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Some Results on the Projective Cone Normed Tensor Product Spaces Over Banach Algebras

Dipankar Das, Nilakshi Goswami and Vishnu Narayan Mishra[1]

ABSTRACT: For two real Banach algebras $\mathbb{A}_1$ and $\mathbb{A}_2$, let K_p be the projective cone in $\mathbb{A}_1 \otimes_\gamma \mathbb{A}_2$. Using this we define a cone norm on the algebraic tensor product of two vector spaces over the Banach algebra $\mathbb{A}_1 \otimes_\gamma \mathbb{A}_2$ and discuss some properties. We derive some fixed point theorems in this projective cone normed tensor product space over Banach algebra with a suitable example. For two self mappings S and T on a cone Banach space over Banach algebra, the stability of the iteration scheme $x_{2n+1} = Sx_{2n}$, $x_{2n+2} = Tx_{2n+1}$, $n = 0, 1, 2, ...$ converging to the common fixed point of S and T is also discussed here.

Key Words: cone normed space, stability of fixed points, projective tensor product.

Contents

1. Introduction

In 2007, Huang and Zhang [19] introduced cone metric spaces and gave application of fixed point theory in such spaces. Since then, a number of researchers ([1], [2], [3], [19], [24], [27], [36]) developed the fixed point theory in cone metric spaces. In 2010, the concept of cone normed spaces was initiated by Turkoglu et al. [39]. In [23], Karapinar derived some fixed point theorems in cone Banach spaces.

However, recently some authors viz. Amini-Harandi et al. [4], Asadi et al. [5], Du [15], Ercan [16], Feng and Mao [17], Khamsi [25] etc., have shown the equivalence of fixed point results between cone metric spaces and metric spaces, and also between cone b-metric spaces and b-metric spaces. So study of fixed point theorems in cone metric spaces is no more interesting in this sense. But in [27],

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[1] Corresponding author.

Liu and Xu introduced cone metric space over Banach algebra and initiated a new study by defining generalized Lipschitz mapping, where the contractive coefficient is a vector instead of usual real constant. They provided an example to explain the non equivalence of fixed point results between the vectorial versions and scalar versions. In present days, some researchers viz., Huang and Radenović ([20], [21]), Huang et al. [22], Xu and Radenović [40] etc., observed the interest and need for research in the field of studying fixed point theorems. They developed many important results in the framework of cone metric spaces and cone b-metric spaces over Banach algebra.

Let $\mathbb{A}_1$ be a real Banach algebra, $\|.\|$ be its norm and e_1 be its unit element. A nonempty closed subset P_1 of $\mathbb{A}_1$ is called a cone if

(i) P_1 is closed, non empty and $\{0, e_1\} \subset P_1$.

(ii) $\alpha P_1 + \beta P_1 \subset P_1$ for all non negative real numbers α, β.

(iii) $P_1^2 = P_1 P_1 \subset P_1$.

(iv) $P_1 \cap (-P_1) = \{0\}$.

For a given a cone $P_1 \subset \mathbb{A}_1$, a partial ordering " $\preceq$ " on $\mathbb{A}_1$ with respect to P_1 is defined by $x \preceq y$ if and only if $y - x \in P_1$. $x \prec y$ will indicate $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in intP_1$(interior of P_1). If $intP_1 \neq \phi$, then P_1 is called a solid cone. Here " $\prec$ " and " $\ll$ " are also partial orderings with respect to P_1.

Definition 1.1 *[27] Let X be a non empty set. Suppose the mapping $d : X \times X \to \mathbb{A}_1$ satisfies:*

(i) $0 \preceq d(x, y)$ *for all* $x, y \in X$,

(ii) $d(x, y) = 0$ *if and only if* $x = y$,

(iii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$ *and*

(iv) $d(x, y) \preceq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.

Then the pair (X, d) is called a cone metric space over Banach algebra.

Definition 1.2 *[39] Let X be a vector space over $\mathbb{R}$ and $\|.\|_{P_1} : X \to \mathbb{A}_1$ be a mapping satisfying:*

(i) $0 \preceq \|x\|_{P_1} \; \forall x \in X$

(ii) $\|x\|_{P_1} = 0 \Leftrightarrow x = 0 \; \forall x \in X$

(iii) $\|kx\|_{P_1} = |k| \|x\|_{P_1} \; \forall \, k \in \mathbb{R}, \; \forall x \in X$

(iv) $\|x+y\|_{P_1} \preceq \|x\|_{P_1} + \|y\|_{P_1}\ \forall\ x, y \in X$

Then the pair $(X, \|.\|_{P_1})$ is called a cone normed space over the Banach algebra $\mathbb{A}_1$ and $\|.\|_{P_1}$ is called a cone norm.
Every cone normed spaces is a cone metric space over Banach algebra with $d(x,y) = \|x-y\|_{P_1}$.

Definition 1.3 *[27] A cone P_1 is called a normal cone if there is a number $K > 0$ such that $\forall x, y \in \mathbb{A}_1$*

$$0 \preceq x \preceq y \Rightarrow \|x\| \leqslant K\|y\|.$$

Definition 1.4 *[19] The cone P_1 is called regular if every increasing sequence in $\mathbb{A}_1$ which is bounded from above is convergent. That is, if $\{x_n\}$ is a sequence such that*

$$x_1 \preceq x_2 \preceq ... \preceq x_n \preceq ... \preceq y$$

for some $y \in \mathbb{A}_1$, then there is $x \in \mathbb{A}_1$ such that $\|x_n - x\| \to 0$ as $n \to \infty$.

Example 1.1 *[27] Let $\mathbb{A}_1 = l^1 = \{x = \{x_n\}_{n\geqslant 1} : \sum_{n=1}^{\infty} |x_n| < \infty\}$ with convolution as multiplication:*

$$xy = \{x_n\}_{n\geqslant 1}\{y_n\}_{n\geqslant 1} = \{\sum_{i+j=n} x_i y_j\}_{n\geqslant 1}$$

Thus $\mathbb{A}_1$ is a Banach algebra with unit $e_1 = \{1, 0, 0, ...\}$. Let $P_1 = \{x = \{x_n\}_{n\geqslant 1} \in \mathbb{A}_1 : x_n \geqslant 0\ \forall\ n\}$, which is a normal cone in $\mathbb{A}_1$. Let $X = l^1$ with the metric $d : X \times X \to \mathbb{A}_1$ defined by

$$d(x,y) = d(\{x_n\}_{n\geqslant 1}, \{y_n\}_{n\geqslant 1}) = \{|x_n - y_n|\}_{n\geqslant 1}$$

Then (X, d) is a cone metric space over the Banach algebra $\mathbb{A}_1$.

Definition 1.5 *[27] Let (X, d) be a cone metric space over Banach algebra $\mathbb{A}_1$, $x \in X$, $\{x_n\}$ a sequence in X. Then*

(i) $\{x_n\}$ converges to x whenever for every $c \in \mathbb{A}_1$ with $0 \ll c$ there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geqslant N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x(n \to \infty)$.

(ii) $\{x_n\}$ is a Cauchy sequence whenever for each $0 \ll c$ there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geqslant N$.

Definition 1.6 *[40] Let P_1 be a solid cone in a Banach algebra $\mathbb{A}_1$. A sequence $\{u_n\} \subset P_1$ is said to be a c-sequence if for each $0 \ll c$ there exists a natural number N such that $u_n \ll c$ for all $n > N$.*

Lemma 1.1 *[34] If $\mathbb{A}_1$ is a real Banach algebra with a solid cone P_1 and if $0 \preceq u \ll c$ for each $0 \ll c$, then $u = 0$.*

Lemma 1.2 *[34] If $\mathbb{A}_1$ is a real Banach algebra with a solid cone P_1 and if $a, b, c \in \mathbb{A}_1$ and $a \preceq b \ll c$, then $a \preceq c$.*

Lemma 1.3 *[34] If $\mathbb{A}_1$ is a real Banach algebra with a solid cone P_1 and if $\|x_n\| \to 0$ $(n \to \infty)$, then for any $0 \ll c$, there exists $N \in \mathbb{N}$ such that, for any $n > N$, we have $x_n \ll c$.*

Lemma 1.4 *[40] Let (X, d) be a complete cone metric space over a Banach algebra $\mathbb{A}_1$ and let P_1 be the underlying solid cone in $\mathbb{A}_1$. Let $\{x_n\}$ be a sequence in X and $0 \ll c$. If $\{x_n\}$ converges to $x \in X$, then we have:*

(i) $\{d(x_n, x)\}$ is a c-sequence.
(ii) For any $p \in \mathbb{N}$, $\{d(x_n, x_{n+p})\}$ is a c-sequence.

