Bol. Soc. Paran. Mat. (3s.) **v. 38** 2 (2020): 41–50.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v38i2.35567

One Sided Generalized (σ,τ)-derivations on Rings

Evrim Guven

Abstract: Let R be a prime ring with characteristic not 2 and $\sigma,\tau,\lambda,\mu,\alpha,\beta$ be automorphisms of R. Let h be a nonzero left (resp. right)-generalized $(\sigma,\tau)-$derivation of R and I,J nonzero ideals of R and $a\in R$. The main object in this article is to study the situations. (1) $h(I)a\subset C_{\lambda,\mu}(J)$ and $ah(I)\subset C_{\lambda,\mu}(J)$, (2) $h(I)\subset C_{\lambda,\mu}(J)$, (3) $[h(I),a]_{\lambda,\mu}=0$, (4) $h(I,a)_{\lambda,\mu}=0$ (or $(h(I),a)_{\lambda,\mu}=0$), (5) $[h(x),x]_{\lambda,\tau}=0,\forall x\in I$, (6) $[h(x)a,x]_{\lambda,\tau}=0,\forall x\in I$.

Key Words: $(\sigma,\tau)-$Lie ideal, Prime ring, Commutativity.

Contents

1. Introduction

Let R be an associative ring with center Z. Recall that R is prime if $aRb=(0)$ implies that $a=0$ or $b=0$. For any $x,y\in R$ the symbol $[x,y]$ represents commutator $xy-yx$ and the Jordan product $(x,y)=xy+xy$. Let σ and τ be any two endomorphisms of R. For any $x,y\in R$ we set $[x,y]_{\sigma,\tau}=x\sigma(y)-\tau(y)x$ and $(x,y)_{\sigma,\tau}=x\sigma(y)+\tau(y)x$. Let h and d be additive mappings of R. If $d(xy)=d(x)y+xd(y),\forall x,y\in R$ then d is called a derivation of R. If there exists a derivation d such that $h(xy)=h(x)y+xd(y),\forall x,y\in R$ then h is called generalized derivation of R (see [3]). If $d(xy)=d(x)\sigma(y)+\tau(x)d(y),\forall x,y\in R$ then d is called a $(\sigma,\tau)-$derivation of R. Obviously every derivation $d:R\to R$ is a $(1,1)-$derivation of R, where $1:R\to R$ is an identity mapping. If $h(xy)=d(x)\sigma(y)+\tau(x)h(y),\forall x,y\in R$ then h is said to be a left-generalized $(\sigma,\tau)-$derivation with d and if $h(xy)=h(x)\sigma(y)+\tau(x)d(y),\forall x,y\in R$ then h is said to be a right-generalized $(\sigma,\tau)-$derivation associated with $(\sigma,\tau)-$derivation d, (see [4]). Every $(\sigma,\tau)-$derivation associated with d is a right (and left)-generalized $(\sigma,\tau)-$derivation associated with d.

The mapping defined by $h(r)=[r,a]_{\sigma,\tau},\forall r\in R$ is a right-generalized derivation associated with derivation $d(r)=[r,\sigma(a)],\forall r\in R$ and left-generalized derivation associated with derivation $d_1(r)=[r,\tau(a)],\forall r\in R$. The mapping $h(r)=(a,r)_{\sigma,\tau},\forall r\in R$ is a left-generalized $(\sigma,\tau)-$ derivation associated with $(\sigma,\tau)-$derivation $d_2(r)=[a,r]_{\sigma,\tau},\forall r\in R$ and right-generalized $(\sigma,\tau)-$derivation associated with $(\sigma,\tau)-$derivation d_2.

2010 *Mathematics Subject Classification:* 16W25, 16U80.

Submitted February 23, 2017. Published September 16, 2017

The following result is proved by Posner in (see [12]). Let R be a prime ring and $d \neq 0$ derivation of R such that $[d(x), x] = 0, \forall x \in R$. Then R is commutative. Ashraf and Rehman (see [1]) generalized Posner's result as follows. Let R be a 2−torsion free prime ring. Suppose there exists a $(\sigma, \tau)-$derivation $d : R \to R$ such that $[d(x), x]_{\sigma,\tau} = 0, \forall x \in R$. Then either $d = 0$ or R is commutative. Taking an ideal of R instead of R, Marubayashi H.and Ashraf M.,Rehman N., Ali Shakir, generalized Rehman's result in (see [10]). On the other hand, Rehman (see [13]) gave another generalization of Posner's Theorem as follows. Let R be a prime ring. If R admits a nonzero generalized derivation h with d such that $[h(x), x] = 0, \forall x \in R$, and if $d \neq 0$, then R is commutative.

In this paper, using left-generalized $(\sigma, \tau)-$derivation of R, we have given another generalization of Ashraf and Rehman's result (see [1]) as in Theorem 3. Also, we discuss the commutativity of prime rings admitting a left-generalized $(\sigma, \tau)-$derivation $h : R \longrightarrow R$ satisfying several conditions on ideals.

