is in* $\mathcal{K}(\alpha)(0 \le \alpha < 1)$, *provided that there exist number* $p, q > 0$ *such that* $\frac{1}{p} + \frac{1}{q} \le 1$ *satisfying the following inequalities:*

$$\sum_{k=1}^{\infty}(pk(\lambda+1)+1-\alpha)|b_k| \le 1-\alpha$$

*and*

$$\begin{aligned}\sum_{k=1}^{\infty}&|k\lambda-1|(qk+1-\alpha)|b_k| \\ &+\left(1+\sum_{k=1}^{\infty}|b_k|\right)\sum_{k=2}^{n}k(1-\alpha+q(k-1))|a_k| \\ &+q\lambda\sum_{k=1}^{\infty}k^2|b_k|\sum_{k=2}^{n}|a_k| \\ &+\sum_{k=1}^{\infty}k|b_k|\sum_{k=2}^{n}\left(q|\lambda+k(1-\lambda)|+\lambda(1-\alpha)\right)|a_k| \le 1-\alpha,\end{aligned}$$

*where* $W_n^\lambda(z)$ *be defined by* (2.24).

It is observed that when $n = 1$, then $W_1^\lambda = F^\lambda(z)$, where $F^\lambda(z)$ is defined by (1.5) and consequently Corollaries 4 and 5 extend and improve the results of Raina and Bansal [6, Theorem 1 and 2].

In particular, if we set $b_k = 0(k = 1, 2, 3, \cdots)$ in (2.24), then $W_n^\lambda(z) = s_n(z)$ and Corollary 4 then yields the following result:

**Corollary 6.** *The function*

$$s_n(z) \in \mathcal{S}^*(A,B) \quad (-1 \le B < A \le 1, B \le 0; z \in \mathbb{U})$$

*if and only if*

$$\sum_{k=2}^{n}|a_k|(k-1+A-Bk) \le A-B,$$

*and when* $-1 \le B < A \le 1, B \le 0, A-B \le 1$, *then*

$$\Re\frac{s_n(z)}{z} > \begin{cases} (1-B)^{\frac{A-B}{B}}, & (B \ne 0; z \in \mathbb{U}), \\ e^{-A}, & (B = 0; z \in \mathbb{U}). \end{cases}$$

If we set

$$g(z) = 1+\sum_{k=1}^{\infty}\left(\frac{\beta+1}{\beta+k}\right)\rho^k z^k \quad (0 < \rho \le 1, \beta > 0; z \in \mathbb{U}),$$

then the function $g(z)$ is convex univalent in $\mathbb{U}$ (see [9, Theorem 5]) and from Corollary 4, we have

**Corollary 7.** *The function*

$$\nu_{\beta,\rho}^{\lambda}(z)=\frac{z}{\left(1+\sum_{k=1}^{\infty}\left(\frac{\beta+1}{\beta+k}\right)\rho^k z^k\right)^{\lambda}}\in \mathcal{S}^*(A,B)\quad(-1\leq B<A\leq 1, B\leq 0,\lambda>0)$$

*if and only if*

$$\sum_{k=1}^{\infty}(k\lambda+|k\lambda-(A-B)|)\left(\frac{\beta+1}{\beta+k}\right)\rho^k\leq A-B,$$

*and when* $-1\leq B<A\leq 1, B\leq 0, A-B\leq 1,$ *then*

$$\Re\frac{1}{\left(1+\sum_{k=1}^{\infty}\left(\frac{\beta+1}{\beta+k}\right)\rho^k z^k\right)^{\lambda}}>\begin{cases}(1-B)^{\frac{A-B}{B}}, & (B\neq 0; z\in\mathbb{U}),\\ e^{-A}, & (B=0; z\in\mathbb{U}).\end{cases}$$

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