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Characterization of Weighted Function Spaces In Terms of Wavelet Transforms

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ABSTRACT: In this paper, we have characterized a weighted function space $B^{p,q}_{\omega,\psi}$, $1 \leq p, q < \infty$ in terms of wavelet transform and shown that the norms on spaces $B^{p,q}_{\omega,\psi}$ and $\bigwedge^{p,q}_{\omega}$ (the space defined in terms of differences $\triangle_x$) are equivalent.

Key Words: Wavelets, Wavelet Transforms, Weight Functions.

Contents

1. Introduction

In this section, we recall some notations and basic definitions, also mention certain weight functions and results given in [2], which we will invoke in the analysis. In Section 2, we define the spaces $\bigwedge^{p,q}_{\omega}$ in terms of differences $\triangle_x$, and $B^{p,q}_{\omega,\psi}$, $1 \leq p, q < \infty$ by means of wavelet transforms. Furthermore, by using the techniques of Ansorena and Blasco [2], we show that the norms on these spaces are equivalent.

Notations: Throughout the paper, $\mathbb{R}^+$ denote the set of positive real numbers, $\mathscr{S}$ denote the Schwartz class of test functions on $\mathbb{R}^n$, $\mathscr{S}'$ the space of tempered distributions, $\mathscr{S}_0$ the set of functions in $\mathscr{S}$ with mean zero and $\mathscr{S}_0'$ its topological dual.

Definition 1.1. *The Fourier transform of a function f is denoted by $\hat{f}$ and defined as*

$$\mathscr{F}(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle \xi, x\rangle} f(x)\, dx, \tag{1.1}$$

provided the integral exists.

Definition 1.2. *The wavelet transform W_ψ of a function f with respect to a wavelet ψ is defined as*

$$(W_\psi f)(a,b) = \langle f, \psi_{a,b}\rangle = \frac{1}{a^n}\int_{\mathbb{R}^n} f(x)\,\overline{\psi\left(\frac{x-b}{a}\right)}dx = (f * h_{a,0})(b), \tag{1.2}$$

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where $a \in \mathbb{R}^+$, $b \in \mathbb{R}^n$, $\psi_{a,b}(x) = \frac{1}{a^n}\psi\left(\frac{x-b}{a}\right)$ *and* $h(x) = \overline{\psi}(-x)$, *provided the integral exists.*

Definition 1.3. *A non-negative bounded measurable function* $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ *is referred to as a weight function or simply a weight.*

Definition 1.4. *A weight function* ω *is said to satisfy Dini's condition if there exists a constant* $C > 0$ *such that*

$$\int_0^s \frac{\omega(t)}{t} dt \leq C\,\omega(s) \quad a.e.\ \ s > 0.$$

Definition 1.5. *Let* $\epsilon \geq 0$, $\delta \geq 0$ *and* ω *be a weight function. Then* ω *is said to be a* d_ϵ*-weight if there exists* $C \geq 0$ *such that*

$$\int_0^s t^\epsilon \omega(t) \frac{dt}{t} \leq \ C\ s^\epsilon \omega(s) \quad a.e.\ \ s > 0 \tag{1.3}$$

and ω *is called a* b_δ*-weight if there exists* $C > 0$ *such that*

$$\int_s^\infty \frac{\omega(t)}{t^\delta} \frac{dt}{t} \ \leq C\ \frac{\omega(s)}{s^\delta} \quad a.e.\ \ s > 0. \tag{1.4}$$

Remark 1.6. *If* (d_ϵ) *denotes the class of* d_ϵ*-weights and* (b_δ) *denotes the class of* b_δ*-weights then we write* $\mathscr{W}_{\epsilon,\delta} = (d_\epsilon) \cap (b_\delta)$.

Some important properties:

1. For any $\epsilon' > \epsilon$, $\omega \in (d_\epsilon) \implies \omega \in (d_{\epsilon'})$.
2. For any $\delta' > \delta$, $\omega \in (b_\epsilon) \implies \omega \in (b_{\delta'})$.
3. Let $\overline{\omega(t)} = \omega(t^{-1})$, then $\omega \in (b_\epsilon)$ if and only if $\overline{\omega} \in (d_\epsilon)$.
4. If $\omega \in \mathscr{W}_{\epsilon,\delta}$, then $\omega(t) \geq C \min\left(t^{-\epsilon}, t^\delta\right)$.

