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Regular Classes Involving a Generalized Shift Plus Fractional Hornich Integral Operator

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abstract: The Hornich space is the set of all locally univalent and analytic functions ϕ on the open unit disk such that $\arg\phi'$ is bounded . Here, we introduce a generalized integral operator in the open unit disk. This operator is defined by the fractional Hornich integral operator joining the shift plus multiplier. In addition, we deal with a new subspace of the Hardy space comprising the normalized analytic functions $\big(\phi(z),\, z\in U\big)$, with the set of all bounded functions $\big(\phi'(z)\neq 0,\, z\neq 0\big)$. We will validate that the new integral operator is closed in the subspace of normalized functions with the bounded first derivative. Formal accounts are renowned in the sequel based on the maximally of Jack Lemma.

Key Words: Fractional calculus, Unit disk, Analytic function, Subordination and superordination.

Contents

1. Introduction

The field of the theory of operators calms with the numerous classes of functions on function spaces, preceded by functional operators (integral and differential). These operators can be enclosed by the classes of the functions straightly or process by exhausting the convolution product. The distinction in this pathway is to examine the boundedness and the compactness of these operators. These formations approved two sets of operators: linear operators and nonlinear operators. The considerable itself carries the topological or geometrical explanations of the spaces of functions. The realities of this class of operators illustrate a dynamic role in mathematics, computer science and physics. To catch an operator using the functional theory, and then observe its properties, is one of the essential purposes of recent studies in the geometric function theory and its connected areas. The aim of the current effort is to bring out a new operator in the open unit disk based on the Hornich integral operator.

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In [1] Hornich expressed an occasional procedure (consistently titled the Hornich operation) on the set of locally univalent (analytic) functions in the open unit disk (or, more generally, a convex dominion). Without vital loss of the generality, we might pressure ourselves to the set of analytic functions in the open unit disk. Then the Hornich procedure is established by

$$I[\phi_1, \phi_2] := \phi_1(z) \oplus \phi_2(z) = \int_0^z \phi_1'(\xi)\phi_2'(\xi)d\xi,$$

$$I_\alpha[\phi] := \alpha \odot \phi(z) = \int_0^z (\phi'(\xi))^\alpha d\xi,$$

where

$$\Big(\alpha \in \mathbb{C},\ z, \xi \in U,\ (\phi')^\alpha(0) \neq 0\Big).$$

Additionally, Hornich [1] stated a norm to a subset of locally univalent set which generates it a separable real Banach space with the exceeding procedure. Cima and Pfaltzgraff [2] delivered further separable real Banach space associated with a few changed set of Hornich's one. These spaces, however, both do not shield the whole set of normalized univalent functions. On the other hand, Campbell et al. [3] reproduced a complex Banach space assembly on the set of locally univalent functions of finite order. Kim et al. [4] considered the integral operators in association with the pre-Schwarzian derivative and the Hornich procedure.

We will introduce a modified integral operator in the open unit disk. This variation designates new classes of analytic functions, identified by the normalized class of analytic functions in the open unit disk. Our inquiries are documented by the geometric explanations and boundedness of the new operator.

2. Processing

Let $U := \{z : |z| < 1\}$ be the open unit disk of the complex plane and $H(U)$ be the space of holomorphic functions on the open unit disk. A holomorphic function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in U$$

on the open unit disk belongs to the Hardy space $H^2(U)$, if its sequence of power series coefficients is square-summable:

$$H^2(U) = \{\phi \in H(U) : \sum_{n=0}^{\infty} |\varphi_n|^2 < \infty\}.$$

Consequently, it can be defined a norm on $H^2(U)$ as follows [5]:

$$\|\phi\|^2_{H^2(U)} = \sum_{n=0}^{\infty} |\varphi_n|^2.$$

Since $L^2(U)$ is Banach space, then $H^2(U)$ is also a Banach space on U. In the sequel, we deal with a subset of analytic function, which are normalized as follows: $f(0) = 0$ and $f'(0) = 1$. Therefore, ϕ is defined as follows:

$$\phi(z) = z + \sum_{n=2}^{\infty} \varphi_n z^n, \quad z \in U.$$

We denote this class by $A(U)$. For $\phi, \psi \in A(U)$, the convolution product, $(*)$ is defined by

$$(\phi * \psi)(z) = (z + \sum_{n=2}^{\infty} \varphi_n \, z^n) * (z + \sum_{n=2}^{\infty} \vartheta_n z^n) = z + \sum_{n=2}^{\infty} \varphi_n \, \vartheta_n z^n.$$