Throughout the paper, R will be a prime ring with characteristic not 2 and $\sigma, \tau, \lambda, \mu, \alpha, \beta$ be automorphisms of R. Let J be an ideal of R We write $C_{\sigma,\tau}(J) = \{r \in R \mid r\sigma(x) = \tau(x)r, \ \forall\ x \in J\}$ and will make extensive use of the following basic commutator identities.

$[xy, z]_{\sigma,\tau} = x[y, z]_{\sigma,\tau} + [x, \tau(z)]y = x[y, \sigma(z)] + [x, z]_{\sigma,\tau}y$
$[x, yz]_{\sigma,\tau} = \tau(y)[x, z]_{\sigma,\tau} + [x, y]_{\sigma,\tau}\sigma(z)$
$(x, yz)_{\sigma,\tau} = \tau(y)(x, z)_{\sigma,\tau} + [x, y]_{\sigma,\tau}\sigma(z) = -\tau(y)[x, z]_{\sigma,\tau} + (x, y)_{\sigma,\tau}\sigma(z)$
$(xy, z)_{\sigma,\tau} = x(y, z)_{\sigma,\tau} - [x, \tau(z)]y = x[y, \sigma(z)] + (x, z)_{\sigma,\tau}y$..

2. Results

We begin with the following known results which will be used to prove our theorems.

Lemma 2.1. *[2, Lemma1] Let R be a prime ring and $d : R \longrightarrow R$ be a $(\sigma, \tau)-$derivation. If U is a nonzero right ideal of R and $d(U) = 0$ then $d = 0$.*

Lemma 2.2. *[11, Lemma3] If a prime ring contains a nonzero commutative right ideal then it is commutative.*

Lemma 2.3. *[6, Lemma5] Let I be a nonzero ideal of R and $a, b \in R$. If $[a, I]_{\alpha,\beta} \subset C_{\lambda,\mu}(R)$ or $(a, I)_{\alpha,\beta} \subset C_{\lambda,\mu}(R)$ then $a \in C_{\alpha,\beta}(R)$ or R is commutative.*

Lemma 2.4. *[5, Corollary 1] If I is a nonzero ideal of R and $a \in R$ such that $[I, a]_{\alpha,\beta} \subset C_{\lambda,\mu}(R)$, then $a \in Z$.*

Lemma 2.5. *[7, Lemma 2.16] Let R be a prime ring and $h : R \longrightarrow R$ be a nonzero left-generalized $(\sigma, \tau)-$ derivation associated with a nonzero $(\sigma, \tau)-$derivation d. If I is a nonzero ideal of R and $a \in R$ such that $(h(I), a)_{\lambda,\mu} = 0$ then $a \in Z$ or $d\tau^{-1}\mu(a) = 0$.*

Lemma 2.6. *[7, Theorem 2.7] Let $h : R \to R$ be a nonzero right-generalized $(\sigma, \tau)-$derivation associated with $(\sigma, \tau)-$derivation d and I, J be nonzero ideals of R. If $a \in R$ such that $ah(I) \subset C_{\lambda,\mu}(J)$ then $a \in Z$ or $d = 0$.*

Lemma 2.7. *Let I be a nonzero ideal of R and $a, b \in R$. If $h : R \longrightarrow R$ is a nonzero left-generalized $(\sigma, \tau)-$derivation associated with $(\sigma, \tau)-$derivation d such that $[h(I)a, b]_{\lambda,\mu} = 0$ then $a[a, \lambda(b)] = 0$ or $d(\tau^{-1}\mu(b)) = 0$.*

Proof. Using hypothesis we have,

$$\begin{aligned} 0 &= [h(\tau^{-1}\mu(b)x)a, b]_{\lambda,\mu} = [d(\tau^{-1}\mu(b))\sigma(x)a + \mu(b)h(x)a, b]_{\lambda,\mu} \\ &= d(\tau^{-1}\mu(b))[\sigma(x)a, \lambda(b)] + [d(\tau^{-1}\mu(b)), b]_{\lambda,\mu}\sigma(x)a \\ &+ \mu(b)[h(x)a, b]_{\lambda,\mu} + [\mu(b), \mu(b)]h(x)a \\ &= d(\tau^{-1}\mu(b))[\sigma(x)a, \lambda(b)] + [d(\tau^{-1}\mu(b)), b]_{\lambda,\mu}\sigma(x)a, \forall x \in I \end{aligned}$$

That is,

$$k[\sigma(x)a, \lambda(b)] + [k, b]_{\lambda,\mu}\sigma(x)a = 0, \forall x \in I \text{ where } k = d(\tau^{-1}\mu(b)). \tag{2.1}$$

Replacing x by $x\sigma^{-1}(a)y$ in (1) and using (1) we get,

$$\begin{aligned} 0 &= k[\sigma(x)a\sigma(y)a, \lambda(b)] + [k, b]_{\lambda,\mu}\sigma(x)a\sigma(y)a \\ &= k\sigma(x)a[\sigma(y)a, \lambda(b)] + k[\sigma(x)a, \lambda(b)]\sigma(y)a + [k, b]_{\lambda,\mu}\sigma(x)a\sigma(y)a \\ &= k\sigma(x)a[\sigma(y)a, \lambda(b)], \forall x, y \in I. \end{aligned}$$