Lemma 1.5 *[40] If $k \in P_1$ with spectral radius $r(k) < 1$, then $\|k^n\| \to 0$ as $n \to \infty$.*

Lemma 1.6 *[20] Let $\mathbb{A}_1$ be a Banach algebra with a unit e_1 and P_1 be a solid cone in $\mathbb{A}_1$. Let $h \in \mathbb{A}_1$ and $u_n = h^n$. If $r(h) < 1$, then $\{u_n\}$ is a c-sequence.*

Lemma 1.7 *[20] Let $\mathbb{A}_1$ be a Banach algebra with a unit e_1 and $u \in \mathbb{A}_1$. If $r(u) < |C|$ and C is a complex constant, then*

$$r(Ce_1 - u)^{-1} \leqslant \frac{1}{|C| - r(u)}.$$

Lemma 1.8 *[40] Let P_1 be a solid cone in a Banach algebra $\mathbb{A}_1$. Suppose that $k \in P_1$ and $\{u_n\}$ is a c-sequence in P . Then $\{ku_n\}$ is a c-sequence.*

Lemma 1.9 *[38] Let $\mathbb{A}_1$ be a Banach algebra with a unit e_1, $k \in \mathbb{A}_1$, then $\lim_{n\to\infty} \|k^n\|^{\frac{1}{n}}$ exists and the spectral radius $r(k)$ satisfies $r(k) = \lim_{n\to\infty} \|k^n\|^{\frac{1}{n}} = \inf \|k^n\|^{\frac{1}{n}}$. If $r(k) < 1$, then $e_1 - k$ is invertible in $\mathbb{A}$, moreover,*

$$(e_1 - k)^{-1} = \sum_{i=0}^{\infty} k^i.$$

Lemma 1.10 *[38] Let $\mathbb{A}_1$ be a Banach algebra with a unit e_1, $a, b \in \mathbb{A}_1$. If a commutes with b, then $r(a + b) \leqslant r(a) + r(b)$, $r(ab) \leqslant r(a)r(b)$.*

Lemma 1.11 *[21] Let $\mathbb{A}_1$ be a Banach algebra with a unit e_1 and P_1 be a solid cone in $\mathbb{A}_1$. Let $u, \alpha, \beta \in P_1$ such that $\alpha \preceq \beta$ and $u \preceq \alpha u$. If $r(\beta) < 1$, then $u = 0$.*

In 2014, Liu and Xu derived the following fixed point theorem with generalized Lipschitz condition:

Theorem 1.1 *[40] Let (X, d) be a cone Banach space over a Banach algebra $\mathbb{A}_1$, and P_1 be the underlying solid cone with $k \in P_1$ and $r(k) < 1$. Suppose the mapping $T : X \to X$ satisfies generalized Lipschitz condition:*

$$d(Tx, Ty) \preceq kd(x, y),\ \forall\, x, y \in X.$$

Then T has a unique fixed point in X and for any $x \in X$, iterative sequence $\{T^n x\}$ converges to the fixed point.

Let $(X, \|.\|_{P_1})$ and $(Y, \|.\|_{P_2})$ be two cone Banach spaces over Banach algebra, where P_1 and P_2 are solid normal cones (with normal constant 1). In this paper, we derive some fixed point theorems for a self mapping T in the projective cone normed tensor product space over Banach algebra. We also discuss the stability of an iteration scheme converging to a common fixed point of two self mappings on a cone Banach space over Banach algebra.

2. Main Results: (a) Projective Cone Normed Tensor Product Space(PCNTPS) over Banach algebra

First, we define a cone norm over Banach algebra for the algebraic tensor product of two vector spaces.

Lemma 2.1 *[7] Let X, Y be normed spaces over $\mathbb{F}$ with dual spaces X^* and Y^* respectively. Given $x \in X, y \in Y$, Let $x \otimes y$ be the element of $BL(X^*, Y^*; \mathbb{F})$ (which is the set of all bounded bilinear forms from $X^* \times Y^*$ to $\mathbb{F}$), defined by*

$$x \otimes y(f, g) = f(x)g(y),\ (f \in X^*, g \in Y^*)$$

The algebraic tensor product of X and Y, $X \otimes Y$ is defined to be the linear span of $\{x \otimes y : x \in X, y \in Y\}$ in $BL(X^*, Y^*; \mathbb{F})$.

Lemma 2.2 *[7] Given normed spaces X and Y, the projective tensor norm γ on $X \otimes Y$ is defined by*

$$\|u\|_\gamma = \inf\{\sum_i \|x_i\|\|y_i\| : u = \sum_i x_i \otimes y_i\}$$

where the infimum is taken over all (finite) representations of u.

The completion of $(X \otimes Y, \|.\|_\gamma)$ is called projective tensor product of X and Y and it is denoted by $X \otimes_\gamma Y$.

Lemma 2.3 *[35] Let X and Y be Banach spaces. Then γ is a cross norm on $X \otimes Y$ and $\|x \otimes y\|_\gamma = \|x\|\|y\|$ for every $x \in X, y \in Y$.*

Lemma 2.4 *[7] $X \otimes_\gamma Y$ can be represented as a linear subspace of $BL(X^*, Y^*; \mathbb{F})$ consisting of all elements of the form $u = \sum_i x_i \otimes y_i$ where $\sum_i \|x_i\|\|y_i\| < \infty$. Moreover, $\|u\|_\gamma = \inf\{\sum_i \|x_i\|\|y_i\|\}$ over all such representations of u.*

Lemma 2.5 *[7] Let X and Y be normed algebras over $\mathbb{F}$. There exists a unique product on $X \otimes Y$ with respect to which $X \otimes Y$ is an algebra and*

$$(a \otimes b)(c \otimes d) = ac \otimes bd \qquad (a, c \in X, b, d \in Y).$$

Lemma 2.6 *[7] Let X and Y be normed algebras over $\mathbb{F}$. Then projective tensor norm on $X \otimes Y$ is an algebra norm.*
Clearly, we can conclude that if X and Y are Banach algebras over $\mathbb{F}$ then $X \otimes_\gamma Y$ becomes a Banach algebra.

Let $\mathbb{A}_1$ and $\mathbb{A}_2$ be two real Banach algebras with the unit elements e_1 and e_2 respectively. P_1 and P_2 be two solid cones in $\mathbb{A}_1$ and $\mathbb{A}_2$ respectively. Let K_p be the projective cone ([31], [32]) in $\mathbb{A}_1 \otimes_\gamma \mathbb{A}_2$ defined by

$$K_p = \{\sum_i x_i \otimes y_i \ : \ x_i \in P_1, \ y_i \in P_2\}$$

[Here,

$$K_p K_p = \sum_i x_i \otimes y_i)(\sum_j c_j \otimes d_j) = \sum_i \sum_j x_i c_j \otimes y_i d_j$$

Since $P_1 P_1 \subset P_1$, $P_2 P_2 \subset P_2$, $x_i, c_j \in P_1$ and $y_i, d_j \in P_2$ so by definition $x_i c_j \in P_1$ and $y_i d_j \in P_2$. Thus, $K_p K_p \subset K_p$.]

For vector spaces X and Y over $\mathbb{R}$, with the cone norms $\|.\|_{P_1} : X \to \mathbb{A}_1$ and $\|.\|_{P_2} : Y \to \mathbb{A}_2$, we define:

$\|.\|_{K_p} : X \otimes Y \to \mathbb{A}_1 \otimes_\gamma \mathbb{A}_2$ *by* $\|u\|_{K_p} = \sum_i \|x_i\|_{P_1} \otimes \|y_i\|_{P_2}, \ u = \sum_i x_i \otimes y_i \in X \otimes Y$

(i) $0 \preceq \|x_i\|_{P_1} \Rightarrow \|x_i\|_{P_1} \in P_1, \ 0 \preceq \|y_i\|_{P_2} \Rightarrow \|y_i\|_{P_2} \in P_2, \ \forall i$
$\Rightarrow \sum_i \|x_i\|_{P_1} \otimes \|y_i\|_{P_2} \in K_p$ (by definition of projective cone).
So, $0 \preceq \sum_i \|x_i\|_{P_1} \otimes \|y_i\|_{P_2}$ i.e., $0 \preceq \|u\|_{K_p}$.

(ii)

$$\begin{aligned} \|u\|_{K_p} = 0 \Rightarrow & \sum_i \|x_i\|_{P_1} \otimes \|y_i\|_{P_2} = 0 \\ \Rightarrow & \|x_1\|_{P_1} \otimes \|y_1\|_{P_2} + \|x_2\|_{P_1} \otimes \|y_2\|_{P_2} + ... = 0 \qquad (1) \end{aligned}$$

Each of the terms in (1) is an element of K_p. We call these as $a_1, a_2, a_3, ...$ etc. So, each $0 \preceq a_i \ \forall i$.
(If $a, b \in K_p$ such that $a + b = 0$, then $a = -b$, i.e., $b, -b \in K_p \Rightarrow b = 0$. So, $a = 0$. Similarly for any n number of terms this holds.)

Therefore, $a_i = 0 \ \forall i$ i.e., $\|x_i\|_{P_1} \otimes \|y_i\|_{P_2} = 0 \ \forall i$

$$\begin{aligned} \Rightarrow & \|x_i\|_{P_1} = 0, \ \|y_i\|_{P_2} = 0 \ \forall i \\ \Rightarrow & x_i = 0, \ y_i = 0 \ \forall i \\ \Rightarrow & \sum_i x_i \otimes y_i = 0 \Rightarrow u = 0 \end{aligned}$$

Conversely, let $u = \sum_i x_i \otimes y_i = 0 \Rightarrow (\sum_i x_i \otimes y_i)(f,g) = 0\ \forall f \in X^*,\ g \in Y^*$
In particular, we take $f : X \to \mathbb{R}^+$, $g : Y \to \mathbb{R}^+ \cup \{0\}$ such that $kerg = \{0\}$.

$$
\begin{aligned}
(\sum_i x_i \otimes y_i)(f,g) = 0 &\Rightarrow \sum_i f(x_i)g(y_i) = 0 \\
&\Rightarrow g(y_i) = 0\ \forall i \Rightarrow y_i = 0\ \forall i \\
&\Rightarrow \|y_i\|_{P_2} = 0\ \forall i \text{ (by cone norm property)} \\
&\Rightarrow \|x_i\|_{P_1} \otimes \|y_i\|_{P_2} = 0\ \forall i \\
&\Rightarrow \sum_i \|x_i\|_{P_1} \otimes \|y_i\|_{P_2} = 0 \\
&\Rightarrow \|u\|_{K_p} = 0
\end{aligned}
$$

(iii) $\|ku\|_{K_p} = |k|\|u\|_{K_p}\ \forall\ u \in X \otimes Y,\ k \in \mathbb{R}$

(iv) $\|u+v\|_{K_p} \preceq \|u\|_{K_p} + \|v\|_{K_p}\ \forall\ u,v \in X \otimes Y$
(follows by definition)

Thus, $\|.\|_{K_p}$ is a cone norm on $X \otimes Y$. We call $(X \otimes Y, \|.\|_{K_p})$ as projective cone normed tensor product space(PCNTPS) over Banach algebra.