It is clear that $A(U) \subset H(U)$ achieving the above norm. The space H^∞ is known as the vector space of bounded holomorphic functions on U, achieving the norm

$$\|\phi\|_{H^\infty} = \sup_{|z|<1} |\phi(z)|, \quad \phi \in H(U).$$

Obviously,

$$H^2(U) \subset H^\infty, \quad \phi \in H(U).$$

We wish to establish a new integral operator of fractional power. First, we define the following operator $J \mapsto A(U)$ as follows:

$$J[\phi_1, ..., \phi_k](z) := z\Big(\phi_1 * ... * \phi_k\Big)(z), \tag{2.1}$$

$$\Big(z \in U, \, \phi_j, \, j = 1, ..., k \in A(U)\Big).$$

This operator is titled the shift operator, when $k = 1$ (see [[5]]). Moreover, we impose the following generalized integral operator:

$$\begin{aligned} I^k_{\alpha_1,...,\alpha_k}[\phi_1, ..., \phi_k](z) &:= (\alpha_1 \odot \phi_1) \oplus ... \oplus (\alpha_k \odot \phi_k)(z) \\ &= \int_0^z \Big(\phi'_1(\xi)\Big)^{\alpha_1} ... \Big(\phi'_k(\xi)\Big)^{\alpha_k} d\xi. \end{aligned} \tag{2.2}$$

The operator $I^k_{\alpha_1,...,\alpha_k}[\phi_1, ..., \phi_k]$ is a linear operator with Hornich space procedures. We proceed to define the generalized integral operator. For $\phi_j \in A(U)$, $j = 1, ..., k$, we deliver the shift plus Hornich operator as follows:

$$\begin{aligned} \mathcal{J}^k_{\alpha_1,...,\alpha_k}[\phi_1, ..., \phi_k](z) :&= J[\phi_1, ..., \phi_k](z) + I^k_{\alpha_1,...,\alpha_k}[\phi_1, ..., \phi_k](z) \\ &= z\Big(\phi_1 * ... * \phi_k\Big)(z) + \int_0^z \Big(\phi'_1(\xi)\Big)^{\alpha_1} ... \Big(\phi'_k(\xi)\Big)^{\alpha_k} d\xi, \end{aligned} \tag{2.3}$$

$$\Big(z, \xi \in U,\ \phi_1 * ... * \phi_k\ \in A(U),\ \alpha_1, ..., \alpha_k \in \mathbb{C}\Big).$$

Obviously, $\mathfrak{J}^k_{\alpha_1,...,\alpha_k}[\phi_1, ..., \phi_k] \in A(U)$. Moreover, since $J[\phi_1, ..., \phi_k]$ is a linear isometry operator and the complex Hornich operator is contraction, then the operator (2.3) is bounded in the Hardy space $H^2(U)$. Let $S^2(U)$ be the space defined by

$$S^2(U) := \Big\{\phi \in H(U) : \phi' \in H^2(U)\Big\}$$

end with the norm

$$\|\phi\|^2_{S^2(U)} := \|\phi\|^2_{H^2(U)} + \|\phi'\|^2_{H^2(U)}.$$

This space is subspace from H^∞, Banach algebra, and every polynomial is dense in it (see [5], Proposition 1). Moreover, let

$$S^2_0(U) := \Big\{\phi \in S^2(U) : \phi(0) = 0\Big\},$$

then

$$S^2_0(U) \subset S^2(U).$$

It has been shown that the range of the Hornich operator is equal to $S^2_0(U)$. Next construction is about dealing with a subspace $S^2_1(U)$ defining by:

$$S^2_1(U) := \Big\{\phi \in S^2(U) : \phi(0) = 0,\ \phi'(0) = 1\Big\}.$$

Then we have the following relation:

$$S^2_1(U) \subset S^2_0(U) \subset S^2(U).$$

Proposition 2.1. *Let* $\mathfrak{J}^k_{\alpha_1,...,\alpha_k}[\phi_1, ..., \phi_k]\ \in H^2(U)$. *Then* $rang\mathfrak{J}^k_{\alpha_1,...,\alpha_k}[\phi_1, ..., \phi_k]\ \subset S^2_1(U)$.