Definition 1.7 (Radial function)**.** *A function defined on Euclidean space* $\mathbb{R}^n$ *whose values at each point depends only on the distance between that points and the origin is called a radial function. For example a radial function* Φ *in two dimensional space has the form* $\Phi(x, y) = \phi(r)$, $\ r = \sqrt{x^2 + y^2}$ *where* ϕ *is a function of a single non-negative real variable.*

Definition 1.8. *In this paper,* $\mathscr{A}$ *and* $\mathscr{A}_1$ *denote the space of the functions defined by*

$$\mathscr{A} = \left(\psi \in \mathscr{S}_0 \colon \int_0^\infty \left(\hat{\psi}(t\xi)\right)^2 \frac{dt}{t} = 1 \text{ for } \xi \ \in \mathbb{R}^n \setminus \{0\}\right),$$

$$\mathscr{A}_1 = \Bigg(\psi \ \in \ \mathscr{A} \colon \psi \text{ radial and real, and supp } \psi \subseteq \{|x| \leq 1\}, \int_{\mathbb{R}^n} x_i \psi(x) dx = 0,\ i = 1, 2, \cdots, n\Bigg).$$

Definition 1.9 (Calderón Reproducing Formula [2]). *Let $\psi \in \mathscr{A}$ and $f \in \mathscr{S}$. For $\xi \in \mathbb{R}^n \setminus \{0\}$, the Fourier transform of f is given by*

$$\hat{f}(\xi) = \int_0^\infty [(\psi_t * \psi_t * f)(\cdot)]\hat{}\,(\xi)\tfrac{dt}{t}; \text{ where } \psi_t(x) = \frac{1}{t^n}\psi\left(\frac{x}{t}\right) \text{ and } x \in \mathbb{R}^n.$$

Furthermore, $f_{\epsilon,\delta}(x) = \int_\epsilon^\delta \psi_t * \psi_t * f(x)\frac{dt}{t}$ converges to ψ in $\mathscr{S}$ as $\epsilon \to 0$ and $\delta \to \infty$.

Lemma 1.10. *[2, p. 8] Let $f \in L^1\left(\mathbb{R}^n, \frac{dx}{(1+|x|)^{n+1}}\right)$ and $\psi \in \mathscr{A}$. For $0 < \epsilon < \delta$ define*

$$f_{\epsilon,\delta}(x) = \int_\epsilon^\delta (\psi_t * \psi_t * f)(x)\frac{dt}{t}.$$

Then $f_{\epsilon,\delta}(x)$ converges to f in $\mathscr{S}_0'$ as $\epsilon \to 0$ and $\delta \to \infty$.

2. Characterization of Function Spaces by Using the Wavelet Transform

Definition 2.1 (The space $\bigwedge_\omega^{p,q}$). *Given a weight function ω and $1 \le p, q \le \infty$, the space $\bigwedge_\omega^{p,q}$ denotes the space of measurable functions $f\colon \mathbb{R}^n \to \mathbb{C}$ such that*

$$\| f \|_{\bigwedge_\omega^{p,q}} = \left(\int_{\mathbb{R}^n} \frac{\| \triangle_x f \|_p^q}{(\omega(|x|))^q}\frac{dx}{|x|^n}\right)^{\frac{1}{q}} < \infty, \text{ for } 1 \le q < \infty,$$

and

$$\| f \|_{\bigwedge_\omega^{p,\infty}} = \inf\{C > 0\colon \| \triangle_x f \|_p \le C\,\omega(|x|) \text{ a.e } x \,\epsilon\, \mathbb{R}^n\} < \infty, \text{ for } q = \infty,$$

where $\| \triangle_x f \|_p = \left(\int_{\mathbb{R}^n} | \triangle_x f(y)|^p dy\right)^{1/p}$ and $\triangle_x f(y) = f(x+y) - f(y)$.

Now, we define a new function space $B_{\omega,\psi}^{p,q}$ by means of the wavelet transform.

Definition 2.2 (The space $B_{\omega,\psi}^{p,q}$). *For $1 \le p, q \le \infty$, $\psi \,\epsilon\, \mathscr{S}_0$ and a weight ω, the space $B_{\omega,\psi}^{p,q}$ denotes the space of functions $f\colon \mathbb{R}^n \to \mathbb{R}$ belonging $L^1\left(\mathbb{R}^n, \frac{dx}{(1+|x|)^{n+1}}\right)$ such that*

$$\| f \|_{B_{\omega,\psi}^{p,q}} = \left(\int_{\mathbb{R}^+} \frac{\| (W_{P\psi}f)(a,\cdot) \|_p^q}{(\omega(a))^q}\frac{da}{a}\right)^{\frac{1}{q}} < \infty, \text{ for } 1 \le q < \infty, \tag{2.1}$$

and

$$\| f \|_{B_{\omega,\psi}^{p,\infty}} = \inf\{C > 0\colon \| (W_{P\psi}f)(a,\cdot) \|_p \le C\omega(a) \text{ a.e } a > 0\} < \infty, \tag{2.2}$$

for $q = \infty$, where P is the parity operator defined by $P\psi(x) = \psi(-x)$ for all $x \in \mathbb{R}^n$.

Theorem 2.3. *Let $1 \leq p \leq \infty, \varrho \geq 0$ and $\psi \in \mathscr{A}$. Then, for any $f \in L^1\left(\mathbb{R}^n, \frac{dx}{(1+|x|)^{n+1}}\right)$, we have*

$$\| W_{P\psi} f(a,\cdot) \|_p \leq \; C \int_{\mathbb{R}^n} \min\left(\left(\frac{|\, x\,|}{a}\right)^n, \left(\frac{a}{|\, x\,|}\right)^{\varrho}\right) \| \triangle_x f \|_p \frac{dx}{|x|^n}, \tag{2.3}$$

and

$$\| \triangle_x f \|_p \leq C \; \int_0^\infty \min\left(1, \frac{|\, x\,|}{a}\right) \| W_{P\psi} f(a,\cdot) \|_p \frac{da}{a}, \tag{2.4}$$

where $C > 0$, is a constant.