Lemma 2.7 *If P_1 and P_2 are normal cones, then K_p is also normal.*

Proof: Let $u, v \in \mathbb{A}_1 \otimes_\gamma \mathbb{A}_2$ be such that $0 \preceq u \preceq v \Rightarrow v - u \in K_p$. Let

$$
v = \sum_{i=1}^{n} x_i \otimes y_i,\ u = \sum_{j=1}^{m} p_j \otimes q_j.
$$

For $n < m$:

$$
\begin{aligned}
v - u \in K_p &\Rightarrow \sum_{i=1}^{n} x_i \otimes y_i - \sum_{j=1}^{m} p_j \otimes q_j \in K_p \\
&\Rightarrow \sum_{i=1}^{n} (x_i - p_i) \otimes y_i + \sum_{i=1}^{n} p_i \otimes (y_i - q_i) + \sum_{j=n+1}^{m} (-p_j) \otimes q_j \in K_p
\end{aligned}
$$

So, by the form of elements of K_p, we get,
$x_i - p_i, p_i \in P_1$; $y_i - q_i, y_i \in P_2\ \forall i = 1, 2, ..., n$ and
$-p_j \in P_1$; $q_j \in P_2\ \forall j = n+1, n+2, ..., m$.
Again,

$$
v - u \in K_p \Rightarrow \sum_{i=1}^{n} x_i \otimes (y_i - q_i) + \sum_{i=1}^{n} (x_i - p_i) \otimes q_i + \sum_{j=n+1}^{m} p_j \otimes (-q_j) \in K_p
$$

So, $x_i - p_i, x_i \in P_1$; $y_i - q_i, q_i \in P_2$ $\forall i = 1, 2, ..., n$ and
$p_j \in P_1$; $-q_j \in P_2$ $\forall j = n+1, n+2, ..., m$.

Hence, $p_j = 0, q_j = 0$ $\forall j = n+1, n+2, ..., m$.

Now, $p_i \preceq x_i$ and $q_i \preceq y_i$. Since P_1 and P_2 are normal cones, so, there exist constants $K_1, K_2 \geqslant 1$ such that

$$\|p_i\| \leqslant K_1\|x_i\|, \ \|q_i\| \leqslant K_2\|y_i\| \ \forall i.$$

Since, $\|u\| = \|\sum_{j=1}^{m} p_j \otimes q_j\| \leqslant \sum_{j=1}^{m} \|p_j\|\|q_j\| \leqslant K_1 K_2 \sum_{j=1}^{m} \|x_j\|\|y_j\|$, so,
for the projective tensor norm in $\mathbb{A}_1 \otimes_\gamma \mathbb{A}_2$, we have $\|u\| \leqslant K_1 K_2 \|v\|$.

For $n \geqslant m$, the condition is obviously satisfied.

Therefore, K_p is a normal cone with the normal constant $K_1K_2(\geqslant 1)$. □

Lemma 2.8 *If P_1 and P_2 are regular cones, then K_p is also regular.*

Proof: Let $\{u_n\}_{n\geqslant 1}$ be a sequence in $\mathbb{A}_1 \otimes_\gamma \mathbb{A}_2$ such that $u_1 \preceq u_2, ..., \preceq y$ for some $y \in \mathbb{A}_1 \otimes_\gamma \mathbb{A}_2$.

To show that $\{u_n\}_{n\geqslant 1}$ is convergent in $\mathbb{A}_1 \otimes_\gamma \mathbb{A}_2$:

Let $u_1 = \sum_i p_{1_i} \otimes q_{1_i}$, $u_2 = \sum_i p_{2_i} \otimes q_{2_i}$, ..., $u_n = \sum_i p_{n_i} \otimes q_{n_i}$, ... and $y = \sum_i a_i \otimes b_i \in \mathbb{A}_1 \otimes_\gamma \mathbb{A}_2$.

Since $u_1 \preceq u_2 \preceq ... \preceq y$, as in above Lemma 2.7, we can show that
$p_{1_i} \preceq p_{2_i} \preceq ... \preceq a_i$ $\forall i$ and $q_{1_i} \preceq q_{2_i} \preceq ... \preceq b_i$ $\forall i$.

For each i, $\{p_{n_i}\}_{n\geqslant 1}$ is a sequence in $\mathbb{A}_1$ (increasing) which is bounded from above, and so also $\{q_{n_i}\}_{n\geqslant 1}$ in $\mathbb{A}_2$. Since P_1 and P_2 are regular, there exist $r_i \in \mathbb{A}_1$ and $s_i \in \mathbb{A}_2$ such that $\lim_{n\to\infty} \|p_{n_i} - r_i\| = 0$ and $\lim_{n\to\infty} \|q_{n_i} - s_i\| = 0$ for each i.

Now, $\sum_i r_i \otimes s_i = u(say) \in \mathbb{A}_1 \otimes_\gamma \mathbb{A}_2$.

$$\begin{aligned}
\|u_n - u\| &= \|\sum_i p_{n_i} \otimes q_{n_i} - \sum_i r_i \otimes s_i\| \\
&\leqslant \sum_i \|p_{n_i} - r_i\|\|q_{n_i}\| + \sum_i \|q_{n_i} - s_i\|\|r_i\| \\
&\to 0 \ as \ n \to \infty
\end{aligned}$$

Hence, K_p is regular. □

Lemma 2.9 *For normal cones P_1 and P_2, if $(X, \|.\|_{P_1})$ and $(Y, \|.\|_{P_2})$ are two cone Banach spaces over Banach algebras $\mathbb{A}_1$ and $\mathbb{A}_2$ respectively, then $(X \otimes Y, \|.\|_{K_p})$ is also a cone Banach space over the Banach algebra $\mathbb{A}_1 \otimes_\gamma \mathbb{A}_2$.*

Proof: Let $\{u_n\}$ (where $u_n = \sum_i x_{n_i} \otimes y_{n_i}$) be a Cauchy sequence in $X \otimes Y$. Since P_1 and P_2 are normal cones, so, by Lemma 2.7, K_p is also normal.

Since $\{u_n\}$ is Cauchy, so, $\|\|u_n - u_m\|_{K_p}\| \to 0 \; as \; m, n \to \infty$. Now,

$$\begin{aligned}
\|\|u_n - u_m\|_{K_p}\| &= \|\| \sum_i x_{n_i} \otimes y_{n_i} - \sum_i x_{m_i} \otimes y_{m_i} \|_{K_p}\| \\
&= \|\| \sum_i (x_{n_i} - x_{m_i}) \otimes y_{n_i} + \sum_i x_{m_i} \otimes (y_{n_i} - y_{m_i}) \|_{K_p}\| \\
&\leqslant \sum_i \|\|x_{n_i} - x_{m_i}\|_{P_1}\| \|\|y_{n_i}\|_{P_2}\| + \sum_i \|\|x_{m_i}\|_{P_1}\| \|\|y_{n_i} - y_{m_i}\|_{P_2}\| \\
&\to 0 \; as \; n, m \to \infty
\end{aligned}$$

$\Rightarrow$ ($\|\|x_{n_i} - x_{m_i}\|_{P_1}\| \to 0 \; as \; m, n \to \infty$ or $\|\|y_{n_i}\|_{P_2}\| \to 0 \; as \; m, n \to \infty$) and ($\|\|x_{m_i}\|_{P_1}\| \to 0 \; as \; m, n \to \infty$ or $\|\|y_{n_i} - y_{m_i}\|_{P_2}\| \to 0 \; as \; m, n \to \infty$) for each i.

$\Rightarrow$ ($\{x_{n_i}\}_n$ is a Cauchy sequence in $(X, \|.\|_{P_1})$ or $\{y_{n_i}\}_n \to 0$ in $(Y, \|.\|_{P_2})$) and ($\{y_{n_i}\}_n$ is a Cauchy sequence in $(Y, \|.\|_{P_2})$ or $\{x_{n_i}\}_n \to 0$ in $(X, \|.\|_{P_1})$) for each i.

$\Rightarrow$ $\{x_{n_i}\}_n$and $\{y_{n_i}\}_n$ are convergent sequences in $(X, \|.\|_{P_1})$ and $(Y, \|.\|_{P_2})$ respectively for each i, since these are cone Banach spaces.

Let $x_{n_i} \to a_i \in X$ and $y_{n_i} \to b_i \in Y$ for each i. We take $u = \sum_i a_i \otimes b_i \in X \otimes Y$. It can be easily shown that $\|\|u_n - u\|_{K_p}\| \to 0 \; as \; n \to \infty$, i.e., $\{u_n\}$ is a convergent sequence in $X \otimes Y$.

Hence, $(X \otimes Y, \|.\|_{K_p})$ is a cone Banach space over the Banach algebra $\mathbb{A}_1 \otimes_\gamma \mathbb{A}_2$. □

Example 2.1 We take X as any normed space and $\mathbb{A}_1 = (l^1, \|.\|)$, over $\mathbb{R}$.
Let $P_1 = \{\{x_n\}_{n \geqslant 1} \in \mathbb{A}_1, x_n \geqslant 0 \; \forall n\}$. Then $\|.\|_{P_1} : X \to \mathbb{A}_1$ defined by:
$\|x\|_{P_1} = \{\dfrac{\|x\|}{2^n}\}_{n \geqslant 1}$ is a cone norm on $\mathbb{R}$. Then clearly, $(X, \|.\|_{P_1})$ is a cone Banach space over $\mathbb{A}_1$.

Next we take, $Y = \mathbb{R}$, $\mathbb{A}_2 = (\mathbb{R}, \|.\|)$, $P_2 = \{y : y \geqslant 0\}$. Then $\|.\|_{P_2} : Y \to \mathbb{A}_2$ defined by:
$\|y\|_{P_2} = |y|$ is a cone norm on $\mathbb{R}$. Clearly, $(\mathbb{R}, \|.\|_{P_2})$ is also a cone Banach space over $\mathbb{A}_2$.

