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# Global Solutions To Nonlinear Second Order Interval Integrodifferential Equations By Fixed Point In Partially Ordered Sets

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abstract: In this paper, we prove the existence and uniqueness of global solution for second order interval valued integrodifferential equation with initial conditions admitting only the existence of a lower solution or an upper solution. In this study, in order to make the global solution on entire $[a, b]$, we use a fixed point in partially ordered sets on the subintervals of $[a, b]$ and obtain local solutions. Also, under weak conditions we show being well-defined a special kind of H-difference involved in this work. Moreover, we compare the results of existence and uniqueness under consideration of two kind of partial ordering on fuzzy numbers.

Key Words: Interval integrodifferential Equations, Method of upper or lower solutions, Fixed point, Partially ordered sets.

## Contents



## 1. Introduction

In many simultaneously occurring processes in modeling of the real world phenomena to obtain data, the field observations are needed. The modeling of a dynamical system based on the field observations becomes uncertain and vagueness or fuzziness, which is inherent in the systems behavior rather than being purely random or deterministic. Motivations for employing interval-valued and fuzzy functions, in general, fall in the fact that models based on the consideration of only two values: $0, 1$ are inadequate for describing real practical problems for which we have to use linguistic variables.The study of interval and fuzzy differential equations is an area of mathematics that has recently received a lot of attention (see e.g. [5,6,13,17]). Recently, there are some papers dealing with the existence of solution for nonlinear set valued and fuzzy differential equations whose methods are based on the monotone method, the method of upper and lower solutions and fixed point theorems [13,7,1,3,2]. Some works also have been done on

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the existence and uniqueness results of solutions for interval-valued second-order differential equations by contraction principle and successive approximations [9,10]. Moreover, author in [3], has proved the existence and uniqueness of global solutions for fuzzy integro-differential equation of Volterra type by means of the fixed point theory, the successive iteration method and Gronwall inequality. Among of them, we can find results on existence of solution for fuzzy differential equations in presence of both lower and upper solutions relative to the problem considered. The contraction mapping theorem and the abstract monotone iterative technique are well known and are applicable to a variety of situations. Recently, there is a fixed point theorem to weaken the requirement on the contraction by considering metric spaces endowed with partial order. The existence of a unique fixed point is based on assuming that the operator considered is monotone in such a setting [11,12,16].

In this study, we consider the following second order fuzzy integrodifferential equation

$$u''(t) = f(t, u(t), u'(t)) + \int_a^t k(t, s, u(s), u'(s))ds, \tag{1.1}$$

together with the initial conditions

$$u(a) = u_{01}, u'(a) = u_{02}, \tag{1.2}$$

where $f : [a, b] \times \mathcal{K}^2 \to \mathcal{K}$, $k : [a, b] \times [a, b] \times \mathcal{K}^2 \to \mathcal{K}$ are continuous in all of their arguments. All initial conditions are supposed to be interval numbers.

Here, we reduce (1.1) to a system of two first order interval integrodifferential equations with the initial condition and use fixed point in partially ordered sets to prove the existence results. This kind of fixed point needs just only a lower solution or an upper solution for the initial value problem of system and also the weak assumptions on the functions $f, k$. In this study, we try to overcome some difficulties mentioned below. Firstly, one of them is being well-defined H-differences appeared in the problem that we show them under weak conditions. Secondly, applying such fixed point gives us a local solution. In order to make the global solution on entire $[0, b]$, we use a fixed point in partially ordered sets on the subintervals of $[0, b]$ and obtain local solutions.

In general, the method can be applied to first order nonlocal systems of fuzzy differential equations.

Our interest in this kind of problem (1.1) may be arisen from its application as a model for population dynamics under uncertainty. The growth and decay of the populations are based on biological principles, reactions, environmental conditions and the parameters obtained from experiment. Therefore there are some uncertainties in determining these parameters. Let $X(t) = (x_1(t), x_2(t))$ be the population of two species; a logistic fuzzy model for populations dynamics is

$$\dot{X}(t) = \beta f(X) - \gamma g(X), \tag{1.3}$$

where, $\beta f(X)$ represents the growth and $\gamma g(X)$ an inhibition term. The growth factor may change with the state of the environment at time $t$ which in turn depends on the past history of the populations, since the populations contributes to the change in the environment. Thus instead of $\beta f(X)$, it becomes admissible to use a variable growth factor incorporating a history-dependent term, for example

$$\beta f(X) = \beta_0 - \int_0^t K(t-s)x(s)ds.$$

The same description can be presented for the inhibition term. Thus Equation (1.3) becomes the Volterra integro-differential equation.

## 2. Preliminaries

In this section we gather together some definitions and results from the literature, which we will use throughout this paper.

$\mathcal{K}$ denotes the spaces of nonempty compact and convex sets of the real line $\mathbb{R}$. For $A \in \mathcal{K}$, we have $A = [a^-, a^+]$ where $a^- \le a^+$. We denote the width of an interval $A$ by $len(A) = a^+ - a^-$. Given two intervals $A, B \in \mathcal{K}$ and $k \in \mathbb{R}$, addition and scalar multiplication are defined by $A + B = [a^- + b^-, a^+ + b^+]$ and

$$kA = \begin{cases} [ka^-, ka^+], & k \ge 0 \\ [ka^+, ka^-], & k < 0. \end{cases} \tag{2.1}$$

Difference is defined as $A - B = A + (-1)B$. It is well known that addition is associative and commutative and with neutral element $\{0\}$. If $A, B \in \mathcal{K}$, and if there exists a unique interval $C \in \mathcal{K}$ such that $B + C = A$, then $C$ is called the H-difference of $A$, $B$ and is denoted by $A \ominus B$ (see e.g. [14]). For intervals $A, B \in \mathcal{K}$ the Hausdorff distance is defined as usual by

$$D(A, B) = max\{|a^- - b^-|, |a^+ - b^+|\}.$$

The following properties of distance $D$ are well-known (see e.g. [18])
For all $A, B, C, E \in \mathcal{K}$ and $\lambda \in \mathbb{R}$, we have

$$\begin{aligned} &D(A + B, A + C) = D(B, C), \\ &D(\lambda A, \lambda B) = |\lambda| D(A, B), \quad \forall \lambda \in \mathbb{R}, \\ &D(A + B, C + E) \le D(A, C) + D(B, E), \end{aligned}$$

also if both of H-differences $A \ominus B$ and $C \ominus E$ exist, we conclude

$$D(A \ominus B, C \ominus E) = D(A + E, B + C),$$

and $(\mathcal{K}, D)$ is a complete metric space.
In vector form, we define

$$\mathcal{D}(A, B) = max\{D(A_1, B_1), D(A_2, B_2)\},$$

for $A = (A_1, A_2), B = (B_1, B_2) \in \mathcal{K} \times \mathcal{K}$.

We recall that if $F : [a, b] \to \mathcal{K}$ is an interval-valued function such that $F(t) = [f^-(t), f^+(t)]$, then $\lim_{t\to t_0} F(t)$ exists, if and only if $lim_{t\to t_0} f^-(t)$ and $lim_{t\to t_0} f^+(t)$ exist as finite numbers. In this case, we have

$$\lim_{t\to t_0} F(t) = [\lim_{t\to t_0} f^-(t), \lim_{t\to t_0} f^+(t)].$$

In particular, $F$ is continuous if and only if $f^-$ and $f^+$ are continuous.

**Definition 2.1.** *(See e.g. [5]) Let $F : (a, b) \to \mathcal{K}$ and $x_0 \in (a, b)$. We say $f$ is strongly generalized differentiable at $x_0$, if there exists an element $F'(x_0) \in \mathcal{K}$, such that for all $h > 0$ sufficiently small,*
*(i) there exist $F(x_0 + h) \ominus F(x_0), F(x_0) \ominus F(x_0 - h)$ and*

$$\lim_{h\searrow 0} \frac{F(x_0 + h) \ominus F(x_0)}{h} = \lim_{h\searrow 0} \frac{F(x_0) \ominus F(x_0 - h)}{h} = F'(x_0),$$

*or (ii) there exist $F(x_0) \ominus F(x_0 + h), F(x_0 - h) \ominus F(x_0)$ and*

$$\lim_{h\searrow 0} \frac{F(x_0) \ominus F(x_0 + h)}{-h} = \lim_{h\searrow 0} \frac{F(x_0 - h) \ominus F(x_0)}{-h} = F'(x_0),$$

*or (iii) there exist $F(x_0 + h) \ominus F(x_0), F(x_0 - h) \ominus F(x_0)$ and*

$$\lim_{h\searrow 0} \frac{F(x_0 + h) \ominus F(x_0)}{h} = \lim_{h\searrow 0} \frac{F(x_0 - h) \ominus F(x_0)}{-h} = F'(x_0),$$