Proof. Since ψ is a wavelet, therefore $\int_{\mathbb{R}^n} \psi(x)dx = 0$ and hence the wavelet transform of f with respect to $P\psi$ may be written as

$$\begin{aligned}
(W_{P\psi} f)(a,b) &= \frac{1}{a^n} \int_{\mathbb{R}^n} f(x)\, \overline{P\psi\left(\frac{x-b}{a}\right)} dx \\
&= \frac{1}{a^n} \int_{\mathbb{R}^n} f(x)\, \overline{\psi\left(\frac{b-x}{a}\right)} dx \quad (\text{Since } P\psi(x) = \psi(-x)) \\
&= \frac{1}{a^n} \int_{\mathbb{R}^n} f(y+b)\, \overline{\psi\left(\tfrac{-y}{a}\right)} dy \\
&= \frac{1}{a^n} \int_{\mathbb{R}^n} f(y+b)\, \overline{\psi\left(\tfrac{-y}{a}\right)} dy - f(b) \int_{\mathbb{R}^n} \overline{\psi(x)} dx \\
&= \frac{1}{a^n} \int_{\mathbb{R}^n} f(y+b)\, \overline{\psi\left(\tfrac{-y}{a}\right)} dy - f(b) \int_{\mathbb{R}^n} \overline{\psi\left(\tfrac{-y}{a}\right)} \frac{dy}{a^n} \\
&= \frac{1}{a^n} \int_{\mathbb{R}^n} [f(y+b) - f(b)]\, \overline{\psi\left(\tfrac{-y}{a}\right)} dy,
\end{aligned}$$

that is,

$$(W_{P\psi} f)(a,b) = \frac{1}{a^n} \int_{\mathbb{R}^n} \triangle_y f(b)\, \overline{\psi\left(\tfrac{-y}{a}\right)}\, dy. \tag{2.5}$$

Using L^p norm and Minkowski's inequality [7, p-41], we get

$$\begin{aligned}
\| (W_{P\psi} f)(a,\cdot) \|_p &= \left(\int_{\mathbb{R}^n} \left| \frac{1}{a^n} \int_{\mathbb{R}^n} \triangle_y f(b)\, \overline{\psi\left(\frac{-y}{a}\right)} dy \right|^p db \right)^{\frac{1}{p}} \\
&\leq \int_{\mathbb{R}^n} \frac{1}{a^n} \left| \psi\left(\frac{-y}{a}\right) \right| \left(\int_{\mathbb{R}^n} |(\triangle_y f)(b)|^p \, db \right)^{\frac{1}{p}} dy \\
&= \int_{\mathbb{R}^n} \frac{1}{a^n} \left| \psi\left(\frac{-y}{a}\right) \right| \| \triangle_y f \|_p \, dy,
\end{aligned}$$

and hence we get the following inequality

$$\| (W_{P\psi} f)(a,\cdot) \|_p \leq \int_{\mathbb{R}^n} \frac{|y|^n}{a^n} \left| \psi\left(\frac{-y}{a}\right) \right| \| \triangle_y f \|_p \frac{dy}{|y|^n}. \tag{2.6}$$

Suppose ψ satisfies the following estimates

$$|\psi(y)| \leq \begin{cases} \frac{C}{|y|^{n+\varrho}}, & \text{if } |y| \geq 1 \\ C, & \text{if } |y| \leq 1. \end{cases} \tag{2.7}$$

Then by using (2.7) in (2.6), we get

$$\| (W_{P\psi} f)(a,\cdot) \|_p \leq C \int_{\mathbb{R}^n} \min\left(\frac{|y|^n}{a^n}, \frac{a^\varrho}{|y|^\varrho} \right) \| \triangle_y f \|_p \frac{dx}{|y|^n}.$$

Now we prove the second part. For $0 < \epsilon < \delta$, we have

$$\triangle_x f_{\epsilon,\delta}(y) = \int_\epsilon^\delta (\triangle_{-x}\psi_a) * \psi_a * f(y) \frac{da}{a}.$$