Now, $\|.\|_{K_p} : X \otimes Y \to \mathbb{A}_1 \otimes_\gamma \mathbb{A}_2$ i.e., $\|.\|_{K_p} : X \otimes \mathbb{R} \to l^1 \otimes_\gamma \mathbb{R}$ is defined by:
$\|u\|_{K_p} = \| \sum_i x_i \otimes y_i \|_{K_p} = \sum_i \|x_i\|_{P_1} \otimes \|y_i\|_{P_2} = \sum_i \{\dfrac{\|x_i\|}{2^n}\}_{n \geqslant 1} \otimes |y_i| = \sum_i \{\dfrac{\|x_i\||y_i|}{2^n}\}_{n \geqslant 1}$.
(Since $l^1 \otimes_\gamma \mathbb{R} = l^1(\mathbb{R})$ [35])
Thus $(X \otimes Y, \|.\|_{K_p})$ is a cone Banach space over the Banach algebra $l^1 \otimes_\gamma \mathbb{R}$.

Lemma 2.10 *For a cone normed space $(X, \|.\|_p)$, where P is a normal cone with normal constant $K = 1$, if $\|\|u\|_P\| < \|\|v\|_P\|$ then $\|u\|_P \prec \|v\|_P$, $u, v \in X$.*

Proof: Given $\|\|u\|_P\| < \|\|v\|_P\|$.
If possible, let $\|v\|_P \preceq \|u\|_P$. Since P is normal, so,

$$\|\|v\|_P\| \leqslant K \|\|u\|_P\| = \|\|u\|_P\|, \text{ a contradiction.}$$

Hence, $\|u\|_P \preceq \|v\|_P$ but $\|u\|_P \neq \|v\|_P$. So, $\|u\|_P \prec \|v\|_P$. □

Now, we want to establish some fixed point theorems in projective cone normed tensor product space over Banach algebra.
(In all the following results, we take $d(x,0) = \|x\|_P$, for the cone P.)

3. Main Results: (b) Some Fixed Point Theorems in PCNTPS

Theorem 3.1 *Let $(X, \|.\|_{P_1})$ and $(Y, \|.\|_{P_2})$ be two cone Banach spaces over Banach algebras $\mathbb{A}_1$ and $\mathbb{A}_2$ respectively, where P_1 and P_2 are solid normal cones (with normal constant 1) and $(X \otimes Y, \|.\|_{K_p})$ is the projective cone Banach space over Banach algebra $\mathbb{A}_1 \otimes_\gamma \mathbb{A}_2$. Let $T_1 : X \otimes Y \to X$ and $T_2 : X \otimes Y \to Y$ be two mappings satisfying:*

(i) $\|\|T_1u - T_1v\|_{P_1}\| \leqslant \dfrac{1}{M_2}\|k\|u - v\|_{K_p}\|$,

(ii) $\|\|T_2u - T_2v\|_{P_2}\| \leqslant \dfrac{1}{M_1}\|k\|u - v\|_{K_p}\|$,

(iii) $\|\|T_1u\|_{P_1}\| < M_1$ *and* $\|\|T_2u\|_{P_2}\| < M_2$, $\forall u, v \in X \otimes Y$.

Then the mapping $T : X \otimes Y \to X \otimes Y$ defined by $Tu = T_1u \otimes T_2u$, $u \in X \otimes Y$ has a unique fixed point in $X \otimes Y$ if $k \in K_p$ with $r(k) < \frac{1}{2}$.

Proof: Let $u, v \in X \otimes Y$. We have,

$$\begin{aligned}\|Tu - Tv\|_{K_p} &= \|T_1u \otimes T_2u - T_1v \otimes T_2v\|_{K_p} \\ &= \|(T_1u - T_1v) \otimes T_2u + T_1v \otimes (T_2u - T_2v)\|_{K_p} \\ &\preceq \|T_1u - T_1v\|_{P_1} \otimes \|T_2u\|_{P_2} + \|T_1v\|_{P_1} \otimes \|T_2u - T_2v\|_{P_2}\end{aligned}$$

Since K_p is normal with normal constant 1, so,

$$\begin{aligned}\|\|Tu - Tv\|_{K_p}\| &\leqslant \|\|T_1u - T_1v\|_{P_1}\|\|\|T_2u\|_{P_2}\| + \|\|T_1v\|_{P_1}\|\|\|T_2u - T_2v\|_{P_2}\| \\ &\qquad \text{(taking projective norm in } \mathbb{A}_1 \otimes_\gamma \mathbb{A}_2\text{).} \\ &< \|k\|u - v\|_{K_p}\| + \|k\|u - v\|_{K_p}\| \\ &= 2\|k\|u - v\|_{K_p}\| \\ \Rightarrow \|\|Tu - Tv\|_{K_p}\| &< \|2k\|u - v\|_{K_p}\| \qquad (2)\end{aligned}$$

So by Lemma 2.10, equation (2) implies

$$\|Tu - Tv\|_{K_p} \prec 2k\|u - v\|_{K_p}$$

Now, by Theorem 1.1 the mapping T has a unique fixed point in $X \otimes Y$. □

Example 3.1 We take $X = l^1$ and $\mathbb{A}_1 = (l^1, \|.\|)$ over $\mathbb{R}$.
Let $P_1 = \{\{x_n\}_{n\geqslant 1} \in l^1, x_n \geqslant 0 \;\forall n\}$. Then $\|.\|_{P_1} : l^1 \to \mathbb{A}_1$ defined by:
$\|a_i\|_{P_1} = \{|a_{i_k}|\}_k$ $(a_i = \{a_{i_k}\}_k)$ is a cone norm. Clearly, $(X, \|.\|_{P_1})$ is also a cone Banach space over the unital Banach algebra $\mathbb{A}_1$.

Next we take, $Y = \mathbb{R}$, $\mathbb{A}_2 = (\mathbb{R}, \|.\|)$, $P_2 = \{y : y \geqslant 0\}$. Then $\|.\|_{P_2} : Y \to \mathbb{A}_2$ defined by:
$\|y\|_{P_2} = |y|$ is a cone norm on $\mathbb{R}$. Clearly, $(\mathbb{R}, \|.\|_{P_2})$ is also a cone Banach space over the unital Banach algebra $\mathbb{A}_2$.

Now, $\|.\|_{K_p} : X \otimes Y \to \mathbb{A}_1 \otimes_\gamma \mathbb{A}_2$ i.e., $\|.\|_{K_p} : l^1 \otimes \mathbb{R} \to l^1 \otimes_\gamma \mathbb{R}$ is defined by:

$$\|u\|_{K_p} = \|\sum_i a_i \otimes y_i\|_{K_p} = \sum_i \|\{a_{i_k}\}_k\|_{P_1} \otimes \|y_i\|_{P_2} = \sum_i \{|a_{i_k}|\}_k \otimes |y_i| = \sum_i \{|a_{i_k}||y_i|\}_k$$

(Since, $l^1 \otimes_\gamma \mathbb{R} = l^1(\mathbb{R})$ [35])
Thus $(l^1 \otimes \mathbb{R}, \|.\|_{K_p})$ is a cone Banach space over $l^1(\mathbb{R})$ with unit $e = \{1, 0, 0, ...\}$.

Let D_{l^1}, $D_\mathbb{R}$ and $D_{l^1\otimes\mathbb{R}}$ (containing $D_{l^1} \otimes D_\mathbb{R}$) be subsets of l^1, $\mathbb{R}$ and $l^1 \otimes \mathbb{R}$ bounded (strictly) by constants c, c and c^2 respectively.

We define $T_1 : D_{l^1\otimes\mathbb{R}} \to D_{l^1}$ by $T_1(\sum_i a_i \otimes y_i) = \dfrac{1}{2c^2} \sum_i \{a_{i_k} y_i\}_k$, where $a_i = \{a_{i_k}\}_k$ and
$T_2 : D_{l^1\otimes\mathbb{R}} \to D_\mathbb{R}$ by $T_2(\sum_i a_i \otimes y_i) = \dfrac{1}{3} \sum_i \|\{|a_{i_k}|\}_k\||y_i|$.
Then

$$\begin{aligned}
\|\|T_1(\sum_i a_i \otimes y_i)\|_{P_1}\| &= \|\|\frac{1}{2c^2} \sum_i \{a_{i_k} y_i\}_k\|_{P_1}\| \leqslant \frac{1}{2c^2} \|\sum_i \{|a_{i_k} y_i|\}_k\| \\
&\leqslant \frac{1}{2c^2} \sum_i (\sum_k |a_{i_k}|)|y_i| \text{ (using norm in } l^1) \\
&= \frac{1}{2c^2} \sum_i \|a_i\||y_i|
\end{aligned}$$

Taking projective tensor norm in $l^1(\mathbb{R})$,

$$\begin{aligned}
\|\|T_1(\sum_i a_i \otimes y_i)\|_{P_1}\| &\leqslant \frac{1}{2c^2} \|\sum_i a_i \otimes y_i\| \\
&< \frac{1}{2c^2}.c^2 = \frac{1}{2} (= M_1)
\end{aligned}$$

and

$$\begin{aligned}
\|\|T_2(\sum_i a_i \otimes y_i)\|_{P_2}\| &= \|\|\frac{1}{3} \sum_i \|\{|a_{i_k}|\}_k\||y_i|\|_{P_2}\| \\
&\leqslant \|\frac{1}{3} \sum_i \|\{|a_{i_k}|\}_k\||y_i|\| \leqslant \frac{1}{3} \sum_i (\sum_k |a_{i_k}|)|y_i| \\
&= \frac{1}{3} \sum_i \|a_i\||y_i|
\end{aligned}$$

So, for projective tensor norm in $l^1(\mathbb{R})$,

$$\|\|T_2(\sum_i a_i \otimes y_i)\|_{P_2}\| \leqslant \frac{1}{3}\|\sum_i a_i \otimes y_i\| < \frac{c^2}{3}(= M_2)$$

