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Multiplicity Results for Kirchhoff Type Elliptic Problems with Hardy Potential

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ABSTRACT: In this paper, we are concerned with the existence of solutions for fourth-order Kirchhoff type elliptic problems with Hardy potential. In fact, employing a consequence of the local minimum theorem due to Bonanno and mountain pass theorem we look into the existence results for the problem under algebraic conditions with the classical Ambrosetti-Rabinowitz (AR) condition on the nonlinear term. Furthermore, by combining two algebraic conditions on the nonlinear term using two consequences of the local minimum theorem due to Bonanno we ensure the existence of two solutions, applying the mountain pass theorem given by Pucci and Serrin we establish the existence of third solution for our problem.

Key Words: p-biharmonic type operators, Navier condition, Hardy potential, Variational methods, Critical point theory.

Contents

1. Introduction

Consider the following p-biharmonic equation with Hardy potential of fourth-order Kirchhoff-type elliptic problem

$$\begin{cases} M\Big(\displaystyle\int_\Omega |\Delta u|^p dx\Big)\Delta_p^2 u - \dfrac{a}{|x|^{2p}}|u|^{p-2}u = \lambda f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N (N \geq 3)$ containing the origin and with smooth boundary $\partial\Omega$, $1 < p < \frac{N}{2}$, $\Delta_p^2 u = \Delta(|\Delta u|^{p-2}\Delta u)$ is the p-biharmonic operator of fourth order, λ is nonnegative parameter, $M : [0, +\infty) \to \mathbb{R}$ is continuous function and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is an L^2-Carathéodory function.

The problem (1.1) is related to the stationary problem

$$\rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$

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for $0 < x < L$, $t \geq 0$, where $u = u(x,t)$ is the lateral displacement at the space coordinate x and the time t, E the Young modulus, ρ the mass density, h the cross-section area, L the length and ρ_0 the initial axial tension, proposed by Kirchhoff [21] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. Kirchhoff model can also be used for describing the dynamics of an axially moving string. In recent years, axially moving string-like continua such as wires, belts, chains, band-saws have been subjects of the study of researchers (see [35]). Similar nonlocal problems also model several physical and biological systems where u describes a process that depends on the average of itself, for example, the population density. Problems of Kirchhoff-type have been widely investigated, we refer the reader to papers [1,2,3,4,5,6,15,16,32,33] and the references therein.

Fourth-order equations can describe the static form change of beam or the sport of rigid body. In [22], Lazer and McKenna have pointed out that this type of nonlinearity furnishes a model to study travelling waves in suspension bridges. Since then more nonlinear biharmonic equations and p-biharmonic equations have been studied. Existence and multiplicity of solutions of nonlinear fourth order differential equations have been deserved a great deal of interest, for instance see [7,9,10,11,20,24,25,26,27].

Recently, combined problems of Kirchhoff-type with p-biharmonic operator have been widely investigated such that many researchers have discussed the existence of at least one solution, or multiple solutions, or even many solutions for such problems with different method. we refer the reader to the papers [14,19,28,38] and references therein. For example, in [28] employing variational methods and critical point theory, Massar et al. ensured the existence of infinitely many solutions the following perturbed p-biharmonic Kirchhoff-type problem

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) - \left[M(\int_\Omega |\nabla u|^p dx)\right]^{p-1} \Delta_p u + \rho|u|^{p-2}u = \lambda f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $p > \max\{1, \frac{N}{2}\}$, $\lambda > 0$ is a real number, $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a bounded smooth domain, $\rho > 0$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is an continues function and $M : [0, +\infty) \to \mathbb{R}$ is continuous function, while in [14] using variational methods and critical point theory, multiplicity results of nontrivial and nonnegative solutions for the same problem were established. Xiu et al. in [38] by employing variational method studied multiplicity of solutions for the following p-biharmonic equation

$$\begin{cases} (a + b\int_\Omega (|\Delta u|^p + |u|^p)\, dx)(\Delta_p^2 u + |u|^{p-2}u) & \\ \quad = h(x)|u|^{r-2}u + H(x)|u|^{q-2}u + g(x), & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $1 < p < \frac{N}{2}$, $\Delta_p^2 u = \Delta(|\Delta u|^{p-2}\Delta u)$ is the p-biharmonic operator of fourth order, $\Omega \subset \mathbb{R}^N$ is an unbounded domain, and $h(x), H(x)$ and $g(x)$ are nonnegative functions with sufficient conditions.