*or (iv) there exist $F(x_0) \ominus F(x_0 + h), F(x_0) \ominus F(x_0 - h)$ and*

$$\lim_{h\searrow 0} \frac{F(x_0) \ominus F(x_0 + h)}{-h} = \lim_{h\searrow 0} \frac{F(x_0) \ominus F(x_0 - h)}{h} = F'(x_0).$$

*( $h$ and $-h$ at denominators mean $\frac{1}{h}\cdot$ and $-\frac{1}{h}\cdot$, respectively).*

**Remark 2.2.** *We say that a function is $(i)$- differentiable if it is differentiable as the case (i) of the definition above, etc.*

Throughout this paper, we consider $J = [a, b]$ and we shall use the notation

$$C(J, \mathcal{K}) = \{\ F : J \to \mathcal{K}|\ F \text{ is continuous}\},$$

where the continuity is one-side at endpoints $a, b$. Also for $k = 1, 2$

$$C^2_{(i)}(J, \mathcal{K}) = \{F : J \to \mathcal{K}|\ F^{(j)} \ \text{ is (i)-differentiable and continuous };j = 0, 1\},$$

$$C^2_{(ii)}(J, \mathcal{K}) = \{F : J \to \mathcal{K}|\ F^{(j)} \ \text{ is (ii)-differentiable and continuous };j = 0, 1\},$$

where differentiability at the endpoints $a$ and $b$, is interpreted right and left gH-differentiability at these points respectively. Define for $F, G \in C(J, \mathcal{K})$

$$H(F, G) = \sup_{t\in J} D(F(t), G(t)).$$

**Remark.** $(C(J,\mathcal{K}),H)$ is a complete metric space.
In vector form, we define

$$\mathcal{H}(\Phi,\Psi)=max\{H(\Phi_1,\Psi_1),H(\Phi_2,\Psi_2)\},$$

for $\Phi=(\Phi_1,\Phi_2),\Psi=(\Psi_1,\Psi_2)\in C(J,\mathcal{K})\times C(J,\mathcal{K})$.
The metric space $(C(J,\mathcal{K})\times C(J,\mathcal{K}),\mathcal{H})$ is a complete space.

**Remark.** In this paper, for the integral concept, we will use the interval Riemann integral introduced in [8]. Let $F:[a,b]\to\mathcal{K}$ be an interval-valued function such that $F(t)=[f^-(t),f^+(t)]$ and $f^-$ and $f^+$ are measurable and Lebesgue integrable on $[a,b]$. Then we define $\int_a^b F(t)dt$ by

$$\int_a^b F(t)dt=\left[\int_a^b f^-(t)dt,\int_a^b f^+(t)dt\right]$$

and we say that $F$ is Lebesgue integrable on [a,b].

**Lemma 2.3.** *(See [15].) Let $F:[a,b]\longrightarrow\mathcal{K}$ be $(i)$-differentiable and $C$ is an interval. Then $C+F$ is (i)-differentiable and $C\ominus f$ is (ii)-differentiable.*

Throughout this work, we will use the following partial orders and we compare the results of existence and uniqueness under them (see e.g. [13]).

**Definition 2.4.** *Suppose $x,\ y\in\mathcal{K}$. We say that $x\le_1 y$ if and only if*

$$x^-\le y^-,\qquad \textit{and}\qquad x^+\le y^+.$$

**Definition 2.5.** *Suppose $x,\ y\in\mathcal{K}$. We say that $x\le_2 y$ if and only if*

$$x^-\ge y^-,\qquad \textit{and}\qquad x^+\le y^+.$$

Let $h_1,\,h_2\in C(J,\mathcal{K})$ be two interval functions, we say that $h_1\le_j h_2$ if $h_1(t)\le_j h_2(t)$ for $t\in J$ $(j=1,2)$.

We recall some properties on the partial ordering $\le_j$ in space of interval functions, which are useful to our procedure.

**Lemma 2.6.** *(See [13].) Let $x,y,z,w\in\mathcal{K}$ and $c\in\mathbb{R}$, $c>0$, $j=1,2$.*

- *$x=y$ if and only if $x\le_j y$ and $x\ge_j y$.*
- *If $x\le_j y$, then $x+z\le_j y+z$.*
- *If $x\le_j y$ and $z\le_j w$, then $x+z\le_j y+w$.*
- *If $x\le_j y$, then $cx\le_j cy$.*

**Lemma 2.7.** *(See [13]) Let $g,\ h\in C(J,\mathcal{K})$ and $g\le_j h$, then*

$$\int_a^t g(s)ds\le_j\int_a^t h(s)ds,\qquad \forall t\in J.$$

**Lemma 2.8.** *Let $g$, $h \in C(J,\mathcal{K})$ and $u_0 \in \mathcal{K}$ and also $g \leq_1 h$, then*

$$u_0 \ominus (-1) \cdot \int_a^t g(s)ds \leq_1 u_0 \ominus (-1) \cdot \int_a^t h(s)ds, \qquad \forall t \in J,$$

*provided $u_0 \ominus (-1) \cdot \int_a^t g(s)ds$ and $u_0 \ominus (-1) \cdot \int_a^t h(s)ds$ are well-defined.*

**Lemma 2.9.** *Let $g$, $h \in C(J,\mathcal{K})$ and $u_0 \in \mathcal{K}$ and also $g \leq_2 h$, then*

$$u_0 \ominus (-1) \cdot \int_a^t g(s)ds \geq_2 u_0 \ominus (-1) \cdot \int_a^t h(s)ds, \qquad \forall t \in J,$$

*provided $u_0 \ominus (-1) \cdot \int_a^t g(s)ds$ and $u_0 \ominus (-1) \cdot \int_a^t h(s)ds$ are well-defined.*

**Lemma 2.10.** *(See [13].) If $\{g_n\} \subseteq C(J,\mathcal{K})$ and $h \in C(J,\mathcal{K})$ are such that*

$$g_n \leq_j h, \qquad \forall n \in \mathbb{N},$$

*and $g_n(t)$ converges to $g(t)$ in $\mathcal{K}$ for all $t \in J$, then $g \leq_j h$.*

**Definition 2.11.** *Let $(X, \leq_j)$ be a partially ordered set and $f : X \longrightarrow X$. We say that $f$ is monotone nondecreasing in $x$ if for any $x, y \in X$,*

$$x \leq_j y \Rightarrow f(x) \leq_j f(y)$$

*and is monotone nonincreasing in $y$, if*

$$x \leq_j y \Rightarrow f(x) \geq_j f(y).$$

The partial ordering in the vector form is defined as follows.

**Definition 2.12.** *Suppose $X = (x_1, x_2), Y = (y_1, y_2) \in \mathcal{K} \times \mathcal{K}$. We say that $X \preceq_j Y$ $(i = 1, 2)$ if and only if*

$$x_1 \leq_j y_1, \qquad \text{and} \qquad x_2 \leq_j y_2, \qquad (j = 1, 2).$$

Throughout this study we will use the following fixed point theorem in the partially ordered set.

**Theorem 2.13.** *(See [11,12].) Let $(X, \leq)$ be a partially ordered set and suppose that $d$ be a metric on $X$ such that $(X, d)$ is a complete metric space. Furthermore, let $T : X \to X$ be a monotone nondecreasing mapping such that*

$$\exists\ 0 \leq k < 1 \ni d(T(x), T(y)) \leq kd(x, y), \qquad \forall x \geq y.$$

*Suppose that either $T$ is continuous or $X$ is such that if $\{x_n\} \to x$ is a nondecreasing (or respectively nonincreasing) sequence in $X$, then $x_n \leq x$ (or respectively $x_n \geq x$) for every $n \in \mathbb{N}$. If there exists $x_0 \in X$ comparable to $T(x_0)$, then $T$ has a fixed point $\bar{x}$ and*

$$\lim_{n\to\infty} T^n(x_0) = \bar{x}.$$

The following lemma shows that a part of assumptions of Theorem 2.13 by considering $X = C(J, \mathcal{K})$ is satisfied.