That is $k\sigma(I)a[\sigma(I)a, \lambda(b)] = 0$. Since $\sigma(I)$ is a nonzero ideal of R then we have

$$d(\tau^{-1}\mu(b) = 0 \text{ or } a[\sigma(I)a, \lambda(b)] = 0. \tag{2.2}$$

If $a[\sigma(I)a, \lambda(b)] = 0$ in (2) then we get,

$$\begin{aligned} 0 &= a[\sigma(\sigma^{-1}(a)x)a, \lambda(b)] = a[a\sigma(x)a, \lambda(b)] \\ &= aa[\sigma(x)a, \lambda(b)] + a[a, \lambda(b)]\sigma(x)a = a[a, \lambda(b)]\sigma(x)a, \forall x \in I. \end{aligned}$$

From the last relation we obtain that $a[a, \lambda(b)] = 0$ for two case. □

Remark 2.8. *Let J be a nonzero ideal of R. If $b \in C_{\lambda,\mu}(J)$ then $b \in C_{\lambda,\mu}(R)$.*

Proof. If $b \in C_{\lambda,\mu}(J)$ then we have $0 = [b, xr]_{\lambda,\mu} = \mu(x)[b, r]_{\lambda,\mu} + [b, x]_{\lambda,\mu}\lambda(r) = \mu(x)[b, r]_{\lambda,\mu}, \forall x \in J, r \in R$. That is $\mu(J)[b, R]_{\lambda,\mu} = 0$. This gives that $b \in C_{\lambda,\mu}(R)$. □

Theorem 2.9. *Let $h : R \longrightarrow R$ be a nonzero left-generalized $(\sigma, \tau)-$derivation associated with nonzero $(\sigma, \tau)-$derivation d and $a, b \in R$. Let I, J be nonzero ideals of R.*

(i) If $h(I)a \subset C_{\lambda,\mu}(J)$ then $a \in Z$.

(ii) If $ah(I) \subset C_{\lambda,\mu}(J)$ then $a \in Z$ or $ad\tau^{-1}(a) = 0$.

Proof. (i) If $h(I)a \subset C_{\lambda,\mu}(J)$ then we have $[h(I)a, x]_{\lambda,\mu} = 0, \forall x \in J$. Using this relation and Lemma 7 we get, for any $x \in J$,

$$a[a, \lambda(x)] = 0 \text{ or } d\tau^{-1}\mu(x) = 0$$

Let $K = \{x \in J \mid a[a, \lambda(x)] = 0\}$ and $L = \left\{x \in J \mid d\tau^{-1}\mu(x) = 0\right\}$.Then K and L are subgroups of J and $J = K \cup L$. A group can not write the union of its proper subgroups. Hence we have $K = J$ or $L = J$. That is,

$$a[a, \lambda(J)] = 0 \text{ or } d(\tau^{-1}\mu(J) = 0$$

Since $d \neq 0$ then $d(\tau^{-1}\mu(J)) \neq 0$ by Lemma 1. If $a[a, \lambda(J)] = 0$ then we get

$$\begin{aligned} 0 = a[a, \lambda(xr)] &= a\lambda(x)[a, \lambda(r)] + a[a, \lambda(x)]\lambda(r) \\ &= a\lambda(x)[a, \lambda(r)], \forall x \in J, r \in R \end{aligned}$$

and so $a\lambda(J)[a, R] = 0$. From this relation we obtain that $a \in Z$.

(ii) If $ah(I) \subset C_{\lambda,\mu}(J)$ then we have $ah(I) \subset C_{\lambda,\mu}(R)$ by Remark 1. Using this relation we get

$$\begin{aligned} 0 &= [ah(\tau^{-1}(a)y), \mu^{-1}(a)]_{\lambda,\mu} = [ad(\tau^{-1}(a))\sigma(y) + aah(y), \mu^{-1}(a)]_{\lambda,\mu} \\ &= ad(\tau^{-1}(a))[\sigma(y), \lambda\mu^{-1}(a)] + [ad(\tau^{-1}(a)), \mu^{-1}(a)]_{\lambda,\mu}\sigma(y) \\ &\quad + a[ah(y), \mu^{-1}(a)]_{\lambda,\mu} + [a, a]ah(y) \\ &= ad(\tau^{-1}(a))[\sigma(y), \lambda\mu^{-1}(a)] + [ad(\tau^{-1}(a)), \mu^{-1}(a)]_{\lambda,\mu}\sigma(y), \forall y \in I, \end{aligned}$$

and so

$$k[\sigma(y), p] + [k, \mu^{-1}(a)]_{\lambda,\mu}\sigma(y) = 0, \forall y \in I, \text{ where } k = ad(\tau^{-1}(a)) \text{ and } p = \lambda\mu^{-1}(a). \tag{2.3}$$