Using Minkowski's inequality [7, p-41] we get the following estimate

$$\begin{aligned} \| \triangle_x f_{\epsilon,\delta} \|_p &= \left(\int_{\mathbb{R}^n} | \triangle_x f_{\epsilon,\delta}|^p dy \right)^{\frac{1}{p}} \\ &= \left(\int_{\mathbb{R}^n} \left| \int_\epsilon^\delta (\triangle_{-x}\psi_a) * \psi_a * f(y) \frac{da}{a} \right|^p dy \right)^{\frac{1}{p}} \\ &\leq \int_\epsilon^\delta \left(\int_{\mathbb{R}^n} |(\triangle_{-x}\psi_a) * \psi_a * f(y)|^p dy \right)^{\frac{1}{p}} \frac{da}{a} \\ &= \int_\epsilon^\delta \left(\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} (\triangle_{-x}\psi_a)(x)(\psi_a * f)(y-x) dx \right|^p dy \right)^{\frac{1}{p}} \frac{da}{a} \\ &\leq \int_\epsilon^\delta \left(\int_{\mathbb{R}^n} |(\triangle_{-x}\psi_a)(x)| \left(\int_{\mathbb{R}^n} |(\psi_a * f)(y-x)|^p dy \right)^{\frac{1}{p}} dx \right) \frac{da}{a} \\ &= \int_\epsilon^\delta \| \triangle_{-x}\psi_a \|_1 \| \psi_a * f \|_p \frac{da}{a} \\ &= \int_\epsilon^\delta \| \triangle_{-x}\psi_a \|_1 \| (W_{P\psi} f)(a,\cdot) \|_p \frac{da}{a}. \end{aligned} \tag{2.8}$$

Now

$$\begin{aligned}\| \triangle_y \psi \|_1 &= \int_{\mathbb{R}^n} | \triangle_y \psi(x)| dx \\ &= \int_{\mathbb{R}^n} |\psi(x+y) - \psi(x)| dx \\ &\leq \int_{\mathbb{R}^n} \left(|\psi(x+y)| + |\psi(x)|\right) dx \\ &= \int_{\mathbb{R}^n} |\psi(x+y)| dx + \int_{\mathbb{R}^n} |\psi(x)| dx \\ &= \int_{\mathbb{R}^n} |\psi(x)| dx + \int_{\mathbb{R}^n} |\psi(x)| dx \\ &= 2 \| \psi \|_1; \text{ if } |y| \geq 1,\end{aligned}$$

and

$$\begin{aligned}\| \triangle_y \psi \|_1 &= \int_{\mathbb{R}^n} |\triangle_y \psi(x)| \, dx \\ &= |y| \int_{\mathbb{R}^n} \left| \frac{\psi(x+y) - \psi(x)}{y} \right| dx \\ &\leq |y| \int_{\mathbb{R}^n} \max_{|z-u|<1} | \bigtriangledown \psi(z)| du; \text{ if } |y| \leq 1,\end{aligned}$$

where $\bigtriangledown$ denotes the gradient $\sum_{j=1}^{n} \widehat{e}_j \left(\frac{\partial}{\partial x_j}\right)$, where $\widehat{e}_j$ is the unit vectors. Hence

$$\begin{aligned}\| \triangle_{-x} \psi_a \|_1 &= \int_{\mathbb{R}^n} | \triangle_{-x} \psi_a(y)| dy \\ &= \int_{\mathbb{R}^n} \left| \frac{1}{a^n} \left(\psi\left(\frac{y-x}{a}\right) - \psi\left(\frac{y}{a}\right)\right)\right| dy \\ &= \int_{\mathbb{R}^n} \left| \left(\psi\left(z - \frac{x}{a}\right) - \psi(z)\right)\right| dz \\ &= \int_{\mathbb{R}^n} \left| \triangle_{\frac{-x}{a}} \psi(z)\right| dz \\ &= \| \triangle_{\frac{-x}{a}} \psi \|_1 \\ &\leq C \; \min\left(1, \frac{|x|}{a}\right),\end{aligned}$$

where $C = \max\left(2 \| \psi \|_1, \int_{\mathbb{R}^n} \max_{|z-u|} | \bigtriangledown \psi(z)| du\right)$. Hence from (2.8), we have

$$\| \triangle_x f \|_p \leq C \int_0^\infty \min\left(1, \frac{|x|}{a}\right) \| (W_{P\psi} f)(a, \cdot) \|_p \frac{da}{a}. \tag{2.9}$$

□

Lemma 2.4. *[2, pp. 11-12] Let $1 \leq p \leq \infty$ and f be a measurable function. If $\| \triangle_x f \|_p \in L^1\left(\mathbb{R}^n, \frac{dx}{(1+|x|)^{n+1}}\right)$ then $f \in L^1\left(\mathbb{R}^n, \frac{dx}{(1+|x|)^{n+1}}\right)$.*

Theorem 2.5. *Let $1 \leq p \leq \infty$, $\psi \in \mathscr{A}$ and $\omega \in \mathscr{W}_{0,1}$. Then $\bigwedge_\omega^{p,\infty} = B_{\omega,\psi}^{p,\infty}$ (with equivalent semi norms).*

Proof. Suppose $f \in \bigwedge_\omega^{p,\infty}$ then

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{\| \triangle_x f \|}{(1+|x|)^{n+1}} dx &\leq C \int_{\mathbb{R}^n} \frac{\omega(|x|)}{(1+|x|)^{n+1}} dx \\ &\leq C' \int_0^\infty \frac{\omega(t) t^{n-1}}{(1+t)^{n+1}} dt \\ &\leq C' \left(\int_0^1 \omega(t) \frac{dt}{t} + \int_1^\infty \omega(t) \frac{dt}{t^2} \right) < \infty.\end{aligned}$$