For $u = \sum_i a_i \otimes y_i,\ v = \sum_i b_i \otimes x_i \in D_{l^1 \otimes \mathbb{R}}$, we have,

$$\begin{aligned}
\|T_1(u) - T_1(v)\|_{P_1} &= \|\frac{1}{2c^2}\sum_i \{a_{i_k} y_i\}_k - \frac{1}{2c^2}\sum_i \{b_{i_k} x_i\}_k\|_{P_1} \\
&= \frac{1}{2c^2}\|\sum_i \{a_{i_k} y_i - b_{i_k} x_i\}_k\|_{P_1} \\
&= \frac{1}{2c^2}\sum_i \{|a_{i_k} y_i - b_{i_k} x_i|\}_k
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \|\|T_1(u) - T_1(v)\|_{P_1}\| &= \|\frac{1}{2c^2}\sum_i \{|a_{i_k} y_i - b_{i_k} x_i|\}_k\| \\
&\leqslant \frac{1}{2c^2}\sum_i \sum_k |a_{i_k} y_i - b_{i_k} x_i| \\
&\leqslant \frac{1}{2c^2}\sum_{i,k} (|a_{i_k} y_i| + |b_{i_k} x_i|) \\
&= \frac{1}{2c^2}\sum_{i,k} (|a_{i_k}||y_i| + |b_{i_k}||x_i|) \qquad (3)
\end{aligned}$$

$$\begin{aligned}
\|\|u - v\|_{K_p}\| &= \|\|\sum_i a_i \otimes y_i - \sum b_i \otimes x_i\|_{K_p}\| \\
&= \|\|\sum_i a_i \otimes y_i + \sum(-b_i) \otimes x_i\|_{K_p}\| \\
&= \|\sum_i \{|a_{i_k}||y_i|\}_k + \sum_i \{|b_{i_k}||x_i|\}_k\| \text{ (by definition of } \|.\|_{K_p}) \\
&= \sum_i \sum_k (|a_{i_k}||y_i| + |b_{i_k}||x_i|) \qquad (4)
\end{aligned}$$

From (3) and (4),

$$\begin{aligned}
\Rightarrow \|\|T_1(u) - T_1(v)\|_{P_1}\| &\leqslant \frac{1}{2c^2}\|\|u - v\|_{K_p}\| \\
&\leqslant \frac{1}{c^2}\|e\|u - v\|_{K_p}\| = \frac{1/3}{c^2/3}\|e\|u - v\|_{K_p}\| \\
&= \frac{1}{M_2}\|k\|u - v\|_{K_p}\|,\ k = \frac{1}{3}e \in K_p
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|T_2(u) - T_2(v)\|_{P_2} &= \|\frac{1}{3}\sum_i \|\{|a_{i_k}|\}_k\||y_i| - \frac{1}{3}\sum_i \|\{|b_{i_k}|\}_k\||x_i|\|_{P_2} \\
&= \frac{1}{3}|\sum_i (\|\{|a_{i_k}|\}_k\||y_i| - \|\{|b_{i_k}|\}_k\||x_i|)| \\
&\preceq \frac{1}{3}\sum_i \|\{|a_{i_k}|\}_k\||y_i| + \sum_i \|\{|b_{i_k}|\}_k\||x_i| \\
\Rightarrow \|\|T_2(u) - T_2(v)\|_{P_2}\| &\leqslant \frac{1}{3}\sum_i\sum_k |a_{i_k}||y_i| + \sum_i\sum_k |b_{i_k}||x_i| \\
&\quad (P_2 \text{ being a normal cone with normal constant } 1) \\
&= \frac{1}{3}\|\|u - v\|_{K_p}\| \text{ (from (4))} \\
&\leqslant \frac{1/3}{1/2}\|e\|u - v\|_{K_p}\| = \frac{1}{M_1}\|k\|u - v\|_{K_p}\|,\ k = \frac{1}{3}e \in K_p. \\
\text{Also } r(k) &= \lim_{n\to\infty} \| \left(\frac{1}{3}e\right)^n \|^{\frac{1}{n}} = \frac{1}{3} < \frac{1}{2}.
\end{aligned}$$

So by Theorem 3.1, the mapping $T : D_{l^1\otimes\mathbb{R}} \to D_{l^1\otimes\mathbb{R}}$ defined by

$$T(\sum_i a_i \otimes x_i) = \frac{1}{6c}\sum_i \{Ma_{i_n}x_i\}_n,\ where\ M = \sum_i \|a_i\|.|x_i|$$

has a unique fixed point in $D_{l^1\otimes\mathbb{R}}$.

Theorem 3.2 *Let $(X, \|.\|_{P_1})$ and $(Y, \|.\|_{P_2})$ be two cone Banach spaces over Banach algebras $\mathbb{A}_1$ and $\mathbb{A}_2$, where P_1 and P_2 are normal cones (with normal constant 1) and $(X \otimes Y, \|.\|_{K_p})$ is the projective cone Banach space over $\mathbb{A}_1 \otimes_\gamma \mathbb{A}_2$. Let $T_1 : X \otimes Y \to X$ and $T_2 : X \otimes Y \to Y$ be two mappings satisfying:*

$$\begin{aligned}
(i)\ \|\|T_1u - T_1v\|_{P_1}\| &\leqslant \frac{1}{M_2}[\|k(\|u - Tu\|_{K_p} + \|v - Tv\|_{K_p})\|] \\
(ii)\ \|\|T_2u - T_2v\|_{P_2}\| &\leqslant \frac{1}{M_1}[\|k(\|u - Tu\|_{K_p} + \|v - Tv\|_{K_p})\|] \\
(iii)\ \|\|T_1u\|_{P_1}\| &< M_1 \ and\ \|\|T_2u\|_{P_2}\| < M_2\ \forall u, v \in X \otimes Y.
\end{aligned}$$

Then the mapping $T : X \otimes Y \to X \otimes Y$ defined by $Tu = T_1u \otimes T_2u$, $u \in X \otimes Y$ has a unique fixed point if $k \in K_p$ with $r(k) < \frac{1}{4}$.

Theorem 3.3 *Let P_1 and P_2 be two solid normal cones (with normal constant 1) in Banach algebras $\mathbb{A}_1$ and $\mathbb{A}_2$ respectively. Let $T_1 : X \otimes Y \to X$ and $T_2 : X \otimes Y \to Y$*

be two mappings satisfying:

$$(i)\ \|T_1u - T_1v\|_{P_1} \otimes e_2 \preceq \frac{k}{M_2}\|u-v\|_{K_p},$$
$$(ii)\ e_1 \otimes \|T_2u - T_2v\|_{P_2} \preceq \frac{k}{M_1}\|u-v\|_{K_p}\ \forall u,v \in X\otimes Y,$$
$$(iii)\ e_1 \in P_1,\ e_2 \in P_2,$$
$$(iv)\ \|\|T_1u\|_{P_1}\| < M_1 \text{ and } \|\|T_2u\|_{P_2}\| < M_2\ \forall u,v \in X\otimes Y.$$

where $k \in K_p$ *and* $r(k) < \frac{1}{2}$*. Then* $T : X\otimes Y \to X\otimes Y$ *defined by* $Tu = T_1u\otimes T_2u$*,* $u \in X\otimes Y$ *has a unique fixed point in* $X\otimes Y$*.*

Proof: Let $u, v \in X\otimes Y$. We have,

$$\begin{aligned}\|Tu - Tv\|_{K_p} &\preceq \|T_1u - T_1v\|_{P_1}\otimes\|T_2u\|_{P_2} + \|T_1v\|_{P_1}\otimes\|T_2u - T_2v\|_{P_2}\\ &= (\|T_1u - T_1v\|_{P_1}\otimes e_2)(e_1\otimes\|T_2u\|_{P_2})\\ &\quad + (\|T_1v\|_{P_1}\otimes e_2)(e_1\otimes\|T_2u - T_2v\|_{P_2})\\ &\preceq \frac{k}{M_2}\|u-v\|_{K_p}(e_1\otimes\|T_2u\|_{P_2}) + (\|T_1v\|_{P_1}\otimes e_2)\frac{k}{M_1}\|u-v\|_{K_p}\end{aligned}$$

Since K_p is normal, so, taking projective norm in $\mathbb{A}_1\otimes_\gamma\mathbb{A}_2$,

$$\begin{aligned}\|\|Tu - Tv\|_{K_p}\| &\leqslant \frac{1}{M_2}\|k\|u-v\|_{K_p}\|\|e_1\otimes\|T_2u\|_{P_2}\| + \|\|T_1v\|_{P_1}\otimes e_2\|\frac{1}{M_1}\|k\|u-v\|_{K_p}\|\\ &= \frac{1}{M_2}\|k\|u-v\|_{K_p}\|\|\|T_2u\|_{P_2}\| + \|\|T_1v\|_{P_1}\|\frac{1}{M_1}\|k\|u-v\|_{K_p}\|\\ &< \|2k\|(u-v)\|_{K_p}\|\end{aligned}$$

Now, proceeding as in Theorem 3.1, we can show that T has a unique fixed point in $X\otimes Y$. □

Theorem 3.4 *In the above theorem, if the conditions* (i) *and* (ii) *are replaced by*

$$(i)\ \|T_1u - T_1v\|_{P_1}\otimes e_2 \preceq \frac{k}{M_2}[\|u - Tv\|_{K_p} + \|v - Tu\|_{K_p}],$$
$$(ii)\ e_1\otimes\|T_2u - T_2v\|_{P_2} \preceq \frac{k}{M_1}[\|u - Tv\|_{K_p} + \|v - Tu\|_{K_p}]\ \forall u,v\in X\otimes Y,$$
$$(iii)\ e_1 \in P_1,\ e_2 \in P_2,$$
$$(iv)\ \|\|T_1u\|_{P_1}\| < M_1 \text{ and } \|\|T_2u\|_{P_2}\| < M_2\ \forall u,v \in X\otimes Y.$$

then T *has a unique fixed point in* $X\otimes Y$*, where* $k\in K_p$ *and* $r(k) < \frac{1}{4}$*.*

4. Main Results: (c) Stability of iteration scheme converging to the fixed point in cone normed spaces over Banach algebra:

In [6], Asadi et al. generalized the results of Qing and Rhoades [33](regarding the T-stability of Picard's iteration scheme in metric spaces) to cone metric spaces. An iteration procedure $x_{n+1} = f(T, x_n)$ is said to be T-stable with respect to T on a metric space X, if $\{x_n\}$ converges to a fixed point q of T and whenever $\{y_n\}$ is a sequence in X with $\lim_{n\to\infty} d(y_{n+1}, f(T, y_n)) = 0$, we have $\lim_{n\to\infty} y_n = q$.