Replacing y by yx, $x \in I$ in (3) we obtain that

$$\begin{aligned} 0 &= k\sigma(y)[\sigma(x), p] + k[\sigma(y), p]\sigma(x) + [k, \mu^{-1}(a)]_{\lambda,\mu}\sigma(y)\sigma(x) \\ &= k\sigma(y)[\sigma(x), p], \forall x, y \in I. \end{aligned}$$

That is,

$$k\sigma(I)[\sigma(I), p] = 0 \tag{2.4}$$

Since $\sigma(I)$ is a nonzero ideal of R then $k = 0$ or $[\sigma(I), p] = 0$ is obtained by the (4). This gives that $ad(\tau^{-1}(a)) = 0$ or $a \in Z$. □

Corollary 2.10. *Let I, J be nonzero ideals of R and $a, b \in R$.*

(i) If $[I, b]_{\sigma,\tau}a \subset C_{\lambda,\mu}(J)$ then $a \in Z$ or $b \in Z$.

(ii) If $[b, I]_{\sigma,\tau}a \subset C_{\lambda,\mu}(J)$ then $a \in Z$ or $b \in C_{\sigma,\tau}(R)$.

(iii) If $a(b, I)_{\sigma,\tau} \subset C_{\lambda,\mu}(J)$ then $a \in Z$ or $b \in C_{\sigma,\tau}(R)$ or $a[b, \tau^{-1}(a)]_{\sigma,\tau} = 0$.

Proof. (i) Let $h(r) = [r, b]_{\sigma,\tau}, \forall r \in R$ and $d(r) = [r, \tau(b)], \forall r \in R$. Since,

$$h(rs) = [rs, b]_{\sigma,\tau} = r[s, b]_{\sigma,\tau} + [r, \tau(b)]s = d(r)s + rh(s), \forall r, s \in R, \quad (2.5)$$

then h is a left-generalized derivation associated with derivation d. If $h = 0$ then $d = 0$ (and so $b \in Z$) is obtained by the relation (5).

If $[I, b]_{\sigma,\tau} a \subset C_{\lambda,\mu}(J)$ then we can write $h(I)a \subset C_{\lambda,\mu}(J)$. If $h \neq 0$ and $d \neq 0$ then we have $a \in Z$ by Theorem 1(i).

(ii) The mapping defined by $d_1(r) = [b, r]_{\sigma,\tau}, \forall r \in R$ is a $(\sigma,\tau)-$derivation and so, left (and right)-generalized $(\sigma,\tau)-$derivation with d_1. If $d_1 = 0$ then we have $b \in C_{\sigma,\tau}(R)$.

Let $d_1 \neq 0$. If $[b, I]_{\sigma,\tau} a \subset C_{\lambda,\mu}(J)$ then we can write $d_1(I)a \subset C_{\lambda,\mu}(J)$. This gives that $a \in Z$ by Theorem 1(i). Finally we obtain that $a \in Z$ or $b \in C_{\sigma,\tau}(R)$.

(iii) The mapping defined by $g(r) = (b, r)_{\sigma,\tau}, \forall r \in R$ is a left-generalized $(\sigma,\tau)-$derivation associated with $(\sigma,\tau)-$derivation $d_1(r) = [b, r]_{\sigma,\tau}, \forall r \in R$. If $g = 0$ then $d_1 = 0$ and so $b \in C_{\sigma,\tau}(R)$ is obtained. Let $g \neq 0$ and $d_1 \neq 0$. If $a(b, I)_{\sigma,\tau} \subset C_{\lambda,\mu}(J)$ then we have $ag(I) \subset C_{\lambda,\mu}(J)$. This implies that $a \in Z$ or $ad_1\tau^{-1}(a) = 0$ by Theorem 1(ii). That is $a \in Z$ or $a[b, \tau^{-1}(a)]_{\sigma,\tau} = 0$. □

Lemma 2.11. *Let I be a nonzero ideal of R and $h : R \longrightarrow R$ be a nonzero left-generalized $(\sigma,\tau)-$derivation associated with a nonzero $(\sigma,\tau)-$derivation d. If $a \in R$ such that $[h(I), a]_{\lambda,\mu} = 0$ then $a \in Z$ or $d(\tau^{-1}\mu(a)) = 0$.*

Proof. Using hypothesis we get,

$$\begin{aligned} 0 &= [h(\tau^{-1}\mu(a)x), a]_{\lambda,\mu} = [d(\tau^{-1}\mu(a))\sigma(x) + \mu(a)h(x), a]_{\lambda,\mu} \\ &= d(\tau^{-1}\mu(a))[\sigma(x), \lambda(a)] + [d(\tau^{-1}\mu(a)), a]_{\lambda,\mu}\sigma(x) \\ &\quad +\mu(a)[h(x), a]_{\lambda,\mu} + [\mu(a), \mu(a)]h(x) \\ &= d(\tau^{-1}\mu(a))[\sigma(x), \lambda(a)] + [d(\tau^{-1}\mu(a)), a]_{\lambda,\mu}\sigma(x), \forall x \in I. \end{aligned}$$