Then from Lemma 2.4, it follows that

$$\int_{\mathbb{R}^n} \frac{|f(x)|}{(1+|x|)^{n+1}} dx < \infty.$$

Putting $\varrho = 1$ in (2.3) we get

$$\begin{aligned}||(W_{P\psi} f)(a,\cdot)||_p &= C\Bigg(\int_{|x|<a} \min\left(\left(\frac{|x|}{a}\right)^n, \frac{a}{|x|} \right) || \triangle_x f||_p \frac{dx}{|x|^n} \\ &\quad + \int_{|x|\geq a} \min\left(\left(\frac{|x|}{a}\right)^n, \frac{a}{|x|} \right) || \triangle_x f||_p \frac{dx}{|x|^n} \Bigg) \\ &\leq C \left(\int_{|x|<a} \frac{|x|^n}{a^n} || \triangle_x f||_p \frac{dx}{|x|^n} + \int_{|x|\geq a} \frac{a}{|x|} || \triangle_x f||_p \frac{dx}{|x|^n} \right) \\ &= C \left(\frac{1}{a^n} \int_{|x|<a} || \triangle_x f||_p dx + a \int_{|x|\geq a} || \triangle_x f||_p \frac{dx}{|x|^{n+1}} \right) \\ &\leq C \left(\frac{1}{a^n} \int_{|x|<a} \omega(|x|) dx + a \int_{|x|\geq a} \omega(|x|) \frac{dx}{|x|^{n+1}} \right) \\ &\leq C \left(\int_0^a \left(\frac{t}{a}\right)^n \omega(t) \frac{dt}{t} + a \int_a^\infty \omega(t) \frac{dt}{t^2} \right).\end{aligned}$$

Using (1.3) and (1.4) we get

$$||(W_{P\psi} f)(a,\cdot)||_p \leq C\omega(a).$$

Now suppose $f \in B^{p,\infty}_{w,\psi}$. Using (1.3), (1.4) and (2.2) in (2.4), we get

$$\begin{aligned}
\| \triangle_x f \|_p &\leq C \left(\int_0^{|x|} \| W_{P\psi} f(a,\cdot) \|_p \frac{da}{a} + \frac{|x|}{a} \int_{|x|}^{\infty} \| W_{P\psi} f(a,\cdot) \|_p \frac{da}{a} \right) \\
&= C \left(\int_0^{|x|} \omega(a) \frac{da}{a} + \frac{|x|}{a} \int_{|x|}^{\infty} \omega(a) \frac{da}{a} \right) \\
&= C \left(\int_0^{|x|} \frac{\omega(a)}{a} da + |x| \int_{|x|}^{\infty} \frac{\omega(a)}{a^2} da \right) \\
&\leq C \left(\omega(|x|) + |x| \frac{\omega(|x|)}{|x|} \right) \\
&= 2C\omega(|x|) \\
&= C'\omega(|x|).
\end{aligned}$$

□

Theorem 2.6. *Let $1 \leq p \leq \infty, \psi \in \mathscr{A}$ and ω such that $\mu(a) = \omega^{-1}(a^{-1}) \in \mathscr{W}_{0,1}$. Then $\bigwedge^{p,1}_{\omega} = B^{p,q}_{\omega,\psi}$.*

Proof. Let us assume that $f \in \bigwedge^{p,1}_{\omega}$. We have to prove that $\int_{\mathbb{R}^n} \frac{|f(x)|}{(1+|x|)^{n+1}} dx < \infty$. Since $\mu(a) \in \mathscr{W}_{0,1}$, therefore

$$\begin{aligned}
\mu(a) &\geq C \min(a^0, a^1) \\
\Longrightarrow \quad \frac{1}{\omega(a^{-1})} &\geq C \min(1, a) \\
\Longrightarrow \quad \frac{1}{\omega(|x|)} &\geq C \min\left(1, \frac{1}{|x|}\right) \\
\Longrightarrow \quad \frac{1}{|x|^n \omega(|x|)} &\geq \frac{C}{|x|^n} \min\left(1, \frac{1}{|x|}\right).
\end{aligned} \tag{2.10}$$

Note that

$$\begin{aligned}
\min\left(1, \frac{1}{|x|}\right) &= \begin{cases} 1, & \text{if } |x| < 1 \\ \frac{1}{|x|}, & \text{if } |x| > 1 \end{cases} \\
&\geq \begin{cases} |x|^n, & \text{if } |x| < 1 \\ \frac{1}{|x|}, & \text{if } |x| > 1. \end{cases}
\end{aligned}$$

From (2.10)

$$\frac{1}{|x|^n \omega(|x|)} \geq \frac{C}{|x|^n} \min\left(1, \frac{1}{|x|}\right) \geq \frac{C}{|x|^n} \min\left(|x|^n, \frac{1}{|x|}\right). \tag{2.11}$$