**Lemma 2.14.** *If a nondecreasing (or nonincreasing) sequence $f_n \to f$ in $C(J, \mathcal{K})$, then $f_n \leq_j f$ (or $f_n \geq_j f$), $\forall n$ respectively.*

*Proof.* Since $f_n$ is nondecreasing sequence in $C(J, \mathbb{R}_{\mathcal{F}})$, $f_n(t)$ is nondecreasing sequence in $\mathcal{K}$ for $t \in J$. Also we have

$$f_1^-(t) \leq_1 ... \leq_1 f_n^-(t) \leq_1 ....$$

Hence $f_n^-(t)$ is a nondecreasing sequence that converges to $f^-(t)$ in $\mathbb{R}$. Therefore $f_n^-(t) \leq_1 f^-(t)$ for every $n$. Similarly we conclude $f_n^+(t) \leq_1 f^+(t)$ for every $n$. Thus $f_n \leq_1 f$ for every $n$. Also we have

$$f_1^-(t) \geq_2 ... \geq_2 f_n^-(t) \geq_2 ....$$

Hence $f_n^-(t)$ is a nonincreasing sequence that converges to $f^-(t)$ in $\mathbb{R}$. Therefore $f_n^-(t) \geq_2 f^-(t)$ for every $n$. Similarly we conclude $f_n^+(t) \leq_2 f^+(t)$ for every $n$. Thus $f_n \leq_2 f$ for every $n$. The similar result can be conclude for nonincreasing function. □

The following lemma guaranties the existence of special kind of H-difference under some conditions that we will be faced it.

**Lemma 2.15.** *Let $x \in \mathcal{K}$ and $f : [a, b] \to \mathcal{K}$ be continuous with respect to $t$. If $x \in \mathcal{K} \setminus \mathbb{R}$ i.e. $x^- < x^+$ or if $x \in \mathbb{R}$ and $f(t) \in \mathbb{R}$ for all $t \in [a, b]$, then there exists $h > a$ such that the H-difference*

$$x \ominus \int_a^t f(s)ds,$$

*exists for any $t \in [a, h]$.*

*Proof.* The proof is given in [4] in fuzzy space. We give it for our special case in intervals space. In order to prove the existence of $x \ominus \int_a^t f(s)ds$, we have to prove that $[x^- - \int_a^t f^-(s)ds, x^+ - \int_a^t f^+(s)ds]$ is an interval. Therefore we have to check

$$\int_a^t f^+(s)ds - \int_a^t f^-(s)ds \leq x^+ - x^-.$$

The above condition is equivalent to

$$\int_a^t len(f(s))ds \leq len(x).$$

By continuity of $f$, there exists $M > 0$ such that $len(f(t)) \leq M$ for all $t \in [a, b]$. Now suppose $x \in \mathcal{K} \setminus \mathbb{R}$ and $t \in [a, a + \frac{len(x)}{M}]$, thus we have

$$\int_a^t len(f(s))ds \leq M(t - a) \leq len(x).$$

If $x, f(t) \in \mathbb{R}$ for all $t \in [a, b]$, then $len(x) = len(f(t)) = 0$ for all $t \in [a, b]$. □

## 3. Main Results

In this section, we consider the following initial value problem for the second order interval integrodifferential equation of Volterra type

$$u''(t) = f(t, u(t), u'(t)) + \int_a^t k(t, s, u(s), u'(s))ds,$$
$$u(a) = u_{01}, u'(a) = u_{02},\ t \in J = [a, b], \tag{3.1}$$

where $f \in C(J \times \mathcal{K}^{(2)}, \mathcal{K})$ and $k \in C(J \times J \times \mathcal{K}^{(2)}, \mathcal{K})$ and $u_{01}$, $u_{02}$ are the interval numbers.
The purpose of current section is finding solutions $u \in C(J, \mathcal{K})$ of (3.1), which are defined as below.

**Definition 3.1.** *We say that $u \in \mathcal{C}^2_{(i)}(J, \mathcal{K})$ is (i)-solution of (3.1), if $u$ and $u'$ satisfy (3.1).*

**Definition 3.2.** *We say that $u \in \mathcal{C}^2_{(ii)}(J, \mathcal{K})$ is (ii)-solution of (3.1), if $u$ and $u'$ satisfy (3.1).*

**Definition 3.3.** *We say that $u$ is (ii,i)-solution of (3.1), if there exists $c \in (a, b)$ such that $u$ is (ii)-solution on $[a, c]$ and (i)-solution on $(c, b]$.*

We can reduce (3.1) to the following system of two first order interval integrodifferential equations

$$\begin{cases} v_1'(t) = v_2(t) & t \in J, \\ v_2'(t) = f(t, v_1(t), v_2(t)) + \int_a^t k(t, s, v_1(s), v_2(s))ds \end{cases} \tag{3.2}$$

together with the initial conditions

$$v_1(a) = u_{01}, v_2(a) = u_{02}. \tag{3.3}$$

For convenience, we apply vector notations $V(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}$, $V'(t) = \begin{bmatrix} v_1'(t) \\ v_2'(t) \end{bmatrix}$ and rewrite the problem (3.2) and (3.3) as

$$V'(t) = \begin{bmatrix} v_2(t) \\ f(t, v_1(t), v_2(t)) + \int_a^t k(t, s, v_1(s), v_2(s))ds \end{bmatrix},$$

$$V(a) = \begin{bmatrix} u_{01} \\ u_{02} \end{bmatrix}. \tag{3.4}$$

We note that two problems (3.1) and (3.4) are equivalent.

**Lemma 3.4.** *(See [3].) The problem (3.4) is equivalent to one of the following integral equations systems*

$(F_1)$ *if $v_1, v_2$ both (i)-differentiable on $J$, then*

$$V(t) = \begin{bmatrix} u_{01} + \int_a^t v_2(s)ds \\ u_{02} + \int_a^t f(s, v_1(s), v_2(s))ds + \int_a^t \int_a^s k(s, r, v_1(r), v_2(r))drds \end{bmatrix}.$$

$(F_2)$ *if $v_1, v_2$ are both (ii)-differentiable on $J$, then*

$$V(t) = \begin{bmatrix} u_{01} \ominus (-1) \cdot \int_a^t v_2(s)ds \\ u_{02} \ominus (-1) \cdot (\int_a^t f(s, v_1(s), v_2(s))ds + \int_a^t \int_a^s k(s, r, v_1(r), v_2(r))drds) \end{bmatrix}.$$

$(F_3)$ *if $v_1, v_2$ are both (ii)-differentiable on $[a, a + c^*]$ and (i)-differentiable on $[a + c^*, b]$, then for $t \in [a, a + c^*]$*

$$V(t) = \begin{bmatrix} u_{01} \ominus (-1) \cdot \int_a^t v_2(s)ds \\ u_{02} \ominus (-1) \cdot (\int_a^t f(s, v_1(s), v_2(s)) + \int_a^s k(s, r, v_1(r), v_2(r))drds) \end{bmatrix},$$

*and for $t \in [a + c^*, b]$*

$$V(t) = \begin{bmatrix} v_1(a + c^*) + \int_{a+c^*}^t v_2(s)ds \\ v_2(a + c^*) + \int_{a+c^*}^t f(s, v_1(s), v_2(s)) + \int_a^s k(s, r, v_1(r), v_2(r))drds \end{bmatrix}.$$

**Remark 3.5.** *The continuous solution, obtained of integral equations $(F_1)$ is corresponding to the (i)-solution, $(F_2)$ is corresponding to the (ii)-solutions and $(F_3)$ is corresponding to the (ii,i)-solution of (3.1).*

### 3.1. Existence of (i)-solution

Now we are in a situation to define the nonlinear mappings $\mathcal{A} : C(J, \mathcal{K}) \times C(J, \mathcal{K}) \longrightarrow C(J, \mathcal{K}) \times C(J, \mathcal{K})$, which plays a main role in our discussion, as following

$$[\mathcal{A}\Phi](t) = \begin{bmatrix} [\mathcal{A}_1\Phi](t) \\ [\mathcal{A}_2\Phi](t) \end{bmatrix} = \begin{bmatrix} u_{01} + \int_a^t \phi_2(s)ds \\ u_{02} + \int_a^t f(s, \Phi(s)) + \int_a^s k(s, r, \Phi(r))drds \end{bmatrix}, \tag{3.5}$$

where $t \in J$ and $\Phi(t) = \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix}$.

In follows, we define upper and lower solution for Problem 3.4 as following:

**Definition 3.6.** *Let $\underline{U} = \begin{bmatrix} \underline{u}_1 \\ \underline{u}_2 \end{bmatrix}, \bar{U} = \begin{bmatrix} \overline{u}_1 \\ \overline{u}_2 \end{bmatrix} \in C(J, \mathcal{K}) \times C(J, \mathcal{K})$, we say that*

*(a) $\underline{U}$ is a lower solution for the problem (3.1) if*

$$\underline{U}(t) \preceq_j [\mathcal{A}\underline{U}](t), \qquad t \in J,$$

*(b) $\overline{U}$ is an upper solution for the problem (3.4) if*

$$\overline{U}(t) \succeq_j [\mathcal{A}\overline{U}](t), \qquad t \in J.$$

We are now in a position to state our main results. We apply fixed point Theorem 2.13 to prove the existence and uniqueness of global solution belonging to $C^2_{(i)}(J, \mathcal{K})$ for the interval initial value problem (3.1) by the existence of just a lower solution or an upper solution. The following theorem states the same results for two kinds of partial ordering $\leq_j$ $(j = 1, 2)$.