That is,

$$k[\sigma(x), \lambda(a)] + [k, a]_{\lambda,\mu}\sigma(x) = 0, \forall x \in I, \text{ where } k = d(\tau^{-1}\mu(a)). \quad (2.6)$$

Replacing x by $xr, r \in R$ in (6) and using (6) we get

$$\begin{aligned} 0 &= k\sigma(x)[\sigma(r), \lambda(a)] + k[\sigma(x), \lambda(a)]\sigma(r) + [k, a]_{\lambda,\mu}\sigma(x)\sigma(r) \\ &= k\sigma(x)[\sigma(r), \lambda(a)], \forall x \in I, r \in R. \end{aligned}$$

and so $k\sigma(I)[R, \lambda(a)] = 0$. Since $\sigma(I) \neq 0$ is an ideal and R is prime then we have $a \in Z$ or $d(\tau^{-1}\mu(a)) = 0$. □

Theorem 2.12. *Let h be a nonzero left-generalized (σ,τ) derivation associated with $(\sigma,\tau)-$ derivation $0 \neq d$ and I, J be nonzero ideals of R.*

(i) If $h(I) \subset C_{\lambda,\mu}(J)$ then R is commutative.

(ii) If $[h(I), J]_{\alpha,\beta} \subset C_{\lambda,\mu}(R)$ or $(h(I), J)_{\alpha,\beta} \subset C_{\lambda,\mu}(R)$ then R is commutative.

(iii) If $[J, h(I)]_{\alpha,\beta} \subset C_{\lambda,\mu}(R)$ then R is commutative.

Proof. (i) If $h(I) \subset C_{\lambda,\mu}(J)$ then we have $[h(I), x]_{\lambda,\mu} = 0, \forall x \in J$. This means that, for any $x \in J$,

$$x \in Z \quad \text{or } d(\tau^{-1}\mu(x)) = 0 \tag{2.7}$$

by Lemma 8. Using (7), let us consider the following sets, $K = \{x \in J \mid x \in Z\}$ and $L = \{x \in J \mid d\tau^{-1}\mu(x) = 0\}$. Considering as in the proof of Theorem 1 we obtain that $J \subset Z$ or $d(\tau^{-1}\mu(J)) = 0$. Since $d \neq 0$ then we have $d(\tau^{-1}\mu(J)) \neq 0$ by Lemma 1. Hence, we obtain that $K = J$ and so $J \subset Z$. This means that R is commutative by Lemma 2.

(ii) If $[h(I), J]_{\alpha,\beta} \subset C_{\lambda,\mu}(R)$ or $(h(I), J)_{\alpha,\beta} \subset C_{\lambda,\mu}(R)$ then we have $h(I) \subset C_{\alpha,\beta}(R)$ or R is commutative by Lemma 3. On the other hand $h(I) \subset C_{\alpha,\beta}(R)$ means that R is commutative by (i).

(iii) If $[J, h(I)]_{\alpha,\beta} \subset C_{\lambda,\mu}(R)$ then we have $h(I) \subset Z$ by Lemma 4 and so R is commutative by (i). □

Corollary 2.13. *[8, Lemma 2] Let U be a nonzero ideal of R. If $d : R \longrightarrow R$ is a nonzero $(\sigma, \tau)-$derivation such that $d(U) \subset C_{\lambda,\mu}(R)$. Then R is commutative.*

Theorem 2.14. *Let $h : R \longrightarrow R$ be a nonzero left-generalized $(\sigma, \tau)-$derivation associated with a nonzero $(\sigma, \tau)-$derivation d. If $I \neq 0$ is an ideal of R such that $[h(x), x]_{\lambda,\tau} = 0, \forall x \in I$ then R is commutative.*

Proof. Linearizing the hypothesis, we get

$$[h(x), y]_{\lambda,\tau} + [h(y), x]_{\lambda,\tau} = 0, \forall x, y \in I. \tag{2.8}$$

Replacing x by yx in (8) and using (8) we have

$$\begin{aligned} 0 &= [h(yx), y]_{\lambda,\tau} + [h(y), yx]_{\lambda,\tau} \\ &= [d(y)\sigma(x) + \tau(y)h(x), y]_{\lambda,\tau} + [h(y), yx]_{\lambda,\tau} \\ &= d(y)[\sigma(x), \lambda(y)] + [d(y), y]_{\lambda,\tau}\sigma(x) + \boldsymbol{\tau}(\mathbf{y})[h(\mathbf{x}), \mathbf{y}]_{\lambda,\tau} \\ &+ [\boldsymbol{\tau}(\mathbf{y}), \boldsymbol{\tau}(\mathbf{y})]h(x) + \tau(y)[\mathbf{h}(\mathbf{y}), x]_{\lambda,\tau} + [h(y), \mathbf{y}]_{\lambda,\tau}\boldsymbol{\lambda}(\mathbf{x}) \\ &= d(y)[\sigma(x), \lambda(y)] + [d(y), y]_{\lambda,\tau}\sigma(x), \forall x, y \in I. \end{aligned}$$