Lemma 4.1 *Let P be a normal cone with normal constant K in a Banach algebra $\mathbb{A}$. Let $\{a_n\}$ and $\{b_n\}$ be two sequences in $\mathbb{A}$ satisfying the following inequality:*

$$a_{n+1} \preceq ha_n + b_n,$$

where $h \in P$ with $r(h) < 1$ and $b_n \to 0$ as $n \to \infty$. Then $a_n \to 0$ as $n \to \infty$.

Proof: Let m be a positive integer. By recursion we have

$$a_{n+1} \preceq b_n + hb_{n-1} + ... + h^m b_{n-m} + h^{m+1} a_{n-m}$$

Since P is normal,

$$\|a_{n+1}\| \leqslant K\|b_n + hb_{n-1} + ... + h^m b_{n-m}\| + K\|h^{m+1}\|\|a_{n-m}\|$$

By Lemma 1.5, and hypothesis $a_n \to 0$ *as* $n \to \infty$. □

Theorem 4.1 *Let $(X, \|.\|_p)$ be a cone Banach space over the unital Banach algebra $\mathbb{A}$, and P be a solid cone (not necessarily normal cone) in $\mathbb{A}$. Suppose S and T be self mappings on X satisfying*

$$\|Sx - Ty\|_p \preceq \alpha\|x - y\|_p + \beta(\|x - Sx\|_p + \|y - Ty\|_p) + \gamma(\|x - Ty\|_p + \|y - Sx\|_p)$$

for all $x, y \in X$, where $\alpha, \beta, \gamma \in P$ commute with each other and $r(\beta+\gamma)+r(\alpha+\beta+\gamma) < 1$. Then S and T have a common unique fixed point q in X.

Proof: Let $x_0 \in X$. We define a sequence $\{x_n\}$ by $x_{2n+1} = Sx_{2n},\ x_{2n+2} = Tx_{2n+1},\ n = 0, 1, 2, ...$. Now,

$$\begin{aligned}
\|x_{2n+1} - x_{2n+2}\|_p &= \|Sx_{2n} - Tx_{2n+1}\|_p \\
&\preceq \alpha\|x_{2n} - x_{2n+1}\|_p + \beta(\|x_{2n} - Sx_{2n}\|_p + \|x_{2n+1} - Tx_{2n+1}\|_p) \\
&\qquad + \gamma(\|x_{2n+1} - Sx_{2n}\|_p + \|x_{2n} - Tx_{2n+1}\|_p) \\
&= (\alpha + \beta + \gamma)\|x_{2n} - x_{2n+1}\|_p + (\beta + \gamma)\|x_{2n+1} - x_{2n+2}\|_p
\end{aligned}$$

$$\Rightarrow \|x_{2n+1} - x_{2n+2}\|_p \preceq (e^* - \beta - \gamma)^{-1}(\alpha + \beta + \gamma)\|x_{2n} - x_{2n+1}\|_p$$

(e^* being the unit element of $\mathbb{A}$.) Similarly it can be shown that

$$\|x_{2n+3} - x_{2n+2}\|_p \preceq (e^* - \beta - \gamma)^{-1}(\alpha + \beta + \gamma)\|x_{2n+2} - x_{2n+1}\|_p$$

Therefore for all n,

$$\|x_{n+1}-x_{n+2}\|_p \preceq (e^*-\beta-\gamma)^{-1}(\alpha+\beta+\gamma)\|x_n-x_{n+1}\|_p$$

Now,

$$\begin{aligned} r((e^*-\beta-\gamma)^{-1}(\alpha+\beta+\gamma)) &\leqslant r(e^*-\beta-\gamma)^{-1})r(\alpha+\beta+\gamma) \\ &\leqslant \frac{r(\alpha+\beta+\gamma)}{1-r(\beta+\gamma)} < 1 \end{aligned}$$

Hence from Lemma 1.2, Lemma 1.3 , Lemma 1.5 and Lemma 1.8(see [22]) we have, $\{x_n\}$ is a Cauchy sequence and converges to some z as $n \to \infty$. Now

$$\begin{aligned} \|z-Tz\|_p &\preceq \|z-x_{2n+1}\|_p + \|x_{2n+1}-Tz\|_p \\ &\preceq \|z-x_{2n+1}\|_p + \|Sx_{2n}-Tz\|_p \\ &\preceq \|z-x_{2n+1}\|_p + \alpha\|x_{2n}-z\|_p + \beta(\|x_{2n}-Sx_{2n}\|_p + \|z-Tz\|_p) \\ &\quad + \gamma(\|z-Sx_{2n}\|_p + \|x_{2n}-Tz\|_p) \\ \Rightarrow \|z-Tz\|_p &\preceq (\beta+\gamma)^{-1}[\|z-x_{2n+1}\|_p + \alpha\|x_{2n}-z\|_p + \beta\|x_{2n}-x_{2n+1}\|_p \\ &\quad + \gamma\|z-x_{2n+1}\|_p + \gamma\|x_{2n}-z\|_p)] \end{aligned}$$

By Lemma 1.1, Lemma 1.4 and Lemma 1.8, it is clear that right hand side of the above inequality is a c-sequence, this means $z = Tz$.

$$\begin{aligned} \|Sz-z\|_p &= \|Sz-Tz\|_p \\ &\preceq \alpha\|z-z\|_p + \beta(\|z-Sz\|_p + \|z-Tz\|_p) \\ &\quad + \gamma(\|z-Sz\|_p + \|z-Tz\|_p) \\ \Rightarrow \|Sz-z\|_p &\preceq (\beta+\gamma)\|z-Sz\|_p \end{aligned}$$

Since, $\beta+\gamma \preceq \alpha+\beta+\gamma$ and $r(\alpha+\beta+\gamma) < 1$. Hence by Lemma 1.11, $\|z-Sz\|_p = 0$, so $Sz = z$.
To show uniqueness: Let z and q be two distinct common fixed points of S and T.

$$\begin{aligned} \|z-q\|_p &= \|Sz-Tq\|_p \\ &\preceq \alpha\|z-q\|_p + \beta(\|z-Sz\|_p + \|q-Tq\|_p) \\ &\quad + \gamma(\|q-Sz\|_p + \|z-Tq\|_p) \\ \Rightarrow \|z-q\|_p &\preceq (\alpha+2\gamma)\|z-q\|_p \end{aligned}$$

Since, $\alpha+2\gamma \preceq (\beta+\gamma)+(\alpha+\beta+\gamma)$, by Lemma 1.10 we have $r(\beta+\gamma+\alpha+\beta+\gamma) \leqslant r(\beta+\gamma)+r(\alpha+\beta+\gamma) < 1$. Hence by Lemma 1.11, $\|z-q\|_p = 0$, so $z = q$. □

Theorem 4.2 *Let $(X, \|.\|_p)$ be a cone Banach space over the unital Banach algebra $\mathbb{A}$ (having unit e^*), and P be a solid cone in $\mathbb{A}$. Suppose T be a self mapping on X satisfying*

$$\|Tx-Ty\|_p \preceq \alpha\|x-y\|_p + \beta(\|x-Tx\|_p + \|y-Ty\|_p) + \gamma(\|x-Ty\|_p + \|y-Tx\|_p)$$

for all $x, y \in X$, where $\alpha, \beta, \gamma \in P$ commute with each other and $r(\beta+\gamma)+r(\alpha+\beta+\gamma)<1$. Then T has a unique fixed point q in X.

Now, we discuss stability of an iteration scheme (see [1]) converging to the common fixed point of S and T on the cone Banach space over Banach algebra. For $x_0 \in X$, we consider the following iteration scheme:

$$x_{2n+1} = Sx_{2n},\ x_{2n+2} = Tx_{2n+1},\ n = 0, 1, 2, ... \tag{5}$$

For the cone Banach space X with a normal cone P (with normal constant 1), the above iteration scheme is said to be stable with respect to S and T, if $\{x_n\}$ converges to the unique common fixed point q of S and T, and whenever $\{y_n\}$ is a sequence in X with

$$\lim_{n\to\infty} \|y_{2n+1} - Sy_{2n}\|_p (= \epsilon_{2n}) = 0, \text{ and} \tag{6}$$

$$\lim_{n\to\infty} \|y_{2n+2} - Ty_{2n+1}\|_p (= \epsilon_{2n+1}) = 0, \tag{7}$$

we have $\lim_{n\to\infty} y_n = q$.

Here we derive the following condition for stability of the iteration scheme (5).

Theorem 4.3 *Let $(X, \|.\|_p)$ be a cone Banach space over the Banach algebra $\mathbb{A}$ (having unit e^*) with the normal cone P(normal constant 1). If S and T are self mappings on X satisfying the condition*

$$\|Su - Tv\|_p \preceq \alpha\|u - v\|_p + \beta(\|u - Su\|_p + \|v - Tv\|_p) + \gamma(\|u - Tv\|_p + \|v - Su\|_p)$$

then the iteration scheme (5) is stable with respect to S and T if $\alpha, \beta, \gamma \in P$ commute with each other and $r(\beta+3\gamma)+r(\alpha+\beta+\gamma)<1$.