On the other hand, singular boundary value problems arise in the context of chemical heterogeneous catalysts and chemical catalyst kinetics, in the theory of heat conduction in eletirically conducting materials, singular minimal surfaces, as well as in the study of non-Newtonian fluids and boundary layer phenomena for viscous fluids. Furthermore, nonlinear singular elliptic equations are also encountered in glocial advance, in transport of coal slurries down conveyor belts and in several other geophysical and industrial contents. For the use of singular problem in the mathematical literature, see [20,23,29]. In recent years, some interesting results for singular p-biharmonic equation of Kirchhoff-type were obtained. For instance, Xu and Bai in [39] by using critical point theory, discussed the existence of infinitely many weak solutions for similar problem to (1.1).

In the present article, we establish the existence of two solutions for the problem (1.1) using a consequence of the local minimum theorem due to Bonanno and mountain pass theorem under some algebraic conditions with the classical Ambrosetti-Rabinowitz (AR) condition on the nonlinear term. Moreover, by combining two algebraic conditions on the nonlinear term employing two consequences of the local minimum theorem due to Bonanno we guarantee the existence of two solutions, applying the mountain pass theorem given by Pucci and Serrin ([30]) we establish the existence of third solution for the problem (1.1).

For a through on the subject, we also refer the reader to [12,18,17].

2. Preliminaries

For a given nonempty set X, and two functionals $\Phi, \Psi : X \to \mathbb{R}$, we define the following functions

$$\beta(r_1, r_2) = \inf_{v\in\Phi^{-1}(r_1,r_2)} \frac{\sup_{u\in\Phi^{-1}(r_1,r_2)} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)},$$

$$\rho_1(r_1, r_2) = \sup_{v\in\Phi^{-1}(r_1,r_2)} \frac{\Psi(v) - \sup_{u\in\Phi^{-1}(-\infty,r_1]} \Psi(u)}{\Phi(v) - r_1}$$

for all $r_1, r_2 \in \mathbb{R}, r_1 < r_2$, and

$$\rho_2(r) = \sup_{v\in\Phi^{-1}(r,+\infty)} \frac{\Psi(v) - \sup_{u\in\Phi^{-1}(-\infty,r]} \Psi(u)}{\Phi(v) - r}$$

for all $r \in \mathbb{R}$.

Theorem 2.1. *[8, Theorem 5.1] Let X be a real Banach space; $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact. Assume that there are $r_1, r_2 \in \mathbb{R}, r_1 < r_2$, such that*

$$\beta(r_1, r_2) < \rho_1(r_1, r_2).$$

Then, setting $I_\lambda := \Phi - \lambda\Psi$, for each $\lambda \in (\frac{1}{\rho_1(r_1,r_2)}, \frac{1}{\beta(r_1,r_2)})$ there is $u_{0,\lambda} \in \Phi^{-1}(r_1, r_2)$ such that $I_\lambda(u_{0,\lambda}) \le I_\lambda(u)$ for all $u \in \Phi^{-1}(r_1, r_2)$ and $I'_\lambda(u_{0,\lambda}) = 0$.

Theorem 2.2. *[8, Theorem 5.3] Let X be a real Banach space; $\Phi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact. Fix $\inf_X \Phi < r < \sup_X \Phi$ and assume that*

$$\rho_2(r) > 0,$$

and for each $\lambda > \frac{1}{\rho_2(r)}$, the functional $I_\lambda := \Phi - \lambda\Psi$ is coercive. Then for each $\lambda \in]\frac{1}{\rho_2(r)}, +\infty[$ there is $u_{0,\lambda} \in \Phi^{-1}(r, +\infty)$ such that $I_\lambda(u_{0,\lambda}) \leq I_\lambda(u)$ for all $u \in \Phi^{-1}(r, +\infty)$ and $I_\lambda'(u_{0,\lambda}) = 0$.