Again

$$\frac{C}{|x|^n}\min\left(|x|^n,\frac{1}{|x|}\right)=\begin{cases}C, & \text{if } |x|<1\\ \frac{C}{|x|^n\,|x|}, & \text{if } |x|>1\end{cases}$$
$$\geq\begin{cases}C, & \text{if } |x|<1\\ \frac{C}{(1+|x|)^{n+1}}, & \text{if } |x|>1,\end{cases}$$

that is,

$$\frac{C}{|x|^n}\min\left(|x|^n,\frac{1}{|x|}\right)\geq\frac{C}{(1+|x|)^{n+1}}\text{ for all } x.$$

Hence

$$\int_{\mathbb{R}^n}\frac{||\triangle_x f||_p}{(1+|x|)^{n+1}}dx\leq\int_{\mathbb{R}^n}\frac{||\triangle_x f||_p}{\omega(|x|)}\frac{dx}{|x|^n}<\infty.$$

Let us now show that $||f||_{B^{p,1}_{\omega,\psi}}\leq C||f||_{\wedge^{p,1}_{\omega}}$. With $\varrho=1$, we have from (2.3),

$$\begin{aligned}
&\int_0^\infty\frac{||W_{P\psi}f(a,\cdot)||_p}{\omega(a)}\frac{da}{a}\\
&\leq C\int_0^\infty\int_{\mathbb{R}^n}\min\left(\left(\frac{|x|}{a}\right)^n,\left(\frac{a}{|x|}\right)\right)\frac{||\triangle_x f||_p}{\omega(a)}\frac{dx}{|x|^n}\frac{da}{a}\\
&\leq C\int_{\mathbb{R}^n}||\triangle_x f||_p\left(\int_0^\infty\min\left(\left(\frac{|x|}{a}\right)^n,\left(\frac{a}{|x|}\right)\right)\mu(a^{-1})\frac{da}{a}\right)\frac{dx}{|x|^n}\\
&\leq C\int_{\mathbb{R}^n}||\triangle_x f||_p\left(\int_0^{|x|}\frac{a\mu(a^{-1})}{|x|}\frac{da}{a}+\int_{|x|}^\infty\frac{|x|^n\mu(a^{-1})}{a^n}\frac{da}{a}\right)\frac{dx}{|x|^n}\\
&\leq C\int_{\mathbb{R}^n}||\triangle_x f||_p\left(\frac{1}{|x|}\int_0^{|x|}\mu(a^{-1})\,da+\int_{|x|}^\infty\mu(a^{-1})\,\frac{da}{a}\right)\frac{dx}{|x|^n}\\
&\leq C\int_{\mathbb{R}^n}||\triangle_x f||_p\left(\frac{1}{|x|}\int_{|x|^{-1}}^\infty\mu(y)\frac{da}{y^2}+\int_0^{|x|^{-1}}\mu(y)\frac{dy}{y}\right)\frac{dx}{|x|^n}\\
&\leq C_1\int_{\mathbb{R}^n}||\triangle_x f||_p\,\mu(|x|^{-1})\frac{dx}{|x|^n}\\
&=C_1\int_{\mathbb{R}^n}\frac{||\triangle_x f||_p}{\omega(|x|)}\frac{dx}{|x|^n}.
\end{aligned}$$

Now consider $f \in B^{p,q}_{\omega,\psi}$. Then from (2.4)

$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{||\triangle_x f||_p}{\omega(|x|)} \frac{dx}{|x|^n} \\
&\leq C \int_0^\infty ||(W_{P\psi} f)(a,\cdot)||_p \left(\int_{\mathbb{R}^n} \mu(|x|^{-1}) \min\left(1, \frac{|x|}{a}\right) \frac{dx}{x^n} \right) \frac{da}{a} \\
&= C \int_0^\infty ||(W_{P\psi} f)(a,\cdot)||_p \left(\int_0^\infty \mu(s) \min\left(1, \frac{1}{sa}\right) \frac{ds}{s} \right) \frac{da}{a} \\
&= C \int_0^\infty ||(W_{P\psi} f)(a,\cdot)||_p \left(\int_0^{a^{-1}} \frac{\mu(s)}{s} ds + \frac{1}{a} \int_{a^{-1}}^\infty \frac{\mu(s)}{s^2} ds \right) \frac{da}{a} \\
&\leq C \int_0^\infty ||(W_{P\psi} f)(a,\cdot)||_p \, \mu(a^{-1}) \frac{da}{a} \\
&= C \int_0^\infty \frac{||(W_{P\psi} f)(a\cdot)||_p}{\omega(a)} \frac{da}{a} < \infty.
\end{aligned}$$