Proof: We have, for the common unique fixed point q of S and T, $\lim_{n\to\infty} x_n = q$. So,

$$\lim_{n\to\infty} \|x_{2n} - Sx_{2n}\|_p = 0 \text{ and } \lim_{n\to\infty} \|x_{2n+1} - Tx_{2n+1}\|_p = 0.$$

Now,

$$\begin{aligned}\|y_{2n+1} - q\|_p &\preceq \|y_{2n+1} - x_{2n+2}\|_p + \|x_{2n+2} - q\|_p \\ &\preceq \|y_{2n+1} - Sy_{2n}\|_p + \|Sy_{2n} - x_{2n+2}\|_p + \|x_{2n+2} - q\|_p \\ &= \epsilon_{2n} + \|Sy_{2n} - Tx_{2n+1}\|_p + \|x_{2n+2} - q\|_p \end{aligned} \tag{8}$$

$$\begin{aligned}\|Sy_{2n} - Tx_{2n+1}\|_p \preceq\ & \alpha\|y_{2n} - x_{2n+1}\|_p + \beta(\|y_{2n} - Sy_{2n}\|_p + \|x_{2n+1} - Tx_{2n+1}\|_p) \\ & + \gamma(\|x_{2n+1} - Sy_{2n}\|_p + \|y_{2n} - Tx_{2n+1}\|_p)\end{aligned}$$

$$\begin{aligned}
&\preceq \alpha\|y_{2n}-x_{2n+1}\|_p+\beta(\|y_{2n}-Sy_{2n}\|_p+\|x_{2n+1}-Tx_{2n+1}\|_p)\\
&\quad+\gamma(\|x_{2n+1}-Tx_{2n+1}\|_p+\|Tx_{2n+1}-Sy_{2n}\|_p+\|y_{2n}-Sy_{2n}\|_p+\|Sy_{2n}-Tx_{2n+1}\|_p)\\
&=\alpha\|y_{2n}-x_{2n+1}\|_p+(\beta+\gamma)\|y_{2n}-Sy_{2n}\|_p+(\beta+\gamma)\|x_{2n+1}-Tx_{2n+1}\|_p\\
&\quad+2\gamma\|Sy_{2n}-Tx_{2n+1}\|_p\\
&\preceq \alpha\|y_{2n}-x_{2n+1}\|_p+(\beta+\gamma)(\|y_{2n}-x_{2n+1}\|_p+\|x_{2n+1}-Tx_{2n+1}\|_p\\
&\quad+\|Tx_{2n+1}-Sy_{2n}\|_p)+(\beta+\gamma)\|x_{2n+1}-Tx_{2n+1}\|_p+2\gamma\|Sy_{2n}-Tx_{2n+1}\|_p\\
&=(\alpha+\beta+\gamma)\|y_{2n}-x_{2n+1}\|_p+(2\beta+2\gamma)\|x_{2n+1}-Tx_{2n+1}\|_p\\
&\quad+(\beta+3\gamma)\|Sy_{2n}-Tx_{2n+1}\|_p\\
&\preceq(\alpha+\beta+\gamma)\|y_{2n}-q\|_p+(\alpha+\beta+\gamma)\|q-x_{2n+1}\|_p\\
&\quad+(2\beta+2\gamma)\|x_{2n+1}-Tx_{2n+1}\|_p+(\beta+3\gamma)\|Sy_{2n}-Tx_{2n+1}\|_p
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \|Sy_{2n}-Tx_{2n+1}\|_p &\preceq \frac{\alpha+\beta+\gamma}{e^*-\beta-3\gamma}(\|y_{2n}-q\|_p+\|q-x_{2n+1}\|_p)\\
&\quad+\frac{2\beta+2\gamma}{e^*-\beta-3\gamma}\|x_{2n+1}-Tx_{2n+1}\|_p
\end{aligned}$$

So, from (8), we have,

$$\begin{aligned}
\|y_{2n+1}-q\|_p \preceq \epsilon_{2n}&+\|x_{2n+2}-q\|_p\\
&+\frac{\alpha+\beta+\gamma}{e^*-\beta-3\gamma}[\|y_{2n}-q\|_p+\|q-x_{2n+1}\|_p]\\
&+\frac{2\beta+2\gamma}{e^*-\beta-3\gamma}\|x_{2n+1}-Tx_{2n+1}\|_p
\end{aligned}$$

We take $a_n=\|y_n-q\|_p$ and

$$\begin{aligned}
b_n=\epsilon_{2n}+\|x_{2n+2}-q\|_p+\frac{\alpha+\beta+\gamma}{e^*-\beta-3\gamma}\|q-x_{2n+1}\|_p\\
+\frac{2\beta+2\gamma}{e^*-\beta-3\gamma}\|x_{2n+1}-Tx_{2n+1}\|_p
\end{aligned}$$

Now, $b_n\to 0$ as $n\to\infty$. Also,

$$\begin{aligned}
r((e^*-\beta-3\gamma)^{-1}(\alpha+\beta+\gamma)) &\leqslant r(e^*-\beta-3\gamma)^{-1}r(\alpha+\beta+\gamma)\\
&\leqslant \frac{r(\alpha+\beta+\gamma)}{1-r(\beta+3\gamma)}<1
\end{aligned}$$

So, by the Lemma 4.1,

$$\lim_{n\to\infty}a_{2n+1}=\lim_{n\to\infty}\|y_{2n+1}-q\|_p=0.$$

Again,

$$\begin{aligned}
\|y_{2n+2}-q\|_p &\preceq \|y_{2n+2}-x_{2n+1}\|_p+\|x_{2n+1}-q\|_p\\
&\preceq \|y_{2n+2}-Ty_{2n+1}\|_p+\|Ty_{2n+1}-Sx_{2n}\|_p+\|x_{2n+1}-q\|_p \qquad (9)
\end{aligned}$$

$$\begin{aligned}\|Sx_{2n}-Ty_{2n+1}\|_p &\preceq \alpha\|x_{2n}-y_{2n+1}\|_p+\beta(\|x_{2n}-Sx_{2n}\|_p+\|y_{2n+1}-Ty_{2n+1}\|_p)\\ &\quad+\gamma(\|y_{2n+1}-Sx_{2n}\|_p+\|x_{2n}-Ty_{2n+1}\|_p)\\ &\preceq \alpha\|x_{2n}-y_{2n+1}\|_p+\beta(\|x_{2n}-Sx_{2n}\|_p+\|y_{2n+1}-x_{2n}\|_p\\ &\quad+\|x_{2n}-Sx_{2n}\|_p+\|Sx_{2n}-Ty_{2n+1}\|_p)\\ &\quad+\gamma(\|y_{2n+1}-x_{2n}\|_p+\|x_{2n}-Sx_{2n}\|_p\\ &\quad+\|x_{2n}-Sx_{2n}\|_p+\|Sx_{2n}-Ty_{2n+1}\|_p)\\ &=(\alpha+\beta+\gamma)\|x_{2n}-y_{2n+1}\|_p+(2\beta+2\gamma)\|x_{2n}-Sx_{2n}\|_p\\ &\quad+(\beta+\gamma)\|Sx_{2n}-Ty_{2n+1}\|_p\\ &\preceq(\alpha+\beta+\gamma)\|x_{2n}-q\|_p+(\alpha+\beta+\gamma)\|q-y_{2n+1}\|_p\\ &\quad+(2\beta+2\gamma)\|x_{2n}-Sx_{2n}\|_p+(\beta+\gamma)\|Sx_{2n}-Ty_{2n+1}\|_p\end{aligned}$$

$$\begin{aligned}\Rightarrow \|Sx_{2n}-Ty_{2n+1}\|_p &\preceq \frac{\alpha+\beta+\gamma}{e^*-\beta-\gamma}(\|x_{2n}-q\|_p+\|q-y_{2n+1}\|_p)\\ &\quad+\frac{2\beta+2\gamma}{e^*-\beta-\gamma}\|x_{2n}-Sx_{2n}\|_p\end{aligned}$$

From (9), we have,

$$\begin{aligned}\|y_{2n+2}-q\|_p &\preceq \epsilon_{2n+1}+\|x_{2n+1}-q\|_p\\ &\quad\frac{\alpha+\beta+\gamma}{e^*-\beta-\gamma}(\|y_{2n+1}-q\|_p+\|q-x_{2n}\|_p)\\ &\quad+\frac{2\beta+2\gamma}{e^*-\beta-\gamma}\|x_{2n}-Sx_{2n}\|_p\end{aligned}$$

Now, as in the first part, we have

$$\lim_{n\to\infty} a_{2n+2}=\lim_{n\to\infty}\|y_{2n+2}-q\|_p=0$$

Thus, for all n we have, $\lim_{n\to\infty}\|y_n-q\|_p=0$. Hence the given iteration is stable with respect to S and T.