Let X denote the space $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ endowed with the norm

$$\|u\| = \left(\int_\Omega |\Delta u|^p dx\right)^{\frac{1}{p}}.$$

We recall the following Rellich inequality [13], which says that, for each $u \in X$,

$$\int_\Omega \frac{|u(x)|^p}{|x|^{2p}} dx \leq \frac{1}{H} \int_\Omega |\Delta u|^p \tag{2.1}$$

where the best constant is

$$H = \left(\frac{(p-1)N(N-2p)}{p^2}\right)^p. \tag{2.2}$$

Now, let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function such that there exists two positive constants m_0 and m_1 such that

$$m_0 \leq M(t) \leq m_1,$$

for all $t \in \mathbb{R}^+$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be an L^2-Carathéodory function, namely, $x \mapsto f(x,t)$ is continuous for almost every $x \in \Omega$, and for every $s > 0$ there exists a function $l_s \in L^2(\Omega)$ such that

$$\sup_{|t| \leq s} |f(x,t)| \leq l_s(x)$$

for almost every $x \in \Omega$. Set $p^* = \frac{pN}{N-p}$. By the Sobolev embedding theorem there exist a positive constant c such that

$$\|u\|_{L^{p^*}(\Omega)} \leq c\|u\|, \quad \forall u \in X,$$

where

$$c := \pi^{-\frac{1}{2}} N^{-\frac{1}{p}} \left(\frac{p-1}{N-p}\right)^{1-\frac{1}{p}} \left[\frac{\Gamma(1+\frac{N}{2})\Gamma(N)}{\Gamma(\frac{N}{p})\Gamma(N+1-\frac{N}{p})}\right]^{\frac{1}{N}},$$

see, [36]. Fixing $q \in [1, p^*)$, again from the Sobolev embedding theorem, there exists a positive constant c_q such that

$$\|u\|_{L^q(\Omega)} \le c_q \|u\|, \qquad \forall u \in X. \tag{2.3}$$

Thus the embedding $X \hookrightarrow L^q(\Omega)$ is compact. By Holder inequality, one has the upper bound

$$c_q \le \pi^{-\frac{1}{2}} N^{-\frac{1}{p}} \left(\frac{p-1}{N-p}\right)^{1-\frac{1}{p}} \left[\frac{\Gamma(1+\frac{N}{2})\Gamma(N)}{\Gamma(\frac{N}{p})\Gamma(N+1-\frac{N}{p})}\right]^{\frac{1}{N}} |\Omega|^{\frac{p^*-q}{p^*q}},$$

where $|\Omega|$ denote the Lebesgue measure of the open set Ω.

Fixing the real parameter λ, a function $u \in W^{1,p}(\Omega)$ is said to be a weak solution of (1.1) if for all $v \in W^{1,p}(\Omega)$,

$$\begin{aligned}&M\Big(\int_\Omega |\Delta u|^p dx\Big) \int_\Omega |\Delta u|^{p-2}\Delta u(x)\Delta v(x)dx - a\int_\Omega \frac{|u(x)|^{p-2}}{|x|^{2p}} u(x)v(x)dx \\ &= \lambda \int_\Omega f(x, u(x))v(x)dx.\end{aligned}$$

By assumption $m_0 > \frac{a}{H}$, we state the following proposition which we need in the proofs of our main result.

Proposition 2.1. *Let $T : X \to X$ be the operator defined by*

$$T(u)h = M\Big(\int_\Omega |\Delta u|^p dx\Big) \int_\Omega |\Delta u|^{p-2}\Delta u(x)\Delta h(x)dx - a\int_\Omega \frac{|u(x)|^{p-2}}{|x|^{2p}} u(x)h(x)dx$$

for every $u, v \in X$. Then, T admits a continuous inverse on X^*.

Proof: Since

$$\begin{aligned} T(u)h &= M\Big(\int_\Omega |\Delta u|^p dx\Big) \int_\Omega |\Delta u|^{p-2}\Delta u(x)\Delta h(x)dx \\ &\quad - a\int_\Omega \frac{|u(x)|^{p-2}}{|x|^{2p}} u(x)h(x)dx \\ &\ge m_0\|u\|^p - \frac{a}{H}\|u\|^p \\ &= \left(m_0 - \frac{a}{H}\right)\|u\|^p, \end{aligned}$$

and since $, m_0 > \frac{a}{H}$, this follows that T is coercive. Taking into account (2.2) of [34] for $p > 1$ there exists a positive constant C_p such that if $p \ge 2$, then