**Theorem 3.7.** *Consider Problem (3.1) with $f$ and $k$ continuous and suppose $f$, $k$ are nondecreasing in two last arguments. Let exist two constant real numbers $l_1, l_2 > 0$ such that*

$$D(f(t, x_1, x_2), f(t, y_1, y_2)) \leq l_1 max\{D(x_1, y_1), D(x_2, y_2)\}, \quad \forall t \in J,$$

$$D(k(t, s, x_1, x_2), k(t, s, y_1, y_2)) \leq l_2 max\{D(x_1, y_1), D(x_2, y_2)\}, \quad \forall t \in J,$$

*for $x_1 \geq_j y_1$ and $x_2 \geq_j y_2$. Then the existence of a lower solution $\underline{U}$ (or an upper solution $\overline{U}$) for Problem (3.1) provides the existence of a fixed point for $\mathcal{A}$ like $U$, and consequently (i)-solution to Problem (3.1) on $[a, b]$. Also, $\lim_{n\to\infty} \mathcal{A}^n(\underline{U}) = U$ (or $\lim_{n\to\infty} \mathcal{A}^n(\overline{U}) = U$). Moreover, if $W \in C(J, \mathcal{K}) \times C(J, \mathcal{K})$ is another fixed point of $\mathcal{A}$ such that is comparable to $\underline{U}$, then $U = W$.*

*Proof.* Since by Lemma 3.4, Problem (3.1) is equivalent to $(F_1)$, we prove that the mapping $\mathcal{A}$ has a unique fixed point under assumption the existence a lower solution $\underline{U}$ for Problem (3.1). Because of similarity we omit the proof under assumption the existence of upper solution. Now we check that hypotheses in Theorem 2.13 are satisfied.
We consider $X = C(J, \mathcal{K}) \times C(J, \mathcal{K})$ that is partially ordered set by the following order relation For $G, F \in C(J, K) \times C(J, \mathcal{K})$,

$$G \preceq_j F \Leftrightarrow G(t) \preceq_j F(t), \ \forall t \in J.$$

Since $f$, $k$ are nondecreasing in their two last arguments, the mapping $\mathcal{A}$, defined by (3.5), is nondecreasing on $J$. Obviously there exists $c > 0$ such that $\frac{b-a}{c} = N \in \mathbb{N}$ and $max\{c, l_1 c + l_2 \frac{c^2}{2}\} < 1$. Firstly We consider the interval $[a, a + c]$. For $\Phi \succeq_j \Psi$, we have

$$D([\mathcal{A}_1\Phi](t), [\mathcal{A}_1\Psi](t)) \leq \int_a^t D(\phi_2(s), \psi_2(s))ds \leq cH(\phi_2, \psi_2), \tag{3.6}$$

and also,

$$\begin{aligned} D([\mathcal{A}_2\Phi](t), [\mathcal{A}_2\Psi](t)) \quad &\leq \quad \int_a^t D(f(s, \Phi(s)), f(s, \Psi(s)))ds \\ &+ \quad \int_a^t \int_a^s D(k(s, r, \Phi(r)), k(s, r, \Psi(r)))drds \\ &\leq \quad l_1 c max\{H(\phi_1, \psi_1), H(\phi_2, \psi_2)\} \\ &+ \quad l_2 \frac{c^2}{2} max\{H(\phi_1, \psi_1), H(\phi_2, \psi_2)\}. \end{aligned} \tag{3.7}$$

Then from (3.6) and (3.7), we have

$$\mathcal{H}(\mathcal{A}\Phi,\mathcal{A}\Psi)\leq L\mathcal{H}(\Phi,\Psi), \tag{3.8}$$

where $L=max\{c,l_1c+l_2\frac{c^2}{2}\}<1$. Applying Theorem 2.13, $\mathcal{A}$ has a fixed point $\mathbb{U}=\begin{bmatrix}\mathbf{u}_1\\ \mathbf{u}_2\end{bmatrix}\in C([a,a+c],\mathcal{K})\times C([a,a+c],\mathcal{K})$ and $\lim_{n\to\infty}\mathcal{A}^n(\underline{U})=\mathbb{U}$ (or $\lim_{n\to\infty}\mathcal{A}^n(\overline{U})=\mathbb{U}$). Now suppose $W\in C([a,a+c],\mathcal{K})\times C([a,a+c],\mathcal{K})$ is another fixed point of $\mathcal{A}$ such that is comparable to $\underline{U}$. It means that $\underline{U}\preceq_j W$ or $W\preceq_j\underline{U}$. We claim that $\mathcal{H}(\mathbb{U},W)=0$. Employing the nondecreasing property of the mapping $\mathcal{A}$, along with Lemma 2.14 and $\underline{U}\preceq_j\mathcal{A}$, we can infer $\underline{U}\preceq_j\mathbb{U}$. Then $\mathcal{A}^n\underline{U}$ is comparable to $\mathcal{A}^n\mathbb{U}=\mathbb{U}$ and $\mathcal{A}^nW=W$ for $n=0,1,2,\ldots$. Utilizing (3.8) we have

$$\begin{aligned}\mathcal{H}(\mathbb{U},W) &= \mathcal{H}(\mathcal{A}^n\mathbb{U},\mathcal{A}^nW)\leq\mathcal{H}(\mathcal{A}^n\mathbb{U},\mathcal{A}^n\underline{U})+\mathcal{H}(\mathcal{A}^nW,\mathcal{A}^n\underline{U})\\ &\leq L^n\mathcal{H}(\mathbb{U},\underline{U})+L^n\mathcal{H}(\underline{U},W).\end{aligned}$$

Since $L<1$, the right-hand side of above equation converges to zero as $n\to\infty$. Then $\mathcal{H}(\mathbb{U},W)=0$. It means that the fixed point is unique on $[a,a+c]$
Now by considering $\mathbb{U}$ as a fixed point for $\mathcal{A}$ on the interval $[a,a+c]$, we define another mapping on the interval $[a+c,a+2c]$ as follows:

$$[\mathcal{T}\Phi](t)=$$
$$\begin{bmatrix}\mathbf{u}_1(a+c)+\int_{a+c}^t\phi_2(s)ds\\ \mathbf{u}_2(a+c)+\int_{a+c}^t\int_a^{a+c}k(s,r,\mathbb{U}(r))drds+\int_{a+c}^t(f(s,\Phi(s))+\int_{a+c}^s k(s,r,\Phi(r))dr)ds\end{bmatrix}.$$

Since $f$, $k$ are nondecreasing in two of last their arguments, the mapping $\mathcal{T}:C([a+c,a+2c],\mathcal{K})\times C([a+c,a+2c],\mathcal{K})\to C([a+c,a+2c],\mathcal{K})\times C([a+c,a+2c],\mathcal{K})$ is nondecreasing.
Now we will show that $\underline{U}(t)\preceq_j[\mathcal{T}\underline{U}](t)$ (or $\overline{U}(t)\preceq_j[\mathcal{T}\overline{U}](t)$) for $t\in[a+c,a+2c]$. Due the fact that $\underline{U}$ is a lower solution of Problem (3.1) for $t\in[a,b]$ and $\underline{U}\preceq_j\mathbb{U}$ for $t\in[a,a+c]$, we have

$$\begin{aligned}\underline{U}&=\begin{bmatrix}\underline{u_1}(t)\\ \underline{u_2}(t)\end{bmatrix}\preceq_j\begin{bmatrix}[\mathcal{A}_1\underline{U}](t)\\ [\mathcal{A}_2\underline{U}](t)\end{bmatrix}=\begin{bmatrix}u_{01}+\int_a^t\underline{u_2}(s)ds\\ u_{02}+\int_a^t f(s,\underline{U}(s))+\int_a^s k(s,r,\underline{U}(r))drds\end{bmatrix}\\
&=\begin{bmatrix}u_{01}+\int_a^{a+c}\underline{u_2}(s)ds+\int_{a+c}^t\underline{u_2}(s)ds\\ u_{02}+\int_a^{a+c}f(s,\underline{U}(s))+\int_a^s k(s,r,\underline{U}(r))drds+\int_{a+c}^t f(s,\underline{U}(s))+\int_a^s k(s,r,\underline{U}(r))drds\end{bmatrix}\\
&\preceq_j\begin{bmatrix}u_{01}+\int_a^{a+c}\mathbf{u}_2(s)ds+\int_{a+c}^t\underline{u_2}(s)ds\\ u_{02}+\int_a^{a+c}f(s,\mathbb{U}(s))+\int_a^s k(s,r,\mathbb{U}(r))drds+\int_{a+c}^t f(s,\underline{U}(s))+\int_a^s k(s,r,\underline{U}(r))drds\end{bmatrix}\\
&=\begin{bmatrix}\mathbf{u}_1(a+c)+\int_{a+c}^t\underline{u_2}(s)ds\\ \mathbf{u}_2(a+c)+\int_{a+c}^t f(s,\underline{U}(s))+\int_a^s k(s,r,\underline{U}(r))drds\end{bmatrix}\\
&=\begin{bmatrix}\mathbf{u}_1(a+c)+\int_{a+c}^t\underline{u_2}(s)ds\\ \mathbf{u}_2(a+c)+\int_{a+c}^t\int_a^{a+c}k(s,r,\underline{U}(r))drds+\int_{a+c}^t f(s,\underline{U}(s))+\int_{a+c}^s k(s,r,\underline{U}(r))drds\end{bmatrix}\\
&\preceq_j\begin{bmatrix}\mathbf{u}_1(a+c)+\int_{a+c}^t\underline{u_2}(s)ds\\ \mathbf{u}_2(a+c)+\int_{a+c}^t\int_a^{a+c}k(s,r,\mathbb{U}(r))drds+\int_{a+c}^t f(s,\underline{U}(s))+\int_{a+c}^s k(s,r,\underline{U}(r))drds\end{bmatrix}\\
&=[\mathcal{T}\underline{U}](t).\end{aligned}$$