Proof: Let $G(z) \in rang\mathfrak{J}^k_{\alpha_1,...,\alpha_k}[\phi_1, ..., \phi_k]$, then there exist normalized functions $\phi_j(z) \in H^2(U), \forall\, j = 0, ..., 1$ such that

$$\begin{aligned} G(z) &= \mathfrak{J}^k_{\alpha_1,...,\alpha_k}[\phi_1, ..., \phi_k](z) \\ &= z\Big(\phi_1 * ... * \phi_k\Big)(z) + \int_0^z \Big(\phi'_1(\xi)\Big)^{\alpha_1} ... \Big(\phi'_k(\xi)\Big)^{\alpha_k} d\xi \\ &= z + \sum_{n=2}^{\infty} \varphi_n z^n, \quad z, \xi \in U. \end{aligned}$$

Then we obtain $G' \in H^2(U)$ with the properties

$$G(0) = 0, \quad G'(0) = 1.$$

Hence, $G \in S^2_1(U)$. □

Proposition 2.2. *Let $\phi_1, ..., \phi_k \in A(U)$. Then*

$$\|\phi_1 * ... * \phi_k\|^2_{S^2_1(U)} \le \|\phi_1\|^2_{S^2_1(U)} ... \|\phi_k\|^2_{S^2_1(U)}, \quad z \in U.$$

Proof: It is clear that $(\phi_1 * ... * \phi_k)(0) = 0$ and $(\phi_1 * ... * \phi_k)'(0) = 1$; thus $(\phi_1 * ... * \phi_k) \in S^2_1(U)$. Moreover, in view of the Young's inequality for convolutions, we have

$$\begin{aligned}
\|\phi_1 * ... * \phi_k\|^2_{S^2_1(U)} &= \|\phi_1 * ... * \phi_k\|^2_{H^2(U)} + \|(\phi_1 * ... * \phi_k)'\|^2_{H^2(U)} \\
&\le 2\sum_{n=0}^{\infty} |a_{1,n} ... a_{k,n}|^2 \\
&\le 2\sum_{n=0}^{\infty} |a_{1,n}|^2 ... |a_{k,n}|^2 \\
&\le \sum_{n=0}^{\infty} |a_{1,n}|^2 + ... + \sum_{n=0}^{\infty} |a_{k,n}|^2 \\
&= \|\phi_1\|^2_{S^2_1(U)} ... \|\phi_k\|^2_{S^2_1(U)}.
\end{aligned}$$

□

In this paper, we focus on the set of locally univalent in $A(U)$, denoting by

$$LA(U) := \{\phi \in A(U) : \phi'(z) \neq 0, \forall\, z \in U\}.$$

Based on the above set we construct the subspace

$$S^2_2(U) := \Big\{\phi \in S^2(U) : \phi(0) = \phi'(0) - 1 = 0,\ \phi'(z) \neq 0,\, z \in U\Big\}.$$

We have the following inclusion relation:

$$S^2_2(U) \subset S^2_1(U) \subset S^2_0(U) \subset S^2(U).$$

We have the following proposition, where the proof is quite similar to Proposition 2.1.

Proposition 2.3. *Let $\phi_1, ..., \phi_k \in S^2_2(U)$. Then*

$$\|\phi_1 * ... * \phi_k\|^2_{S^2_2(U)} \le \|\phi_1\|^2_{S^2_2(U)} ... \|\phi_k\|^2_{S^2_2(U)}, \quad z \in U.$$

Recall that the function $\phi \in A(U)$ is called starlike of order $\delta \in [0, 1)$ if and only if

$$\Re\Big(\frac{z\phi'(z)}{\phi(z)}\Big) > \delta, \quad z \in U;$$

this class is denoted by $S^*(\delta)$. And $\phi \in A(U)$ is called convex of order $\delta \in [0,1)$ if and only if

$$\Re\big(1+\frac{z\phi''(z)}{\phi'(z)}\big) > \delta, \quad z \in U;$$

this class is denoted by $K(\delta)$. Finally, The function $\phi \in A(U)$ is called bounded turning of order δ if and only if $\Re\big(\phi'(z)\big) > \delta$; this class is symbolized by $B(\delta)$. A function $\phi \in A(U)$ is called strongly starlike ($SS(\delta)$) of order α if

$$|arg(z\phi'(z)/\phi(z))| \le (\delta\pi)/2, \quad z \in U.$$

It is well known that (see [4], [6])

$$I_\alpha[\phi] \in SS(\alpha), \quad \alpha \in [0,1]$$

and

$$\phi \in B \Rightarrow I_\alpha[\phi] \in SS(\alpha), \quad \alpha \in [0,2].$$