□

Lemma 2.7. *[2, pp. 8-9] Let $1 < q < \infty$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be two σ - finite measure spaces and $K\colon \Omega_1 \times \Omega_2 \to \mathbb{R}^+$ be a measurable function and define an operator T_K by*

$$T_K(f)(w_2) = \int_{\Omega_1} K(w_1, w_2) f(w_1) d\mu_1(w_1). \tag{2.12}$$

If there exist $C > 0$ and measurable functions $h_i\colon \Omega_i \to \mathbb{R}^+ (i = 1, 2)$ such that

$$\int_{\Omega_1} K(w_1, w_2) h_1^{q'}(w_1) d\mu_1(w_1) \leq C h_2^{q'}(w_2) \mu_2 \ (a.e.), \tag{2.13}$$

and

$$\int_{\Omega_2} K(w_1, w_2) h_2^q(w_2) d\mu_2(w_2) \leq C h_2^q(w_1) \mu_1 \ (a.e.), \tag{2.14}$$

then T_K is a bounded operator from $L^q(\Omega_1, \mu_1)$ into $L^q(\Omega_2, \mu_2)$.

Lemma 2.8. *[2, p. 10] Given $0 \leq \epsilon, \delta < \infty, 1 < q < \infty$, and w a weight, let us consider $R_{\epsilon,\delta}(s,t) = \frac{\omega(s)}{\omega(t)} \min\left(\left(\frac{s}{t}\right)^\epsilon, \left(\frac{t}{s}\right)^\delta\right)$. If $\omega(s) = \lambda^{\frac{1}{q'}}(s) \mu^{\frac{-1}{q}}(s^{-1})$ for some pair of weights $\lambda, \mu \in \mathscr{W}_{\epsilon,\delta}$, then there exist $C > 0$ and $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ measurable such that*

$$\int_0^\infty R_{\epsilon,\delta}(s,t) g^{q'}(s) \frac{ds}{s} \leq C g^{q'}(t) \tag{2.15}$$

and

$$\int_0^\infty R_{\epsilon,\delta}(s,t) g^q(t) \frac{dt}{t} \leq C g^q(t). \tag{2.16}$$

Theorem 2.9. *Let* $1 \leq p < \infty, 1 < q < \infty$, $\psi \in \mathscr{A}$ *and* ω *be a weight such that* $\omega(t) = \lambda^{\frac{1}{q'}}(t)\mu^{\frac{-1}{q}}(t^{-1})$, $\frac{1}{q} + \frac{1}{q'}$ *for some pair of weights* $\lambda, \mu \in \mathscr{W}_{0,1}$. *Then* $\bigwedge_{\omega}^{p,q} = B_{\omega,\psi}^{p,q}$ *(with equivalent semi norms).*

Proof. Let $f \in \bigwedge_{\omega}^{p,q}$. Let us first show that

$$\int_{\mathbb{R}^n} \frac{|f(x)|}{(1+|x|)^{n+1}} dx < \infty. \tag{2.17}$$

Let us denote $\Phi(|x|) = \omega(|x|)|x|^n/(1+|x|)^{n+1}$. Then,

$$\int_0^\infty \Phi^{q'}(t)\frac{dt}{t} = \int_0^\infty \lambda(t)\mu^{\frac{-q'}{q}}(t^{-1})\frac{t^{nq'}}{(1+t)^{q'(n+1)}}\frac{dt}{t}.$$

Since $\mu \in \mathscr{W}_{0,1} = (d_0) \cap (b_1)$, then

$$\begin{aligned}
& \mu(s) \geq C' \min(1, s) \\
\Longrightarrow\ & \frac{1}{\mu(s)} \leq \frac{1}{C'} \max\left(1, \frac{1}{s}\right) \\
\Longrightarrow\ & \mu^{-\frac{q'}{q}}(s) \leq \left(\frac{1}{C'}\right)^{\frac{q'}{q}} \max\left(1, \left(\frac{1}{s}\right)^{\frac{q'}{q}}\right) \\
\Longrightarrow\ & \mu^{-\frac{q'}{q}}\left(t^{-1}\right) \leq C \max\left(1, t^{\frac{q'}{q}}\right) \\
\Longrightarrow\ & \mu^{-\frac{q'}{q}}\left(t^{-1}\right) \leq C \max\left(1, t^{(q'-1)}\right), \text{ since } \frac{1}{q'} + \frac{1}{q} = 1.
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_0^\infty \Phi^{q'}(t)\frac{dt}{t} &\leq C\int_0^\infty \lambda(t)\max(1, t^{(q'-1)})\frac{t^{nq'}}{(1+t)^{q'(n+1)}}\frac{dt}{t} \\
&\leq C\left(\int_0^1 \lambda(t)\frac{dt}{t} + \int_1^\infty \frac{\lambda(t)}{t}\frac{dt}{t}\right).
\end{aligned}$$