For $\Phi \succeq_j \Psi$, we conclude

$$D([\mathcal{T}_1\Phi](t), [\mathcal{T}_1\Psi](t)) \leq \int_{a+c}^{t} D(\phi_2(s), \psi_2(s))ds \leq cH(\phi_2, \psi_2), \tag{3.9}$$

and also,

$$\begin{aligned} D([\mathcal{T}_2\Phi](t), [\mathcal{T}_2\Psi](t)) &\leq \int_{a+c}^{t} D(f(s,\Phi(s)), f(s,\Psi(s)))ds \\ &+ \int_{a+c}^{t}\int_{a+c}^{s} D(k(s,r,\Phi(r)), k(s,r,\Psi(r)))drds \\ &\leq l_1 c max\{H(\phi_1,\psi_1), H(\phi_2,\psi_2)\} \\ &+ l_2\frac{c^2}{2} max\{H(\phi_1,\psi_1), H(\phi_2,\psi_2)\}. \end{aligned}$$

Then we have

$$\mathcal{H}(\mathcal{T}\Phi, \mathcal{T}\Psi) \leq L\mathcal{H}(\Phi, \Psi), \tag{3.10}$$

where $L = max\{c, l_1c + l_2\frac{c^2}{2}\} < 1$. All the conditions in Theorem 2.13 are satisfied, therefore the mapping $\mathcal{T}$ has a fixed point $\mathbb{V} \in C([a+c, a+2c], \mathcal{K}) \times C([a+c, a+2c], \mathcal{K})$ and $\lim_{n\to\infty} \mathcal{T}^n(\underline{U}) = \mathbb{V}$.
If we suppose $W \in C([a+c, a+2c], \mathcal{K}) \times C([a+c, a+2c], \mathcal{K})$ is another fixed point of $\mathcal{T}$ such that is comparable to $\underline{U}$ on $[a+c, a+2c]$, then it is clear that $\mathcal{H}(\mathbb{V}, W) = 0$.
Obviously $U$ as defined

$$U = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{cases} \mathbb{U} & t \in [a, a+c] \\ \mathbb{V} & t \in [a+c, a+2c] \end{cases} \tag{3.11}$$

is a fixed point of $\mathcal{A}$ defined by (3.5) on $[a, a+2c]$. By Lemma 2.3, $u_1, u_2$ are (i)-differentiable on $[a, a+2c]$. In the same trend we can make a fixed point of $\mathcal{A}$ defined by (3.5) on $[a, a+Nc] = [a, b]$. Let $U = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in C(J, \mathcal{K}) \times C(J, \mathcal{K})$ is a fixed point of $\mathcal{A}$ where $J = [a, b]$. Therefore $U$ is a solution of integral equation $(F_1)$. By Remark 3.5 and System (3.2), we can conclude $u_1$ is a (i)-solution of Problem (3.1).
Now suppose $W \in C(J, \mathcal{K}) \times C(J, \mathcal{K})$ is another fixed point of $\mathcal{A}$ such that is comparable to $\underline{U}$ on $J = [a, b]$. It is clear that $\mathcal{H}(U, W) = 0$.

□

### 3.2. Existence of (ii)-solution

Let $x_0 \in \mathcal{K}$. We denote by $\bar{B}(x_0) = \{x \in \mathcal{K} : len(x) \leq len(x_0)\}$, a closed subset in $\mathcal{K}$. Now we are in a situation to define the nonlinear mappings $\mathcal{B}$, which plays a main role in our discussion, as following

$$[\mathcal{B}\Phi](t) = \begin{bmatrix} [\mathcal{B}_1\Phi](t) \\ [\mathcal{B}_2\Phi](t) \end{bmatrix} = \begin{bmatrix} u_{01} \ominus (-1)\int_a^t \phi_2(s)ds \\ u_{02} \ominus (-1)\int_a^t f(s,\Phi(s)) + \int_a^s k(s,r,\Phi(r))drds \end{bmatrix} \tag{3.12}$$

where $t \in J$ and $\Phi(t) = \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix}$. In general the mapping $\mathcal{B} : C(J, \mathcal{K}) \times C(J, \mathcal{K}) \longrightarrow C(J, \mathcal{K}) \times C(J, \mathcal{K})$ is not well-defined. The following lemma guaranties the existence of H-differences involving in the mapping $\mathcal{B}$.

**Lemma 3.8.** *Let $u_{01}, u_{0,2} \in \mathcal{K} \setminus \mathbb{R}$ and $len(f(t,x,y))$, $len(k(t,s,x,y))$ for all $x \in \bar{B}(u_{01}), y \in \bar{B}(u_{02})$, $\forall t, s \in [a,b]$ are bounded. Then there exists $c^* > 0$ such that the mapping $\mathcal{B} : C([a, a+c^*], \bar{B}(u_{01})) \times C([a, a+c^*], \bar{B}(u_{02})) \longrightarrow C([a, a+c^*], \bar{B}(u_{01})) \times C([a, a+c^*], \bar{B}(u_{02}))$ is well-defined.*

*Proof.* By Lemma 2.15, for $t \in [a, a + \frac{len(u_{01})}{len(u_{02})}]$ we have

$$\int_a^t len(\phi_2(s))ds \leq (t-a)len(u_{02}) \leq len(u_{01}). \tag{3.13}$$

Now let $len(f(t,x,y)) \leq M_1$, $len(k(t,s,x,y)) \leq M_2$ for all $x \in \bar{B}(u_{01}), y \in \bar{B}(u_{02})$, $\forall t, s \in [a,b]$. Thus for $t \in [a, a + \frac{2len(u_{02})}{2M_1 + M_2(b-a)}]$, we can conclude

$$\begin{aligned} \int_a^t len(f(s,\Phi(s))) + \int_a^s len(k(s,r,\Phi(r)))drds &\leq M_1(t-a) + M_2\frac{(t-a)^2}{2} \\ &\leq len(u_{02}). \end{aligned} \tag{3.14}$$

Let consider $c^* = min\{\frac{len(u_{01})}{len(u_{02})}, \frac{2len(u_{02})}{2M_1+M_2(b-a)}, b-a\}$. Then from the relations (3.13) and (3.14), H-differences involving in the mapping $\mathcal{B}$ exist on $t \in [a, a+c^*]$. $\square$

**Definition 3.9.** *Let $\underline{U} = \begin{bmatrix} \underline{u}_1 \\ \underline{u}_2 \end{bmatrix}, \bar{U} = \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{bmatrix} \in C(J, \bar{B}(u_{01})) \times C(J, \bar{B}(u_{02}))$, we say that*
*(a) $\underline{U}$ is a lower solution for the problem (3.4) if*

$$\underline{U}(t) \preceq_j [\mathcal{B}\underline{U}](t), \qquad t \in J,$$

*(b) $\bar{U}$ is an upper solution for the problem (3.4) if*

$$\bar{U}(t) \succeq_j [\mathcal{B}\bar{U}](t), \qquad t \in J.$$

**Remark 3.10.** *If $u_{01}, u_{02} \in \mathcal{K} \setminus \mathbb{R}$, then Definition 3.13 is well-defined.*

The following theorem gives (ii)-solution to Problem (3.1) considering partial ordering $\leq_1$, along with nondecreasing property of $f, k$.