That is

$$d(y)[\sigma(x), \lambda(y)] + [d(y), y]_{\lambda,\tau}\sigma(x) = 0, \forall x, y \in I. \tag{2.9}$$

Taking $xr, r \in R$ instead of x in (9) and using (9) then we arrive

$$\begin{aligned} 0 &= d(y)\sigma(x)[\sigma(r), \lambda(y)] + d(y)[\sigma(x), \lambda(y)]\sigma(r) + [d(y), y]_{\lambda,\tau}\sigma(x)\sigma(r) \\ &= d(y)\sigma(x)[\sigma(r), \lambda(y)], \forall x, y \in I, r \in R \end{aligned}$$

which leads to

$$d(y)\sigma(I)[R, \lambda(y)] = 0, \forall y \in I. \tag{2.10}$$

Since $\sigma(I) \neq 0$ an ideal then, for any $y \in I$, we have $[R, \lambda(y)] = 0$ or $d(y) = 0$ by (10) and so $y \in Z$ or $d(y) = 0$.

Let $K=\{y\in I\mid y\in Z\}$ and $L=\{y\in I\mid d(y)=0\}$. Considering as in the proof of Theorem 1 we have, $I\subset Z$ or $d(I)=0$. Since $I\neq 0$ an ideal and $d\neq 0$ then we obtain that $K=I$ by Lemma 1 and so $I\subset Z$. This means that R is commutative by Lemma 2. □

Corollary 2.15. *[1, Theorem 1] Let R be a prime ring and I be a nonzero ideal of R. If R admits a nonzero $(\alpha,\beta)-$derivation d such that $[d(x),x]_{\alpha,\beta}=0,\forall x\in I$, then R is commutative.*

Theorem 2.16. *Let R be a prime ring and $0\neq a\in R$. If $h:R\longrightarrow R$ is a nonzero left-generalized $(\sigma,\tau)-$derivation associated with a nonzero $(\sigma,\tau)-$derivation d and $I\neq 0$ an ideal of R such that $[h(x)a,x]_{\lambda,\tau}=0,\forall x\in I$ then R is commutative.*

Proof. Replacing x by $x+y$ in hypothesis we have

$$[h(x)a,y]_{\lambda,\tau}+[h(y)a,x]_{\lambda,\tau}=0,\forall x,y\in I. \tag{2.11}$$

If we take yx instead of x in (11) and using (11) we get

$$\begin{aligned}0&=[h(yx)a,y]_{\lambda,\tau}+[h(y)a,yx]_{\lambda,\tau}\\&=[d(y)\sigma(x)a+\tau(y)h(x)a,y]_{\lambda,\tau}+[h(y)a,yx]_{\lambda,\tau}\\&=d(y)[\sigma(x)a,\lambda(y)]+[d(y),y]_{\lambda,\tau}\sigma(x)a+\boldsymbol{\tau}(\mathbf{y})[h(x)a,\mathbf{y}]_{\lambda,\tau}\\&+[\boldsymbol{\tau}(\mathbf{y}),\boldsymbol{\tau}(\mathbf{y})]h(x)a+\tau(y)[\mathbf{h}(\mathbf{y})a,x]_{\lambda,\tau}+[h(y)a,y]_{\lambda,\tau}\lambda(x),\forall x,y\in I.\end{aligned}$$

That is

$$d(y)[\sigma(x)a,\lambda(y)]+[d(y),y]_{\lambda,\tau}\sigma(x)a\ =0,\forall x,y\in I. \tag{2.12}$$

Replacing x by $x\sigma^{-1}(a)$ in (12) and using (12) we have

$$\begin{aligned}0&=d(y)[\sigma(x)aa,\lambda(y)]+[d(y),y]_{\lambda,\tau}\sigma(x)aa\\&=d(y)\sigma(x)a[a,\lambda(y)]+d(y)[\sigma(x)a,\lambda(y)]a+[d(y),y]_{\lambda,\tau}\sigma(x)aa\\&=d(y)\sigma(x)a[a,\lambda(y)],\forall x,y\in I.\end{aligned}$$

That is

$$d(y)\sigma(I)a[a,\lambda(y)]=0,\forall y\in I. \tag{2.13}$$

Since $\sigma(I)$ a nonzero ideal of R then, for any $y\in I$, we obtain that

$$a[a,\lambda(y)]=0 \text{ or } d(y)=0$$

by (13). Hence, the additive group I is a union of subgroups $K=\{y\in I\mid a[a,\lambda(y)]=0\}$ and $L=\{y\in I\mid d(y)=0\}$. Considering as in the proof of the Theorem 1, we obtain that $K=I$ and so $a[a,\lambda(I)]=0$. Using this result we get,

$$\begin{aligned}0&=a[a,\lambda(yr)]=a\lambda(y)[a,\lambda(r)]+a[a,\lambda(y)]\lambda(r)\\&=a\lambda(y)[a,\lambda(r)],\forall r\in R,y\in I.\end{aligned}$$