We can also show that, if
$\lim_{n\to\infty}\|y_n-q\|_p=0$, then

$$\lim_{n\to\infty}\|y_{2n+1}-Sy_{2n}\|_p(=\epsilon_{2n},say)=0,\ and\ \lim_{n\to\infty}\|y_{2n+2}-Ty_{2n+1}\|_p(=\epsilon_{2n+1},say)=0$$

$$\|y_{2n+1}-Sy_{2n}\|_p\preceq\|y_{2n+1}-q\|_p+\|q-Sy_{2n}\|_p$$

$$\begin{aligned}\|q-Sy_{2n}\|_p=\|Sy_{2n}-Tq\|_p &\preceq \alpha\|y_{2n}-q\|_p+\beta(\|q-Tq\|_p+\|y_{2n}-Sy_{2n}\|_p)\\ &\quad+\gamma(\|q-Sy_{2n}\|_p+\|y_{2n}-Tq\|_p)\\ &\preceq \alpha\|y_{2n}-q\|_p+\beta\|y_{2n}-q\|_p+\gamma\|y_{2n}-Tq\|_p\\ &\quad+\beta\|q-Sy_{2n}\|_p+\gamma\|q-Sy_{2n}\|_p\\ \Rightarrow \|q-Sy_{2n}\|_p &\preceq \frac{\alpha+\beta+\gamma}{e^*-\beta-\gamma}\|y_{2n}-q\|_p\end{aligned}$$

$$\begin{aligned} r(e^* - (\beta+\gamma)^{-1}(\alpha+\beta+\gamma)) &\leqslant r(e^* - (\beta+\gamma))^{-1} r(\alpha+\beta+\gamma) \\ &\leqslant \frac{r(\alpha+\beta+\gamma)}{1 - r(\beta+\gamma)} < 1 \end{aligned}$$

Now, using normality condition we get, $\|q - Sy_{2n}\|_p \to 0$ *as* $n \to \infty$. So, $\epsilon_{2n} \to 0$ *as* $n \to \infty$. Again,

$$\|y_{2n+2} - Ty_{2n+1}\|_p \preceq \|y_{2n+2} - q\|_p + \|q - Ty_{2n+1}\|_p$$

$$\begin{aligned} \|q - Ty_{2n+1}\|_p &= \|Sq - Ty_{2n+1}\|_p \\ &\preceq \alpha\|q - y_{2n+1}\|_p + \beta[\|q - Sq\|_p + \|y_{2n+1} - Ty_{2n+1}\|_p] \\ &+ \gamma[\|q - Ty_{2n+1}\|_p + \|y_{2n+1} - Sq\|_p] \\ &\preceq \alpha\|q - y_{2n+1}\|_p + \beta\|y_{2n+1} - q\|_p + \gamma\|y_{2n+1} - Sq\|_p \\ &+ \gamma[\|q - Ty_{2n+1}\|_p + \beta\|q - Ty_{2n+1}\|_p \\ \Rightarrow \|q - Ty_{2n+1}\|_p &\preceq \frac{\alpha+\beta+\gamma}{e^* - \beta - \gamma}\|y_{2n+1} - q\|_p \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

So, $\epsilon_{2n+1} \to 0$ *as* $n \to \infty$. □

Theorem 4.4 *For the cone Banach space* $(X \otimes Y, \|.\|_{K_p})$ *over the unital Banach algebra* $\mathbb{A}_1 \otimes_\gamma \mathbb{A}_2$ *with the normal cone* K_p *(normal constant 1), let* T *be the self mapping on* $X \otimes Y$ *satisfying the condition:*

$$\begin{aligned} \|Tu - Tv\|_{K_p} &\preceq \alpha\|u - v\|_{K_p} + \beta(\|u - Tu\|_{K_p} + \|v - Tv\|_{K_p}) \\ &+ \gamma(\|u - Tv\|_{K_p} + \|v - Tu\|_{K_p}) \end{aligned} \tag{10}$$

for all $u, v \in X \otimes Y$*, where* $\alpha, \beta, \gamma \in K_p$ *commute with each other and* $r(\beta + 3\gamma) + r(\alpha+\beta+\gamma) < 1$*. Then the iteration scheme:*

$$\begin{aligned} &x_0 \in X \otimes_\gamma Y, \\ &x_{n+1} = Tx_n,\ n = 0, 1, 2, ... \end{aligned}$$

converging to the fixed point of T *is stable with respect to* T*.*

Example 4.1 We take $X = l^1$ and $\mathbb{A}_1 = (l^1, \|.\|)$ over $\mathbb{R}$.
Let $P_1 = \{\{x_n\}_{n\geqslant 1} \in \mathbb{A}_1, x_n \geqslant 0\ \forall n\}$. Then $\|.\|_{P_1} : l^1 \to \mathbb{A}_1$ defined by $\|a_i\|_{P_1} = \{|a_{i_k}|\}_k$, is a cone norm on X. Clearly, $(X, \|.\|_{P_1})$ is also a cone Banach space over $\mathbb{A}_1$.
Next we take $Y = [0,1]$, $\mathbb{A}_2 = ([0,1], \|.\|)$, $P_2 = \{y \in [0,1] : y \geqslant 0\}$. Then $\|.\|_{P_2} : Y \to \mathbb{A}_2$ defined by $\|y\|_{P_2} = y$ is a cone norm on Y. Clearly, $(Y, \|.\|_{P_2})$ is also a cone Banach space over $\mathbb{A}_2$.
Now, $\|.\|_{K_p} : X \otimes Y \to \mathbb{A}_1 \otimes_\gamma \mathbb{A}_2$ i.e., $\|.\|_{K_p} : l^1 \otimes [0,1] \to l^1 \otimes_\gamma [0,1]$ is defined by

$\|u\|_{K_p} = \| \sum_i a_i \otimes y_i\|_{K_p} = \sum_i \|\{a_{i_k}\}\|_{P_1} \otimes \|y_i\|_{P_2} = \sum_i \{|a_{i_k}|\}_k \otimes y_i = \sum_i \{|a_{i_k}| y_i\}_k$
Thus $(l^1 \otimes [0,1], \|.\|_{K_p})$ is a cone Banach space over the Banach algebra $\mathbb{A}_1 \otimes_\gamma \mathbb{A}_2$ i.e., $l^1([0,1])$ with unit $e = \{1, 0, 0, ...\}$.
We define $T_1 : l^1 \otimes [0,1] \to l^1$ by $T_1(\sum_i a_i \otimes y_i) = \dfrac{1}{4} \sum_i \{a_{i_k} y_i\}_k$, and $T_2 : l^1 \otimes [0,1] \to [0,1]$ by $T_2(\sum_i a_i \otimes y_i) = \dfrac{1}{2}$. Let T be the self mapping on $l^1 \otimes [0,1]$ such that

$$\begin{aligned} T(\sum_i a_i \otimes y_i) &= T_1(\sum_i a_i \otimes y_i) \otimes T_2(\sum_i a_i \otimes y_i) \\ &= \frac{1}{4} \sum_i \{a_{i_k} y_i\}_k \otimes \frac{1}{2} \\ &= \frac{1}{8} \sum_i a_i \otimes y_i \text{ (taking norm in } l^1([0,1])) \end{aligned}$$

$$\begin{aligned} \|Tu - Tv\|_{K_p} \preceq \|\frac{u}{8} - \frac{v}{8}\|_{K_p} &= 2\|\frac{u}{16} - \frac{v}{16}\|_{K_p} \\ &= 2[\|\frac{u}{16} - \frac{v}{16} + \frac{v}{16} - \frac{v}{128} + \frac{u}{16} - \frac{u}{128} + \frac{u}{128} - \frac{u}{16} + \frac{v}{128} - \frac{v}{16}\|_{K_p}] \\ &\preceq \frac{1}{8}\|u - v\|_{K_p} + \frac{1}{8}[\|u - \frac{u}{8}\|_{K_p} + \|v - \frac{v}{8}\|_{K_p}] \\ &\quad + \frac{1}{8}[\|v - \frac{u}{8}\|_{K_p} + \|u - \frac{v}{8}\|_{K_p}] \\ &= \frac{1}{8} e\|u - v\|_{K_p} + \frac{1}{8}[e\|u - Su\|_{K_p} + e\|v - Tv\|_{K_p}] \\ &\quad + \frac{1}{8}[e\|v - Su\|_{K_p} + e\|u - Tv\|_{K_p}] \end{aligned}$$

Thus T satisfies (10) with $\alpha = \beta = \gamma = \frac{1}{8}e$. Also $\alpha, \beta, \gamma \in K_p$, and they commute with each other.
$r(\beta + 3\gamma) = \lim_{n\to\infty} \| \left(\frac{1}{2}e\right)^n \|^{\frac{1}{n}} = \dfrac{1}{2}$ and
$r(\alpha + \beta + \gamma) = \lim_{n\to\infty} \| \left(\frac{3}{8}e\right)^n \|^{\frac{1}{n}} = \dfrac{3}{8}$. Hence $r(\beta + 3\gamma) + r(\alpha + \beta + \gamma) < 1$.
Let $b_n = \sum_i \{\dfrac{a_i}{n}\}_{n \geqslant 1}, y_n = \frac{1}{n+1}\ \forall n = 1, 2, ...$
$Tb_n = \dfrac{1}{8} \sum_i \{\dfrac{a_i}{n}\}_{n\geqslant 1}$. Now, $q = 0$ is the unique fixed point of T and $\lim_{n\to\infty} y_n = 0$. Also we have,

$$\lim_{n\to\infty} \|b_{n+1} - Tb_n\|_{K_p} \to 0$$

Hence the iteration scheme is T-stable for the cone Banach space $l^1 \otimes [0,1]$ over the Banach algebra $l^1 \otimes_\gamma [0,1]$.

Remark 4.1 *There are many other iteration schemes viz., Picard-Mann hybrid iteration, Mann iteration, Ishikawa iteration etc. [26], for which we can discuss* *the stability with respect to some self mappings on cone normed spaces as well as PCNTPS over Banach algebra.*

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Dipankar Das
Department of Mathematics,
Gauhati University,
Guwahati-781014,
Assam, India
dipankardasguw@yahoo.com

and

Nilakshi Goswami
Department of Mathematics,
Gauhati University,
Guwahati-781014,
Assam, India
nila_g2003@yahoo.co.in

and

Vishnu Narayan Mishra
Department of Mathematics,
Indira Gandhi National Tribal University,
Lalpur, Amarkantak, Anuppur 484 887, India

L. 1627 Awadh Puri Colony Beniganj,
Phase -III, Opposite - Industrial Training Institute (I.T.I.),
Ayodhya Main Road Faizabad 224 001,
Uttar Pradesh, India
vishnunarayanmishra@gmail.com
vishnu_narayanmishra@yahoo.co.in