**Theorem 3.11.** *Consider Problem (3.1) with $f$ and $k$ continuous and suppose $f$, $k$ are nondecreasing in all their arguments except for the first. Let $u_{01}, u_{0,2} \in \mathcal{K} \setminus \mathbb{R}$ and $len(f(t,x,y))$, $len(k(t,s,x,y))$ for all $x \in \bar{B}(u_{01}), y \in \bar{B}(u_{02})$, $\forall t, s \in [a,b]$ are bounded with bounds of $M_1, M_2$ respectively. Assume*

$$c^* = min\{\frac{len(u_{01})}{len(u_{02})}, \frac{2len(u_{02})}{2M_1 + M_2(b-a)}, b-a\}.$$

*Let exist $l_1, l_2 > 0$ such that*

$$D(f(t,x_1,x_2), f(t,y_1,y_2)) \le l_1 max\{D(x_1,y_1), D(x_2,y_2)\}, \qquad \forall t \in [a, a+c^*],$$

$$D(k(t,s,x_1,x_2), k(t,s,y_1,y_2)) \le l_2 max\{D(x_1,y_1), D(x_2,y_2)\}, \qquad \forall t \in [a, a+c^*],$$

*for $x_1 \succeq_1 y_1$ and $x_2 \succeq_1 y_2$. Then the existence of a lower solution $\underline{U}$ (or an upper solution $\bar{U}$) for Problem (3.1) provides the existence of a fixed point for $\mathcal{B}$ like $U$, and consequently (ii)-solution to Problem (3.1) on $[a, a+c^*]$. Also, $\lim_{n\to\infty} \mathcal{B}^n(\underline{U}) = U$ (or $\lim_{n\to\infty} \mathcal{B}^n(\bar{U}) = U$). Moreover, if $W \in C([a, a+c^*], \bar{B}(u_{01})) \times C([a, a+c^*], \bar{B}(u_{02}))$ is another fixed point of $\mathcal{B}$ such that is comparable to $\underline{U}$ in the partial ordering $\preceq_1$, then $U = W$.*

*Proof.* Since by Lemma 3.4, Problem (3.1) is equivalent to $(F_2)$, we prove that the mapping $\mathcal{B}$ has a unique fixed point under assumption the existence of a lower solution $\underline{U}$ for Problem (3.1). Because of similarity we omit the proof under assumption the existence of upper solution. Now we check that hypotheses in Theorem 2.13 are satisfied.
We consider $X = C([a, a+c^*], \bar{B}(u_{01})) \times C([a, a+c^*], \bar{B}(u_{02}))$ that is partially ordered set by the following order relation For $G, F \in C([a, a+c^*], \bar{B}(u_{01})) \times C([a, a+c^*], \bar{B}(u_{02}))$,

$$G \preceq_1 F \Leftrightarrow G(t) \preceq_1 F(t), \ \forall t \in [a, a+c^*].$$

Obviously there exists $c > 0$ such that $\frac{c^*}{c} = N \in \mathbb{N}$ and $max\{c, l_1 c + l_2 \frac{c^2}{2}\} < 1$. Firstly We consider the interval $[a, a+c]$. By Lemma 3.8, the mapping $\mathcal{B} : C([a, a+c], \bar{B}(u_{01})) \times C([a, a+c], \bar{B}(u_{02})) \to C([a, a+c], \bar{B}(u_{01})) \times C([a, a+c], \bar{B}(u_{02}))$ defined by (3.21), is well-defined and since $f$, $k$ are nondecreasing in their two last arguments, the mapping $\mathcal{B}$ is nondecreasing on $[a, a+c]$. For $\Phi \succeq_1 \Psi$, we have

$$D([\mathcal{B}_1\Phi](t), [\mathcal{B}_1\Psi](t)) \le \int_a^t D(\phi_2(s), \psi_2(s))ds \le cH(\phi_2, \psi_2), \tag{3.15}$$

and also,

$$\begin{aligned} D([\mathcal{B}_2\Phi](t), [\mathcal{B}_2\Psi](t)) &\le \int_a^t D(f(s,\Phi(s)), f(s,\Psi(s)))ds \\ &+ \int_a^t \int_a^s D(k(s,r,\Phi(r)), k(s,r,\Psi(r)))drds \\ &\le l_1 c max\{H(\phi_1,\psi_1), H(\phi_2,\psi_2)\} \\ &+ l_2 \frac{c^2}{2} max\{H(\phi_1,\psi_1), H(\phi_2,\psi_2)\}. \end{aligned} \tag{3.16}$$

Then from (3.15) and (3.16), we have

$$\mathcal{H}(\mathcal{B}\Phi, \mathcal{B}\Psi) \le L\mathcal{H}(\Phi, \Psi), \tag{3.17}$$

where $L = max\{c, l_1c + l_2\frac{c^2}{2}\} < 1$. Applying Theorem 2.13, $\mathcal{B}$ has a fixed point $\mathbb{U} = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} \in C([a, a+c], \bar{B}(u_{01})) \times C([a, a+c], \bar{B}(u_{02}))$ and $\lim_{n\to\infty} \mathcal{B}^n(\underline{U}) = \mathbb{U}$ (or $\lim_{n\to\infty} \mathcal{B}^n(\overline{U}) = \mathbb{U}$). Now suppose $W \in C([a, a+c], \bar{B}(u_{01})) \times C([a, a+c], \bar{B}(u_{02}))$ is another fixed point of $\mathcal{B}$ such that is comparable to $\underline{U}$ with respect to partial ordering $\preceq_1$. It means that $\underline{U} \preceq_1 W$ or $W \preceq_1 \underline{U}$. We claim that $\mathcal{H}(\mathbb{U}, W) = 0$. Employing the nondecreasing property of the mapping $\mathcal{B}$, along with Lemma 2.14 and $\underline{U} \preceq_1 \mathcal{B}\underline{U}$, we can infer $\underline{U} \preceq_1 \mathbb{U}$. Then $\mathcal{B}^n\underline{U}$ is comparable to $\mathcal{B}^n\mathbb{U} = \mathbb{U}$ and $\mathcal{B}^nW = W$ for $n = 0, 1, 2, ....$ Utilizing (3.17) we have

$$\begin{aligned} \mathcal{H}(\mathbb{U}, W) = \mathcal{H}(\mathcal{B}^n\mathbb{U}, \mathcal{B}^nW) &\leq \mathcal{H}(\mathcal{B}^n\mathbb{U}, \mathcal{B}^n\underline{U}) + \mathcal{H}(\mathcal{B}^nW, \mathcal{B}^n\underline{U}) \\ &\leq L^n\mathcal{H}(\mathbb{U}, \underline{U}) + L^n\mathcal{H}(\underline{U}, W). \end{aligned}$$

Since $L < 1$, the right-hand side of above equation converges to zero as $n \to \infty$. Then $\mathcal{H}(\mathbb{U}, W) = 0$. It means that the fixed point is unique on $[a, a+c]$
Now by considering $\mathbb{U}$ as a fixed point for $\mathcal{B}$ on the interval $[a, a+c]$, we define another mapping on the interval $[a+c, a+2c]$ as follows:

$$[\mathcal{T}\Phi](t) =$$
$$\begin{bmatrix} \mathbf{u}_1(a+c) \ominus (-1)\int_{a+c}^t \phi_2(s)ds \\ \mathbf{u}_2(a+c) \ominus (-1)(\int_{a+c}^t \int_a^{a+c} k(s,r,\mathbb{U}(r))dr + (f(s,\Phi(s)) + \int_{a+c}^s k(s,r,\Phi(r))drds) \end{bmatrix}.$$

The mapping $\mathcal{T} : C([a+c, a+2c], \bar{B}(u_{01})) \times C([a+c, a+2c], \bar{B}(u_{02})) \to C([a+c, a+2c], \bar{B}(u_{01})) \times C([a+c, a+2c], \bar{B}(u_{02}))$ is well-defined, since for $t \in [a+c, a+2c]$

$$\begin{aligned} \int_a^{a+c} len(\mathbf{u_2}(s))ds + \int_{a+c}^t len(\phi_2(s))ds &\leq len(u_{02})c + len(u_{02})(t-a-c) \\ &\leq len(u_{02})c^* \leq len(u_{01}). \end{aligned}$$

That means

$$\int_{a+c}^t len(\phi_2(s))ds \leq len(u_{01}) - \int_a^{a+c} len(\mathbf{u_2}(s))ds = len(\mathbf{u_1}(a+c)).$$

Also for being well-defined the second component of $\mathcal{T}$ we have

$$\begin{aligned} &\int_{a+c}^t (\int_a^{a+c} len(k(s,r,\mathbb{U}(r)))dr + len(f(s,\Phi(s))) + \int_{a+c}^s len(k(s,r,\Phi(r)))dr)ds \\ &+ \int_a^{a+c} (len(f(s,\mathbb{U}(s))) + \int_a^s len(k(s,r,\mathbb{U}(r)))dr)ds \\ &\leq M_2c(t-a-c) + M_1(t-a-c) + M_2\frac{(t-a-c)^2}{2} + M_1c + M_2\frac{c^2}{2} \\ &= M_1(t-a) + M_2\frac{(t-a)^2}{2} \leq c^*(M_1 + M_2\frac{(b-a)}{2}) \leq len(u_{02}). \end{aligned}$$