$$\big\langle |x|^{p-2}x - |y|^{p-2}y, x - y\big\rangle \ge C_p|x-y|^p,$$

if $1 < p < 2$, then

$$\big\langle |x|^{p-2}x - |y|^{p-2}y, x - y\big\rangle \ge C_p \frac{|x-y|^2}{(|x|+|y|)^{2-p}}$$

where $\langle .,. \rangle$ denotes the usual inner product in $\mathbb{R}^N$.Then, we observe that

$$\langle T(u) - T(v), u - v \rangle \geq C\|u - v\|^p > 0$$

for some $C > 0$ for every $u, v \in X$, which means that T is strictly monotone. Furthermore, since X is reflexive, for $u_n \to u$ strongly in X as $n \to +\infty$, one has $T(u_n) \to T(u)$ weakly in X^* as $n \to +\infty$. Hence, T is demicontinuous, so by [40, Theorem 26.A(d)], the inverse operator T^{-1} of T exists. T^{-1} is continuous. Indeed, let (ν_n) be a sequence of X^* such that $\nu_n \to \nu$ strongly in X^* as $n \to +\infty$. Let u_n and u in X such that $T^{-1}(\nu_n) = u_n$ and $T^{-1}(\nu) = u$. Taking in to account that T is coercive, one has that the sequence u_n is bounded in the reflexive space X. For a suitable subsequence, we have $u_n \to \widehat{u}$ weakly in X as $n \to +\infty$, which concludes

$$\lim_{n \to +\infty} \langle T(u_n) - T(u), u_n - \widehat{u} \rangle = \langle \nu_n - \nu, u_n - \widehat{u} \rangle = 0.$$

Note that if $u_n \to \widehat{u}$ weakly in X as $n \to +\infty$ and $T(u_n) \to T(\widehat{u})$ strongly in X^* as $n \to +\infty$, one has $u_n \to \widehat{u}$ strongly in X as $n \to +\infty$, and since T is continuous, we have $u_n \to \widehat{u}$ weakly in X as $n \to +\infty$ and $T(u_n) \to T(\widehat{u}) = T(u)$ strongly in X^* as $n \to +\infty$. Hence, taking into account that T is an injective, we have $u = \widehat{u}$. $\square$

3. Main results

In this section, we formulate our main results as follow.

Put

$$\widehat{M}(t) = \int_0^t M(s)ds, \qquad t \geq 0$$

and

$$F(x,t) = \int_0^t f(x,\xi)d\xi, \qquad (x,t) \in \Omega \times \mathbb{R}.$$

Choose $s > 0$ such that $B(0,s) \subset \Omega$, where $B(0,s)$ denotes the open ball in $\mathbb{R}^N$ of radius s with center at 0. Put

$$L = \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \int_{\frac{s}{2}}^{s} \Big| \frac{12r(N+1)}{s^3} - \frac{24N}{s^2} + \frac{9(N-1)}{sr} \Big|^p r^{N-1} dr.$$

For a nonnegative constant η and a positive constant δ with

$$(m_0 H - a)\eta^p \neq m_1 H L (c_q \delta)^p$$

we set

$$a_\eta(\delta) := p \frac{\int_\Omega \sup_{\|t\|_{L^q(\Omega)} \leq \eta} F(x,t)dx - \int_{B(0,\frac{s}{2})} F(x,\delta)dx}{(m_0 H - a)\eta^p - m_1 H L (c_q \delta)^p}.$$

We now present our main result as follows.

Theorem 3.1. *Suppose that* $0 < a < m_0H$ *(with* H *is as in (2.2)). Moreover, assume that there exist a nonnegative constant* η_1 *and two positive constants* η_2 *and* δ *with*

$$\frac{1}{c_q L^{\frac{1}{p}}}\eta_1 < \delta < \left(\frac{m_0H - a}{m_1LH}\right)^{\frac{1}{p}} \frac{\eta_2}{c_q} \tag{3.1}$$

such that

(A1) $F(x,t) \geq 0$ *for each* $(x,t) \in (B(0,s) \setminus B(0,\frac{s}{2})) \times \mathbb{R}$*;*