That is $a\lambda(I)[a,R]=0$. This means that $a\in Z$. On the other hand, considering that $a\in Z$ and hypothesis, we get

$$\begin{aligned}0 &= [h(x)a,x]_{\lambda,\tau} = h(x)[a,\lambda(x)] + [h(x),x]_{\lambda,\tau}a \\ &= [h(x),x]_{\lambda,\tau}a \text{ for all } x\in I.\end{aligned}$$

That is $[h(x),x]_{\lambda,\tau}a=0, \forall x\in I$. Since $a\in Z$ and $a\neq 0$ we have $[h(x),x]_{\lambda,\tau}=0$ for all $x\in I$.This gives that R is commutative by Theorem 3. □

Remark 2.17. *Let I be a nonzero ideal of R and $a,b\in R$. If $(I,a)_{\lambda,\mu}b=0$ or $b(I,a)_{\lambda,\mu}=0$ then $a\in Z$ or $b=0$.*

Proof. If $(I,a)_{\lambda,\mu}b=0$ then we have

$0=(rx,a)_{\lambda,\mu}b=r(x,a)_{\lambda,\mu}b-[r,\mu(a)]xb=-[r,\mu(a)]xb, \forall r\in R, x\in I$. That is $[R,\mu(a)]Ib=0$. This gives that $a\in Z$ or $b=0$.

Let $b(I,a)_{\lambda,\mu}=0$. Then $0=b(xr,a)_{\lambda,\mu}=bx[r,\lambda(a)]+b(x,a)_{\lambda,\mu}r=bx[r,\lambda(a)], \forall r\in R$, $x\in I$.

This gives that $bI[R,\lambda(a)]=0$ and so $a\in Z$ or $b=0$. □

Lemma 2.18. *Let I be a nonzero ideal of R and a be a noncentral element of R. Let $h:R\longrightarrow R$ be a nonzero right-generalized derivation associated with d. If $h(I,a)_{\lambda,\mu}=0$ or $(h(I),a)_{\lambda,\mu}=0$ then $d\lambda(a)=0$.*

Proof. If $h(I,a)_{\lambda,\mu}=0$ then using that h is a right generalized derivation we get

$$\begin{aligned}0 &= h(x\lambda(a),a)_{\lambda,\mu} = h\{x[\lambda(a),\lambda(a)]+(x,a)_{\lambda,\mu}\lambda(a)\} = h\{(x,a)_{\lambda,\mu}\lambda(a)\} \\ &= h(x,a)_{\lambda,\mu}\lambda(a)+(x,a)_{\lambda,\mu}d\lambda(a) = (x,a)_{\lambda,\mu}d\lambda(a), \forall x\in I,\end{aligned}$$

which leads to

$$(I,a)_{\lambda,\mu}d\lambda(a)=0. \tag{2.14}$$

Using Remark 2 and (14) we have $a\in Z$ or $d\lambda(a)=0$. Since a be a noncentral then $d\lambda(a)=0$ is obtained.

If $(h(I),a)_{\lambda,\mu}=0$ then we have

$$\begin{aligned}0 &= (h(x\lambda(a)),a)_{\lambda,\mu} = (h(x)\lambda(a)+xd\lambda(a),a)_{\lambda,\mu} \\ &= h(x)[\boldsymbol{\lambda}(\mathbf{a}),\lambda(a)]+(h(x),a)_{\lambda,\mu}\lambda(a)+x(d\lambda(a),a)_{\lambda,\mu}-[x,\mu(a)]d\lambda(a) \\ &= x(d\lambda(a),a)_{\lambda,\mu}-[x,\mu(a)]d\lambda(a), \forall x\in I.\end{aligned}$$

That is,

$$x(d\lambda(a),a)_{\lambda,\mu}-[x,\mu(a)]d\lambda(a)=0, \forall x\in I. \tag{2.15}$$

Replacing x by $xy, y\in I$ in (15) and using (15) we get

$$\begin{aligned}0 &= xy(d\lambda(a),a)_{\lambda,\mu}-x[y,\mu(a)]d\lambda(a)-[x,\mu(a)]yd\lambda(a) \\ &= -[x,\mu(a)]yd\lambda(a), \forall x,y\in I.\end{aligned}$$

and so $[I,\mu(a)]Id\lambda(a)=0$. Since R is prime and a be a noncentral element then we obtain that $d\lambda(a)=0$. □

Lemma 2.19. *Let I be a nonzero ideal of R and a is a noncentral element of R. Let $h : R \longrightarrow R$ be a nonzero left generalized derivation associated with derivation $d_1 : R \longrightarrow R$. If $h((I, a)_{\lambda,\mu}) = 0$ or $(h(I), a)_{\lambda,\mu} = 0$ then $d_1\mu(a) = 0$.*