Since $f$, $k$ are nondecreasing in two of last their arguments, the mapping $\mathcal{T}$ : $C([a+c, a+2c], \bar{B}(u_{01})) \times C([a+c, a+2c], \bar{B}(u_{02})) \to C([a+c, a+2c], \bar{B}(u_{01})) \times$

$C([a+c,a+2c],\bar{B}(u_{02}))$ is nondecreasing.
Now we will show that $\underline{U}(t) \preceq_1 [\mathcal{T}\underline{U}](t)$ (or $\overline{U}(t) \preceq_1 [\mathcal{T}\overline{U}](t)$) for $t \in [a+c,a+2c]$.
Due the fact that $\underline{U}$ is a lower solution of Problem (3.1) for $t \in [a,b]$ and $\underline{U} \preceq_1 \mathbb{U}$ for $t \in [a,a+c]$, we have

$$\underline{U} = \begin{bmatrix} \underline{u_1}(t) \\ \underline{u_2}(t) \end{bmatrix} \preceq_1 \begin{bmatrix} [\mathcal{B}_1\underline{U}](t) \\ [\mathcal{B}_2\underline{U}](t) \end{bmatrix} = \begin{bmatrix} u_{01} \ominus (-1)\int_a^t \underline{u_2}(s)ds \\ u_{02} \ominus (-1)(\int_a^t f(s,\underline{U}(s)) + \int_a^s k(s,r,\underline{U}(r))drds) \end{bmatrix} \preceq_1$$
$$\begin{bmatrix} u_{01} \ominus (-1)\int_a^{a+c} \mathbf{u}_2(s)ds \ominus (-1)\int_{a+c}^t \underline{u_2}(s)ds \\ u_{02} \ominus (-1)(\int_a^{a+c} f(s,\mathbb{U}(s)) + \int_a^s k(s,r,\mathbb{U}(r))drds + \int_{a+c}^t f(s,\underline{U}(s)) + \int_a^s k(s,r,\underline{U}(r))drds) \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{u}_1(a+c) \ominus (-1)\int_{a+c}^t \underline{u_2}(s)ds \\ \mathbf{u}_2(a+c) \ominus (-1)\int_{a+c}^t f(s,\underline{U}(s)) + \int_a^s k(s,r,\underline{U}(r))drds \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{u}_1(a+c) \ominus (-1)\int_{a+c}^t \underline{u_2}(s)ds \\ \mathbf{u}_2(a+c) \ominus (-1)(\int_{a+c}^t\int_a^{a+c} k(s,r,\underline{U}(r))drds + \int_{a+c}^t f(s,\underline{U}(s)) + \int_{a+c}^s k(s,r,\underline{U}(r))drds) \end{bmatrix}$$
$$\preceq_1 \begin{bmatrix} \mathbf{u}_1(a+c) \ominus (-1)\int_{a+c}^t \underline{u_2}(s)ds \\ \mathbf{u}_2(a+c) \ominus (-1)(\int_{a+c}^t\int_a^{a+c} k(s,r,\mathbb{U}(r))drds + \int_{a+c}^t f(s,\underline{U}(s)) + \int_{a+c}^s k(s,r,\underline{U}(r))drds) \end{bmatrix}$$
$$= [\mathcal{T}\underline{U}](t).$$

For $\Phi \succeq_1 \Psi$, we conclude

$$D([\mathcal{T}_1\Phi](t),[\mathcal{T}_1\Psi](t)) \le \int_{a+c}^t D(\phi_2(s),\psi_2(s))ds \le cH(\phi_2,\psi_2), \tag{3.18}$$

and also,

$$\begin{aligned} D([\mathcal{T}_2\Phi](t),[\mathcal{T}_2\Psi](t)) &\le \int_{a+c}^t D(f(s,\Phi(s)),f(s,\Psi(s)))ds \\ &+ \int_{a+c}^t\int_{a+c}^s D(k(s,r,\Phi(r)),k(s,r,\Psi(r)))drds \\ &\le l_1 c max\{H(\phi_1,\psi_1),H(\phi_2,\psi_2)\} \\ &+ l_2\frac{c^2}{2} max\{H(\phi_1,\psi_1),H(\phi_2,\psi_2)\}. \end{aligned}$$

Then we have

$$\mathcal{H}(\mathcal{T}\Phi,\mathcal{T}\Psi) \le L\mathcal{H}(\Phi,\Psi), \tag{3.19}$$

where $L = max\{c, l_1c + l_2\frac{c^2}{2}\} < 1$. All the conditions in Theorem 2.13 are satisfied, therefore the mapping $\mathcal{T}$ has a fixed point $\mathbb{V} \in C([a+c,a+2c],\bar{B}(u_{01})) \times C([a+c,a+2c],\bar{B}(u_{02}))$ and $\lim_{n\to\infty}\mathcal{T}^n(\underline{U}) = \mathbb{V}$.
If we suppose $W \in C([a+c,a+2c],\bar{B}(u_{01})) \times C([a+c,a+2c],\bar{B}(u_{02}))$ is another fixed point of $\mathcal{T}$ such that is comparable to $\underline{U}$ on $[a+c,a+2c]$, then it is clear that $\mathcal{H}(\mathbb{V},W) = 0$.
Obviously $U$ as defined

$$U = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{cases} \mathbb{U} & t \in [a,a+c] \\ \mathbb{V} & t \in [a+c,a+2c] \end{cases} \tag{3.20}$$

is a fixed point of $\mathcal{B}$ defined by (3.21) on $[a, a + 2c]$. By Lemma 2.3, $u_1, u_2$ are (ii)-differentiable on $[a, a + 2c]$. In the same trend we can make a fixed point of $\mathcal{B}$ defined by (3.21) on $[a, a + Nc] = [a, a + c^*]$. Let $U = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in C([a, a + c^*], \bar{B}(u_{01})) \times C([a, a + c^*], \bar{B}(u_{02}))$ is a fixed point of $\mathcal{B}$. Therefore $U$ is a solution of integral equation $(F_2)$. By Remark 3.5 and System (3.2), we can conclude $u_1$ is a (ii)-solution of Problem (3.1). Now suppose $W \in C(J, \mathcal{K}) \times C(J, \mathcal{K})$ is another fixed point of $\mathcal{A}$ such that is comparable to $\underline{U}$ on $[a, a + c^*]$. It is clear that $\mathcal{H}(U, W) = 0$. $\square$

The following theorem gives (ii)-solution to Problem (3.1) considering partial ordering $\leq_2$, along with nonincreasing property of $f, k$ which is different from assumptions of Theorem 3.11.

**Theorem 3.12.** *Consider Problem (3.1) with $f$ and $k$ continuous and suppose $f$, $k$ are nonincreasing in all their arguments except for the first. Let $u_{01}, u_{0,2} \in \mathcal{K} \setminus \mathbb{R}$ and $len(f(t, x, y))$, $len(k(t, s, x, y))$ for all $x \in \bar{B}(u_{01}), y \in \bar{B}(u_{02})$, $\forall t, s \in [a, c^*]$ are bounded with bounds of $M_1, M_2$ respectively, where*

$$c^* = min\{\frac{len(u_{01})}{len(u_{02})}, \frac{2len(u_{02})}{2M_1 + M_2(b - a)}, b - a\}.$$

*Let exist $l_1, l_2 > 0$ such that*

$$\begin{aligned} D(f(t, x_1, x_2), f(t, y_1, y_2)) &\leq l_1 max\{D(x_1, y_1), D(x_2, y_2)\}, \quad \forall t \in [a, a + c^*], \\ D(k(t, s, x_1, x_2), k(t, s, y_1, y_2)) &\leq l_2 max\{D(x_1, y_1), D(x_2, y_2)\}, \quad \forall t \in [a, a + c^*], \end{aligned}$$

*for $x_1 \geq_2 y_1$ and $x_2 \geq_2 y_2$. Then the existence of a lower solution $\underline{U}$ (or an upper solution $\bar{U}$) for Problem (3.1) provides the existence of a fixed point for $\mathcal{B}$ like $U$, and consequently (ii)-solution to Problem (3.1) on $[a, a + c^*]$. Also, $\lim_{n\to\infty} \mathcal{B}^n(\underline{U}) = U$ (or $\lim_{n\to\infty} \mathcal{B}^n(\bar{U}) = U$). Moreover, if $W \in C([a, a + c^*], \bar{B}(u_{01})) \times C([a, a + c^*], \bar{B}(u_{02}))$ is another fixed point of $\mathcal{B}$ such that is comparable to $\underline{U}$ in the partial ordering $\preceq_1$, then $U = W$.*