We need the following result in the sequel:

Lemma 2.4. *[7] Let $\phi(z)$ be analytic in U with $\phi(0)=0$. Then if $|\phi(z)|$ approaches its maximization at a point $z_0 \in U$ when $|z|=r$, then $z_0\phi'(z_0) = c\,\phi(z_0)$, where $c \ge 1$ is a real number.*

Furthermore, we request the subordination category (see [8]), which is introduced as follows: assume that $f(\zeta)$ and $g(\zeta)$ are analytic in U. Then $f(\zeta)$ is called subordinate to $g(\zeta)$ if for analytic function $\phi(\zeta)$ in U achieving $\phi(0)=0, |\phi(\zeta)|<1$, $(\zeta \in U)$ and $f(\zeta) = g(\phi(\zeta))$. This subordination is denoted by

$$f(\zeta) \prec g(\zeta), \quad \zeta \in U.$$

3. Findings

In this section, we introduce sufficient conditions to study the geometric properties of the operator

For this purpose, we let $\Phi(z) := \Big(\phi_1 * ... * \phi_k\Big)(z)$ then (2.3) becomes

$$\mathfrak{J}^k_{\alpha_1,...,\alpha_k}[\phi_1,...,\phi_k](z) = \mathfrak{J}^k_{\alpha_1,...,\alpha_k}[\Phi](z)$$

Theorem 3.1. *Consider the operator (2.3). If*

$$\Re\Big(\frac{z\mathfrak{J}^k_{\alpha_1,...,\alpha_k}[\Phi]''(z)}{\mathfrak{J}^k_{\alpha_1,...,\alpha_k}[\Phi]'(z)}\Big) < 0, \quad z \in U, \ \Phi \in A(U),$$

then

$$\mathfrak{J}^k_{\alpha_1,...,\alpha_k}[\Phi] \in S^*.$$

Proof: Let $0 < \nu < 1$ be a real positive constant satisfying

$$\frac{z\mathfrak{J}^k_{\alpha_1,\ldots,\alpha_k}[\Phi]'(z)}{\mathfrak{J}^k_{\alpha_1,\ldots,\alpha_k}[\Phi](z)} = \frac{1+\nu\omega(z)}{1-\nu\omega(z)}, \quad \omega(z) \neq \frac{1}{\nu}, \, \nu \neq 0,$$

where $\omega(z)$, $z \in U$ is a function in the open unit disk. Obviously $\omega(z)$ is analytic in U such that $\omega(0) = 0$. We aim to show that $|\omega(z)| < 1$ in U. Differentiating both side logarithmically, we obtain

$$1 + \frac{z\mathfrak{J}^k_{\alpha_1,\ldots,\alpha_k}[\Phi]''(z)}{\mathfrak{J}^k_{\alpha_1,\ldots,\alpha_k}[\Phi]'(z)} = \frac{2\nu z\omega'(z)}{1-\nu^2\omega^2(z)} + \frac{1+\nu\omega(z)}{1-\nu\omega(z)}.$$

Thus, by the assumption we have

$$\begin{aligned}\Re\Big(1 + \frac{z\mathfrak{J}^k_{\alpha_1,\ldots,\alpha_k}[\Phi]''(z)}{\mathfrak{J}^k_{\alpha_1,\ldots,\alpha_k}[\Phi]'(z)}\Big) &= \Re\Big(\frac{2\nu z\omega'(z)}{1-\nu^2\omega^2(z)} + \frac{1+\nu\omega(z)}{1-\nu\omega(z)}\Big) \\ &< 1, \quad z \in U, \, \Phi \in A(U).\end{aligned}$$