Proof. If $h(I, a)_{\lambda,\mu} = 0$ then using that h is a left-generaized derivation we get

$$\begin{aligned} 0 &= h(\mu(a)x, a)_{\lambda,\mu} = h\{\mu(a)(x, a)_{\lambda,\mu} - [\mu(a), \mu(a)]x\} \\ &= h\{\mu(a)(x, a)_{\lambda,\mu}\} = d_1(\mu(a))(x, a)_{\lambda,\mu} + \mu(a)h((x, a)_{\lambda,\mu}) \\ &= d_1(\mu(a))(x, a)_{\lambda,\mu}, \forall x \in I. \end{aligned}$$

That is,

$$d_1(\mu(a))(I, a)_{\lambda,\mu} = 0. \tag{2.16}$$

Since a be noncentral then using Remark 2 and (16) we obtain that$d_1(\mu(a)) = 0$.

On the other hand, If $(h(I), a)_{\lambda,\mu} = 0$ then we have $d_1(\mu(a)) = 0$ by Lemma 5.
□

Theorem 2.20. *Let I be a nonzero ideal of R and a is a noncentral element of R. Let $h : R \longrightarrow R$ be a nonzero right-generalized derivation associated with d and left-generalized derivation associated with d_1 . Then $h((I, a)_{\lambda,\mu}) = 0$ if and only if $(h(I), a)_{\lambda,\mu} = 0$.*

Proof. If $h((I, a)_{\lambda,\mu}) = 0$ or $(h(I), a)_{\lambda,\mu} = 0$ then $d(\lambda(a)) = 0$ and $d_1(\mu(a)) = 0$ are obtained by Lemma 9 and Lemma 10.

Using these results we get

$$\begin{aligned} h((I, a)_{\lambda,\mu}) &= 0 \Longleftrightarrow h(x\lambda(a) + \mu(a)x) = 0, \forall x \in I. \\ &\Longleftrightarrow h(x)\lambda(a) + xd(\lambda(a)) + d_1(\mu(a))x + \mu(a)h(x) = 0, \forall x \in I. \\ &\Longleftrightarrow h(x)\lambda(a) + \mu(a)h(x) = 0, \forall x \in I. \\ &\Longleftrightarrow (h(I), a)_{\lambda,\mu} = 0. \end{aligned}$$

□

Corollary 2.21. *[9, Theorem 7] Let R be a prime ring of characteristic different from two, $d : R \longrightarrow R$ be a nonzero derivation and $a \in R$. Then $(d(R), a) = 0$ if and only if $d(R, a) = 0$.*

References

1. Ashraf, M. and Rehman, N., *On (σ,τ)-derivations in prime rings*, Archivum Mathematicum,(BRNO), Tomus38, 259-264, (2002).
2. Aydın, N. and Kaya, K., *Some Generalizations in Prime Rings with $(\sigma, \tau)-$Derivation*, Doğa-Tr. J. of Math. 16, (1992).
3. Bresar, M., *On the distance of the compositions of two derivations to the generalized derivations*, Glasgow Math. J. 33, 80-93, (1991).
4. Chang, J. C., *On the identity $h(x) = af(x)+g(x)b$*, Taiwanese J. Math., 7(1), 103-113, (2003).

5. Güven, E., Kaya, K. and Soytürk, M., *Some Results on* $(\sigma, \tau)-$*Lie Ideals*, Math. J. Okayama Univ. 49, (2007).

6. Güven,E ,*Some Results on Generalized* $(\sigma, \tau)-$*derivations in Prime Rings*, Beitrage zur Algebra and Geometry, 54, ISSN 0138-4821, (2013).

7. Güven, E., *One Sided* $(\sigma, \tau)-$*Lie Ideals and Generalized Derivations in Prime Rings*, Palestine Journal of Mathematics, (2018 to appear).

8. Güven, E. and Soytürk, M., *On* $(\sigma, \tau)-$*derivations in Prime Rings*, Int, J. Contemp Math. Sci. (2008).

9. Kaya, K., Gölbaşı,Ö., Aydın, N., *Some Results for Generalized Lie Ideals in Prime Rings with Derivation II*, Applied Mathematics E-Notes, 1, (2001).

10. Marubayashi H. and Ashraf M., Rehman N., Ali Shakir., *On Generalized* $(\alpha, \beta)-$*Derivations in Prime Rings* , Algebra Colloquium 17 (Spec 1), 865-874, (2010).

11. Mayne, J. H., *Centralizing Mappings of Prime Rings*, Canad. Math. Bull, 27, 122-126, (1984).

12. Posner, E. C., *Derivations in prime rings*, Proc. Amer. Math. Soc. 8, 1093-1100, (1957).

13. Rehman, N., *On Commutativity of Rings with Generalized Derivations*, Math. J. Okayama Univ. Volume 44, Issue 1, (2002).

Evrim Guven,
Kocaeli University,
Department of Mathematics,
Faculty of Science and Art,
TURKEY.
E-mail address: evrim@kocaeli.edu.tr