(A2) $a_{\eta_1}(\delta) < a_{\eta_2}(\delta)$*;*

(A3) there exist two constants $\xi > p$ *and* $R > 0$ *such that*

$$0 < \xi F(x,t) \leq tf(x,t), \tag{3.2}$$

for all $|t| \geq R$ *and for all* $x \in \Omega$*.*

Then for each $\lambda \in \left(\frac{1}{Hc_q^p}\frac{1}{a_{\eta_1}(\delta)}, \frac{1}{Hc_q^p}\frac{1}{a_{\eta_2}(\delta)}\right)$*, the problem (1.1) admits at least two nontrivial weak solutions* u_1 *and* u_2 *in* x*, such that*

$$\frac{m_0H - a}{m_1Hc_q^p}\eta_1^p < \|u_1\|^p < m_0\eta_2^p$$

.

Proof: Our aim is to apply Theorem 2.1 to the problem (1.1). Let Φ and Ψ be the the functionals defined by

$$\Phi(u) = \frac{1}{p}\widehat{M}(\|u\|^p) - \frac{a}{p}\int_\Omega \frac{|u(x)|^p}{|x|^{2p}}dx, \tag{3.3}$$

and

$$\Psi(u) = \int_\Omega F(x,u(x))dx. \tag{3.4}$$

Put $I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$ for all $u \in X$. It is easy to show that the functionals Φ and Ψ are well define and continuously Gâteaux differentiable. Moreover, we introduce the functional $I_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ associated with problem (1.1),

$$I_\lambda(u) := \frac{1}{p}\widehat{M}(\|u\|^p) - \frac{a}{p}\int_\Omega \frac{|u(x)|^p}{|x|^{2p}}dx - \lambda\int_\Omega F(x,u(x))dx.$$

Clearly Φ and Ψ are continuously Gâteaux differentiable and

$$\Phi^{'}(u)(v) = M\Big(\int_\Omega |\Delta u|^p dx\Big)\int_\Omega |\Delta u|^{p-2}\Delta u(x)\Delta v(x)dx - a\int_\Omega \frac{|u(x)|^{p-2}}{|x|^{2p}}u(x)v(x)dx,$$

and

$$\Psi^{'}(u)(v) = \int_\Omega f(x,u(x))v(x)dx$$

for every $v \in X$. Proposition 2.1 follows that Φ' admits a continuous inverse on X^*. Moreover, Ψ' is a compact operator. Note that the critical points of I_λ are exactly the weak solutions of problem (1.1). Since $m_0 \leq M(t) \leq m_1$ for all $t \in \mathbb{R}^+$, we see that

$$\frac{m_0 H - a}{pH}\|u\|^p \leq \Phi(u) \leq \frac{m_1}{p}\|u\|^p. \tag{3.5}$$

Put

$$r_1 = \frac{m_0 H - a}{pHc_q^p}\eta_1^p \text{ and } \quad r_2 = \frac{m_0 H - a}{pHc_q^p}\eta_2^p, \tag{3.6}$$

and define $w_\delta \in X$ by

$$w_\delta(x) := \begin{cases} 0, & x \in \overline{\Omega} \setminus B(0,s), \\ \delta(\frac{4}{s^3}l^3 - \frac{12}{s^2}l^2 + \frac{9}{s}l - 1), & x \in B(0,s) \setminus B(0,\frac{s}{2}), \\ \delta, & x \in B(0,\frac{s}{2}), \end{cases} \tag{3.7}$$

with $l = dist(x,0) = \sqrt{\sum_{i=1}^N x_i^2}$. Then

$$\frac{\partial w_\delta(x)}{\partial x_i} := \begin{cases} 0, & x \in \overline{\Omega} \setminus B(0,s) \cap B(0,\frac{s}{2}), \\ \delta(\frac{12lx_i}{s^3} - \frac{24x_i}{s^2} + \frac{9x_i}{sl}), & x \in B(0,s) \setminus B(0,\frac{s}{2}), \end{cases}$$

and

$$\frac{\partial^2 w_\delta(x)}{\partial x_i^2} := \begin{cases} 0, & x \in \overline{\Omega} \setminus B(0,s) \cap B(0,\frac{s}{2}), \\ \delta(\frac{12(x_i^2 + l^2)}{s^3 l} - \frac{24}{s^2} + \frac{9(l^2 - x_i^2)}{sl^3}), & x \in B(0,s) \setminus B(0,\frac{s}{2}). \end{cases}$$