If there exists a point $z_0 \in U$ such that

$$\max_{|z|\leq|z_0|} |\omega(z)| = |\omega(z_0)| = 1,$$

then Lemma 2.4 implies that $\omega(z_0) = e^{i\theta}$ and

$$z_0\omega'(z_0) = c\,\omega(z_0), \; c \geq 1.$$

Thus, we obtain

$$\begin{aligned}1 + \frac{z_0\mathfrak{J}^k_{\alpha_1,\ldots,\alpha_k}[\Phi]''(z_0)}{\mathfrak{J}^k_{\alpha_1,\ldots,\alpha_k}[\Phi]'(z_0)} &= \frac{2\nu z_0\omega'(z_0)}{1-\nu^2\omega^2(z_0)} + \frac{1+\nu\omega(z_0)}{1-\nu\omega(z_0)} \\ &= \frac{2\nu c\, e^{i\theta}}{1-\nu^2e^{2i\theta}} + \frac{1+\nu e^{i\theta}}{1-\nu e^{i\theta}}\end{aligned}$$

Since

$$\Re\big(\frac{1}{1-\nu\, e^{i\theta}}\big) = \frac{1}{1+\nu},$$

therefore, we conclude that

$$\begin{aligned}\Re\Big(1 + \frac{z_0\mathfrak{J}^k_{\alpha_1,\ldots,\alpha_k}[\Phi]''(z_0)}{\mathfrak{J}^k_{\alpha_1,\ldots,\alpha_k}[\Phi]'(z_0)}\Big) &= \Re\Big(\frac{2\nu z_0\omega'(z_0)}{1-\nu^2\omega^2(z_0)} + \frac{1+\nu\omega(z_0)}{1-\nu\omega(z_0)}\Big) \\ &= \Re\Big(\frac{2\nu\, c\, e^{i\theta}}{1-\nu^2e^{2i\theta}} + \frac{1+\nu e^{i\theta}}{1-\nu e^{i\theta}}\Big) \\ &\geq \frac{(1+\nu)^2}{1+\nu^2}, \quad c = 1 \\ &> 1.\end{aligned}$$

Hence,

$$\Re\Big(\frac{z_0\mathfrak{J}^k_{\alpha_1,\ldots,\alpha_k}[\Phi]''(z_0)}{\mathfrak{J}^k_{\alpha_1,\ldots,\alpha_k}[\Phi]'(z_0)}\Big) > 0,$$

which contradicts the assumption of the theorem. This leads that there is no $z_0 \in U$ such that $|\omega(z_0)| = 1$ for all $z \in U$ *i.e*

$$\frac{z\mathfrak{J}^k_{\alpha_1,\ldots,\alpha_k}[\Phi]'(z)}{\mathfrak{J}^k_{\alpha_1,\ldots,\alpha_k}[\Phi](z)} \prec \frac{1+\nu z}{1-\nu z}, \quad z \in U,\ \Phi \in A(U).$$

This completes the proof. □

Theorem 3.2. *Consider the integral operator* (2.3). *If for* $1 < \wp < 2$, *such that*

$$\Re\{\frac{z\mathfrak{J}^k_{\alpha_1,\ldots,\alpha_k}[\Phi]''(z)}{\mathfrak{J}^k_{\alpha_1,\ldots,\alpha_k}[\Phi]'(z)}\} > \frac{\wp}{2},, \quad z \in U,\ \phi \in A(U),$$

then $\mathfrak{J}^k_{\alpha_1,\ldots,\alpha_k}[\Phi](z) \in B$.

Proof: Define a function $\ell(z)$, $z \in U$ as follows:

$$\mathfrak{J}^k_{\alpha_1,\ldots,\alpha_k}[\Phi]'(z) = (1-\ell(z))^{\wp}, \quad z \in U,$$

where, $\ell(z)$ is analytic with $\ell(0) = 0$. We need only to show that $|\ell(z)| < 1$. From the definition of ψ, we have

$$\frac{z\mathfrak{J}^k_{\alpha_1,\ldots,\alpha_k}[\Phi]''(z)}{\mathfrak{J}^k_{\alpha_1,\ldots,\alpha_k}[\Phi]'(z)} = \wp\frac{-z\ell'(z)}{1-\ell(z)}.$$

Thus, we attain

$$\begin{aligned}\Re\{\frac{z\mathfrak{J}^k_{\alpha_1,\ldots,\alpha_k}[\Phi]''(z)}{\mathfrak{J}^k_{\alpha_1,\ldots,\alpha_k}[\Phi]'(z)}\} &= \wp\Re\{\frac{-z\ell'(z)}{1-\ell(z)}\} \\ &> \frac{\wp}{2}, \quad \wp \in (1,2).\end{aligned}$$