*Proof.* Since $f$, $k$ are nonincreasing, by Lemma 2.9, the mapping $\mathcal{B}$ is nondecreasing. Continuing the similar trend with the proof of Theorem 3.11, we prove our results. $\square$

### 3.3. Existence of (ii,i)-solution

Now we define the nonlinear mappings $\mathcal{L}$, which plays a main role in our discussion, as following

$$[\mathcal{L}\Phi](t) = \begin{bmatrix} [\mathcal{L}_1\Phi](t) \\ [\mathcal{L}_2\Phi](t) \end{bmatrix} \tag{3.21}$$

$$= \begin{cases} \begin{bmatrix} u_{01} \ominus (-1) \cdot \int_a^t \phi_2(s)ds \\ u_{02} \ominus (-1) \cdot (\int_a^t f(s, \Phi(s)) + \int_a^s k(s, r, \Phi(r))drds) \end{bmatrix}, t \in [a, a + c^*] \\ \begin{bmatrix} \phi_1(a + c^*) + \int_{a+c^*}^t \phi_2(s)ds \\ \phi_2(a + c^*) + \int_{a+c^*}^t f(s, \Phi(s)) + \int_a^s k(s, r, \Phi(r))drds \end{bmatrix}, t \in [a + c^*, b] \end{cases}$$

where $\Phi(t) = \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix}$. In general the $\mathcal{L} : C(J,\mathcal{K}) \times C(J,\mathcal{K}) \longrightarrow C(J,\mathcal{K}) \times C(J,\mathcal{K})$ is not well defined, but under the conditions of the next theorem, it will be well-defined.

**Definition 3.13.** *Let* $\underline{U} = \begin{bmatrix} \underline{u}_1 \\ \underline{u}_2 \end{bmatrix}, \bar{U} = \begin{bmatrix} \overline{u}_1 \\ \overline{u}_2 \end{bmatrix} \in C(J, \bar{B}(u_{01})) \times C(J, \bar{B}(u_{02}))$, *we say that*
*(a)* $\underline{U}$ *is a lower solution for the problem (3.4) if*

$$\underline{U}(t) \preceq_j [\mathcal{L}\underline{U}](t), \qquad t \in J,$$

*(b)* $\bar{U}$ *is an upper solution for the problem (3.4) if*

$$\bar{U}(t) \succeq_j [\mathcal{L}\bar{U}](t), \qquad t \in J.$$

The following theorem gives (ii,i)-solution to Problem (3.1)with considering just only partial ordering $\leq_1$.

**Theorem 3.14.** *Consider Problem (3.1) with* $f$ *and* $k$ *continuous and suppose* $f$, $k$ *are nondecreasing in all their arguments except for the first. Let* $u_{01}, u_{0,2} \in \mathcal{K} \setminus \mathbb{R}$. *And also, let* $len(f(t,x,y))$, $len(k(t,s,x,y))$ *for all* $x \in \bar{B}(u_{01}), y \in \bar{B}(u_{02})$, $\forall t, s \in [a, a + c^*]$ *be bounded with bounds of* $M_1, M_2$ *respectively, where*

$$c^* = min\{\frac{len(u_{01})}{len(u_{02})}, \frac{2len(u_{02})}{2M_1 + M_2(b-a)}, b-a\}.$$

*Let exist* $l_1, l_2 > 0$ *such that*

$$D(f(t,x_1,x_2), f(t,y_1,y_2)) \leq l_1 max\{D(x_1,y_1), D(x_2,y_2)\}, \qquad \forall t \in [a,b],$$

$$D(k(t,s,x_1,x_2), k(t,s,y_1,y_2)) \leq l_2 max\{D(x_1,y_1), D(x_2,y_2)\}, \qquad \forall t \in [a,b],$$

*for* $x_1 \geq_1 y_1$ *and* $x_2 \geq_1 y_2$. *Then the existence of a lower solution* $\underline{U}$ *(or an upper solution* $\bar{U}$*) for Problem (3.1) provides the existence of a fixed point for* $\mathcal{L}$ *like* $U$, *and consequently (ii,i)-solution to Problem (3.1) on* $[a,b]$. *Also,* $\lim_{n\to\infty} \mathcal{L}^n(\underline{U}) = U$ *(or* $\lim_{n\to\infty} \mathcal{B}^n(\bar{U}) = U$*). Moreover, if* $W \in C([a, a + c^*], \bar{B}(u_{01})) \times C([a, a + c^*], \bar{B}(u_{02}))$ *is another fixed point of* $\mathcal{B}$ *such that is comparable to* $\underline{U}$ *in the partial ordering* $\preceq_1$, *then* $U = W$.

*Proof.* By Theorem 3.11, the mapping $\mathcal{L}$ is well-defined and there exists a fixed point for the mapping $\mathcal{L}$ like $\mathbb{U} = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} \in C([a, a + c^*], \bar{B}(u_{01})) \times C([a, a + c^*], \bar{B}(u_{02}))$ and consequently (ii)-solution for Problem (3.1) on $[a, a + c^*]$. Now by considering $\mathbb{U}$ as a fixed point for $\mathcal{B}$ on the interval $[a, a + c^*]$, we define the other mapping $\mathcal{T} : C([a+c^*, b], \mathcal{K}) \times C([a+c^*, b], \mathcal{K}) \to C([a+c^*, b], \mathcal{K}) \times C([a+c^*, b], \mathcal{K})$ as follows:

$$[\mathcal{T}\Phi](t) =$$
$$\begin{bmatrix} \mathbf{u}_1(a+c^*) + \int_{a+c^*}^t \phi_2(s)ds \\ \mathbf{u}_2(a+c^*) + (\int_{a+c^*}^t \int_a^{a+c^*} k(s,r,\mathbb{U}(r))dr + (f(s,\Phi(s)) + \int_{a+c^*}^s k(s,r,\Phi(r))drds) \end{bmatrix}.$$

Now we will show that $\underline{U}(t) \preceq_1 [\mathcal{T}\underline{U}](t)$ (or $\overline{U}(t) \succeq_1 [\mathcal{T}\overline{U}](t)$) for $t \in [a+c^*, b]$. Due the fact that $\underline{U}$ is a lower solution of Problem (3.1) for $t \in [a,b]$ and $\underline{U} \preceq_1 \mathbb{U}$ for $t \in [a, a+c^*]$, we have for $t \in [a+c^*, b]$

$$\underline{U} = \begin{bmatrix} \underline{u_1}(t) \\ \underline{u_2}(t) \end{bmatrix} \preceq_1 \begin{bmatrix} [\mathcal{L}_1\underline{U}](t) \\ [\mathcal{L}_2\underline{U}](t) \end{bmatrix} = \begin{bmatrix} \underline{u_1}(a+c^*) + \int_{a+c^*}^t \underline{u_2}(s)ds \\ \underline{u_2}(a+c^*) + \int_{a+c^*}^t f(s,\underline{U}(s)) + \int_a^s k(s,r,\underline{U}(r))drds \end{bmatrix}$$
$$\preceq_1 \begin{bmatrix} \mathbf{u}_1(a+c^*) + \int_{a+c^*}^t \underline{u_2}(s)ds \\ \mathbf{u}_2(a+c^*) + \int_{a+c^*}^t f(s,\underline{U}(s)) + \int_a^s k(s,r,\underline{U}(r))drds \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{u}_1(a+c^*) + \int_{a+c^*}^t \underline{u_2}(s)ds \\ \mathbf{u}_2(a+c^*) + \int_{a+c^*}^t \int_a^{a+c^*} k(s,r,\underline{U}(r))drds + \int_{a+c^*}^t f(s,\underline{U}(s)) + \int_{a+c^*}^s k(s,r,\underline{U}(r))drds \end{bmatrix}$$
$$\preceq_1 \begin{bmatrix} \mathbf{u}_1(a+c) + \int_{a+c}^t \underline{u_2}(s)ds \\ \mathbf{u}_2(a+c^*) + \int_{a+c^*}^t \int_a^{a+c^*} k(s,r,\mathbb{U}(r))drds + \int_{a+c^*}^t f(s,\underline{U}(s)) + \int_{a+c^*}^s k(s,r,\underline{U}(r))drds \end{bmatrix}$$
$$= [\mathcal{T}\underline{U}](t).$$

By Theorem 3.7, the mapping $\mathcal{T}$ has a fixed point $\mathbb{V} \in C([a+c^*, b], \mathcal{K}) \times C([a+c^*, b], \mathcal{K})$. Obviously $U$ as defined

$$U = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{cases} \mathbb{U} & t \in [a, a+c^*] \\ \mathbb{V} & t \in [a+c^*, b] \end{cases} \tag{3.22}$$

is a fixed point of $\mathcal{L}$ on $[a,b]$. □

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