Since $\lambda \in \mathscr{W}_{0,1} = (d_0) \cap (b_1)$ then right hand side of above inequality is bounded a.e., and hence

$$\int_0^\infty \Phi^{q'}(t)\frac{dt}{t} < \infty.$$

Now, using Holder's inequality, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^n} \frac{||\triangle_x f||_p}{(1+|x|)^{n+1}} dx\right| &= \left|\int_{\mathbb{R}^n} \frac{||\triangle_x f||_p}{\omega(|x|)} \Phi(|x|) \frac{dx}{|x|^n}\right| \\ &\le \left(\int_{\mathbb{R}^n} \frac{||\triangle_x f||_p^q}{\omega(|x|)^q} \frac{dx}{|x|^n}\right)^{\frac{1}{q}} \times \left(\int_{\mathbb{R}^n} |\Phi(|x|)|^{q'} \frac{dx}{|x|^n}\right)^{\frac{1}{q'}} \\ &\le C \left(\int_{\mathbb{R}^n} |\Phi(|x|)|^{q'} \frac{dx}{|x|^n}\right)^{\frac{1}{q'}} \\ &\le C \int_0^\infty |\Phi(t)|^{q'} \frac{t^{n-1}}{t^n} dt \\ &= C \int_0^\infty |\Phi(t)|^{q'} \frac{dt}{t} < \infty,\end{aligned}$$

and hence by lemma (2.4) the result (2.17) is proved.

Let us now prove that

$$||f||_{B^{p,q}_{\omega,\psi}} \le C||f||_{\Lambda^{p,q}_{\omega}}.$$

From (2.3) with $\varrho = 1$, it follows that

$$\begin{aligned}\frac{||W_\psi f(a,\cdot)||_p}{\omega(a)} &\le C \int_{\mathbb{R}^n} K(x,a) \frac{||\triangle_x f||_p}{\omega(|x|)} \frac{dx}{|x|^n}, \\ &= CT_K\left(\frac{||\triangle_x f||_p}{\omega(|x|)}\right),\end{aligned}$$

where

$$K(x,a) = \frac{\omega(|x|)}{\omega(a)} \min\left(1, \frac{a}{|x|}\right).$$

Consider two measurable spaces as

$$(\Omega_1, \Sigma_1, \mu_1) = \left(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n), \frac{dx}{|x|^n}\right)$$

and

$$(\Omega_2, \Sigma_2, \mu_2) = \left((0,\infty), \mathscr{B}(0,\infty), \frac{dx}{|x|^n}\right).$$

Since $K(x,a) = R_{0,1}(|x|, a)$, defined in Lemma 2.8. By applying Lemma 2.8 with $\epsilon = 0$ and $\delta = 1$ we can find a measurable function g that satisfies the conditions (2.15) and (2.16). If we take $h_1(x) = g(|x|)$ and $h_2(a) = g(a)$ and using polar coordinates, (2.15) and (2.16) gives (2.13) and (2.14) in lemma(2.7). Hence T_k define in (2.12) is bounded operator from $L^q\left(\mathbb{R}^n, \frac{dx}{|x|^n}\right)$ into $L^q\left((0,\infty), \frac{da}{a}\right)$. Therefore

$$\begin{aligned} ||f||_{B^{p,q}_{\omega,\psi}} &\le C \left|\left| T_k \left(\frac{||\triangle_x f||_p}{\omega(|x|)} \right) \right|\right|_{L^q((0,\infty),\frac{dt}{t})} \\ &\le C \left|\left| \left(\frac{||\triangle_x f||_p}{\omega(|x|)} \right) \right|\right|_{L^q(\mathbb{R}^n,\frac{dx}{|x|^n})} \\ &= C||f||_{\wedge^{p,q}_{\omega}}. \end{aligned}$$

Now, let us prove that $||f||_{\wedge^{p,q}_{\omega}} \le ||f||_{B^{p,q}_{\omega,\psi}}$.

Suppose $f \in B^{p,q}_{\omega,\psi}$. Then from (2.4) we obtain

$$\frac{||\triangle_x f||_p}{\omega(|x|)} \le C \int_0^\infty R(x,a) \frac{||W_{p\psi} f(a,\cdot)||_p}{\omega(a)} \frac{da}{a},$$

where

$$R(x,a) = \frac{\omega(a)}{\omega(|x|)} \min\left(1, \frac{|x|}{a}\right).$$

Now take

$$(\Omega_1, \Sigma_1, \mu_1) = \left((0,\infty), \mathscr{B}((0,\infty)), \frac{dx}{|x|^n}\right)$$

and

$$(\Omega_2, \Sigma_2, \mu_2) = \left(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n), \frac{dx}{|x|^n}\right).$$

Using lemma (2.8) and (2.7) we get the boundedness of T_k from $L^q\left((0,\infty), \frac{da}{a}\right)$ into $L^q\left(\mathbb{R}^n, \frac{dx}{|x|^n}\right)$. Therefore

$$\begin{aligned} ||f||_{\wedge^{p,q}_{w}} &\le C \left|\left| T_R \left(\frac{||W_{p\psi} f(a,\cdot)||_p}{\omega(a)} \right) \right|\right|_{L^q\left(\frac{dx}{|x|^n}\right)} \\ &\le C \left|\left| \frac{||W_{p\psi} f(a,\cdot)||_p}{\omega(a)} \right|\right|_{L^q\left(\frac{da}{a}\right)} \\ &\le C||f||_{B^{p,q}_{w,\psi}}. \end{aligned}$$

□

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