In view of Lemma 2.4, there exists a complex number $z_0 \in U$ such that $\ell(z_0) = e^{i\theta}$ and

$$z_0\ell'(z_0) = c\,\ell(z_0) = c\,e^{i\theta},\ c \geq 1.$$

Since

$$\Re\Big(\frac{1}{1-\ell(z_0)}\Big) = \Re\Big(\frac{1}{1-e^{i\theta}}\Big) = \frac{1}{2}$$

then, we attain

$$\begin{aligned}\Re\{\frac{z\mathfrak{J}^k_{\alpha_1,\dots,\alpha_k}[\Phi]''(z_0)}{\mathfrak{J}^k_{\alpha_1,\dots,\alpha_k}[\Phi]'(z_0)}\} &= \wp\Re\{\frac{-c\,\ell(z_0)}{1-\ell(z_0)}\}\\ &= \wp\Re\{\frac{-c\,e^{i\theta}}{1-e^{i\theta}}\}\\ &\le \frac{\wp}{2},\quad c=1,\end{aligned}$$

which contradicts the assumption of the theorem. Hence, there is no $z_0 \in U$ with $|\ell(z_0)| = 1$, which yields that $|\ell(z)| < 1$. Moreover, we have

$$\mathfrak{J}^k_{\alpha_1,\dots,\alpha_k}[\Phi]'(z) \prec (1-z)^{\wp},$$

which means that $\Re[\mathfrak{J}^k_{\alpha_1,\dots,\alpha_k}[\Phi]'(z)] > 0$, equivalently, $\mathfrak{J}^k_{\alpha_1,\dots,\alpha_k}[\Phi]'(z) \in B$. This completes the proof. □

Theorem 3.3. *Consider the integral operator (3). If for $\beta > 1/2$, such that*

$$\Re\{\frac{z\mathfrak{J}^k_{\alpha_1,\dots,\alpha_k}[\Phi]'(z)}{\mathfrak{J}^k_{\alpha_1,\dots,\alpha_k}[\Phi](z)}\} > \frac{2\beta-1}{2\beta},$$

then

$$\frac{\mathfrak{J}^k_{\alpha_1,\dots,\alpha_k}[\Phi](z)}{z} \prec (1-z)^{1/\beta}.$$

Proof: Define a function $w(z)$, $z \in U$ as follows:

$$\frac{\mathfrak{J}^k_{\alpha_1,\dots,\alpha_k}[\Phi](z)}{z} = (1-w(z))^{1/\beta},\quad z\in U,$$

where, $w(z)$ is analytic with $w(0) = 0$. We need only to show that $|w(z)| < 1$. From the definition of w, we have

$$\frac{z\mathfrak{J}^k_{\alpha_1,\dots,\alpha_k}[\Phi]'(z)}{\mathfrak{J}^k_{\alpha_1,\dots,\alpha_k}[\Phi](z)} = 1 - \frac{zw'(z)}{\beta(1-w(z))}.$$

Hence, we obtain

$$\begin{aligned}\Re\Big\{\frac{z\mathfrak{J}^k_{\alpha_1,\dots,\alpha_k}[\Phi]'(z)}{\mathfrak{J}^k_{\alpha_1,\dots,\alpha_k}[\Phi](z)}\Big\} &= \Re\Big\{1-\frac{zw'(z)}{\beta(1-w(z))}\Big\}\\ &> \frac{2\beta-1}{2\beta},\quad \beta>1/2.\end{aligned}$$

In view of Lemma 2.4, there exists a complex number $z_0 \in U$ such that $w(z_0) = e^{i\theta}$ and

$$z_0w'(z_0) = c\,w(z_0) = c\,e^{i\theta},\ c \ge 1.$$

Therefore, we arrive at

$$\begin{aligned}\Re\Big\{\frac{z_0\mathfrak{J}^k_{\alpha_1,\ldots,\alpha_k}[\Phi]'(z_0)}{\mathfrak{J}^k_{\alpha_1,\ldots,\alpha_k}[\Phi](z_0)}\Big\} &= \Re\Big\{1-\frac{z_0\,w'(z_0)}{\beta(1-w(z_0))}\Big\}\\ &= \Re\Big\{1-\frac{c\,w(z_0)}{\beta(1-w(z_0))}\Big\}\\ &= 1-\Re\Big\{\frac{c\,e^{i\theta}}{\beta(1+e^{i\theta})}\Big\}\\ &= \frac{2\beta-1}{2\beta},\end{aligned}$$

and this is a contradiction with the assumption of the theorem. Hence, there is no $z_0 \in U$ with $|w(z_0)| = 1$, which yields that $|w(z)| < 1$. This completes the proof. □

Theorem 3.4. *Let* $\Phi \in A(U)$. *Then* $\mathfrak{J}^k_{\alpha_1,\ldots,\alpha_k}[\Phi] \in B$.

Proof: Let $\Phi \in A(U)$. Dividing (2.3) by $\Phi(z)$, $z \in U \setminus \{0\}$ and differentiating logarithmic, we have

$$\frac{z\mathfrak{J}^k_{\alpha_1,\ldots,\alpha_k}[\Phi]'(z)}{\mathfrak{J}^k_{\alpha_1,\ldots,\alpha_k}[\Phi](z)} - 1 = \frac{zI^k_{\alpha_1,\ldots,\alpha_k}[\Phi]'(z)}{I^k_{\alpha_1,\ldots,\alpha_k}[\Phi](z)}.$$

By letting

$$\vartheta(z) = I^k_{\alpha_1,\ldots,\alpha_k}[\Phi](z) \Rightarrow \vartheta'(z) = I^k_{\alpha_1,\ldots,\alpha_k}[\Phi]'(z),$$

we obtain

$$\frac{zI^k_{\alpha_1,\ldots,\alpha_k}[\Phi]'(z)}{I^k_{\alpha_1,\ldots,\alpha_k}[\Phi](z)} = \frac{z\vartheta'(z)}{\vartheta(z)}.$$

In view of Lemma 2.1, there exists a complex number $z_0 \in U$ such that $\vartheta(z_0) = e^{i\theta}$ and

$$z_0\vartheta'(z_0) = c\,\vartheta(z_0) = c\,e^{i\theta},\ c \geq 1.$$

Therefore, we conclude that

$$\begin{aligned}\Big|\frac{z\mathfrak{J}^k_{\alpha_1,\ldots,\alpha_k}[\Phi]'(z)}{\mathfrak{J}^k_{\alpha_1,\ldots,\alpha_k}[\Phi](z)} - 1\Big| &= \Big|\frac{zI^k_{\alpha_1,\ldots,\alpha_k}[\Phi]'(z)}{I^k_{\alpha_1,\ldots,\alpha_k}[\Phi](z)}\Big|\\ &= \Big|\frac{z\vartheta'(z)}{\vartheta(z)}\Big|, \quad z \in U\\ &< \Big|\frac{z\vartheta'(z_0)}{\vartheta(z_0)}\Big|, \quad |z| < |z_0|\\ &\leq c, \quad c \geq 1.\end{aligned}$$

But c is arbitrary, then for $c = 1$, and in virtue of Theorem 5.5g P299 in [8], we obtain

$$\left|\mathcal{J}^k_{\alpha_1,\dots,\alpha_k}[\Phi]'(z) - 1\right| < 1 \Rightarrow \mathcal{J}^k_{\alpha_1,\dots,\alpha_k}[\Phi] \in B.$$

□

4. Conclusion and discussion

Here, we provided a complex integral in the open unit disk based on the Hornich operator ($\mathcal{J}^k_{\alpha_1,\dots,\alpha_k}[\Phi]$). The new operator is achieved the differential, shift plus operator $J[\Phi](z) = z\Phi(z)$, $\Phi \in A(U)$. Boundedness of the new integral operator is suggested in new extended space ($S^2_2(U)$). In addition, some regular classes; such as univalent, starlike and bounded turning are investigated. Our main tool is based on the Jack Lemma. It has checked that $\mathcal{J}^k_{\alpha_1,\dots,\alpha_k} : A(U) \to A(U)$. For future work, one can employ the new operator to define new classes of analytic functions. Furthermore, for further investigations, one can deal with the subordination and superordination idea by employing the above integral operator. Additionally, it can be studied the connection between closed ideals of a Banach algebra together with closed invariant subspaces of the operator $J[\Phi]$.

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