Therefore, we have

$$\sum_{i=1}^N \frac{\partial^2 w_\delta(x)}{\partial x_i^2} := \begin{cases} 0, & x \in \overline{\Omega} \setminus B(0,s) \cap B(0,\frac{s}{2}), \\ \delta(\frac{12l(N+1)}{s^3} - \frac{24N}{s^2} + \frac{9(N-1)}{sl}), & x \in B(0,s) \setminus B(0,\frac{s}{2}). \end{cases}$$

and

$$\int_\Omega |\Delta w_\delta(x)|^p dx = \delta^p \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \int_{\frac{s}{2}}^s \Big|(\frac{12r(N+1)}{s^3} - \frac{24N}{s^2} + \frac{9(N-1)}{sr})\Big|^p r^{(N-1)} dr = L\delta^p. \tag{3.8}$$

So, from (3.5), we have

$$\frac{m_0 H - a}{pH} L\delta^p \leq \Phi(w_\delta) \leq \frac{m_1}{p} L\delta^p. \tag{3.9}$$

From the condition (3.1), we obtain $r_1 < \Phi(u) < r_2$. Then, for all $u \in X$, we see that

$$\begin{aligned} \Phi^{-1}(-\infty, r_2) &= \{u \in X, \Phi(u) \leq r_2\} \\ &\subseteq \{u \in X, \|u\|_{L^q(\Omega)} \leq \eta_2\} \end{aligned}$$

and it follows that

$$\sup_{u\in\Phi^{-1}(-\infty,r_2)}\Psi(u)\le\int_\Omega\sup_{\|t\|_{L^q(\Omega)}\le\eta_2}F(x,t)dx.$$

Therefore, by (A1) one has

$$\begin{aligned}
\beta(r_1,r_2) &\le \frac{\sup_{u\in\Phi^{-1}(-\infty,r_2)}\Psi(u)-\Psi(w_\delta)}{r_2-\Phi(w_\delta)}\\
&\le \frac{\int_\Omega\sup_{\|t\|_{L^q(\Omega)}\le\eta_2}F(x,t)dx-\int_\Omega F(x,w_\delta(x))dx}{\frac{m_0H-a}{pc_q^pH}\eta_2^p-\frac{m_1}{p}L\delta^p}\\
&\le pHk^p\frac{\int_\Omega\sup_{\|t\|_{L^q(\Omega)}\le\eta_2}F(x,t)dx-\int_{B(0,\frac{s}{2})}F(x,\delta)dx}{(m_0H-a)\eta_2^p-m_1HL(c_q\delta)^p}\\
&= Hc_q^pa_{\eta_2}(\delta).
\end{aligned}$$

On the other hand, arguing as before, one has

$$\begin{aligned}
\rho_2(r_1,r_2) &\ge \frac{\Psi(w_\delta)-\sup_{u\in\Phi^{-1}(-\infty,r_1]}\Psi(u)}{\Phi(w_\delta)-r_1}\\
&\ge \frac{\int_\Omega F(x,w_\delta(x))dx-\int_\Omega\sup_{\|t\|_{L^q(\Omega)}\le\eta_2}F(x,t)dx}{\frac{m_1}{p}L\delta^p-\frac{m_0H-a}{pc_q^pH}\eta_1^p}\\
&\ge pHk^p\frac{\int_{B(0,\frac{s}{2})}F(x,\delta)dx-\int_\Omega\sup_{\|t\|_{L^q(\Omega)}\le\eta_2}F(x,t)dx}{m_1HL(c_q\delta)^p-(m_0H-a)\eta_1^p}\\
&= Hc_q^pa_{\eta_1}(\delta).
\end{aligned}$$

Hence, from assumption (A2), one has $\beta(r_1,r_2)<\rho_2(r_1,r_2)$. Therefore, from Theorem (2.1), for each $\lambda\in\left(\frac{1}{c_q^pH}\frac{1}{a_{\eta_1}(\delta)},\frac{1}{c_q^pH}\frac{1}{a_{\eta_2}(\delta)}\right)$, the functional I_λ admits at least one nontrivial critical point u_1 such that

$$r_1<\Phi(u_1)<r_2,$$

that is,

$$\frac{m_0H-a}{m_1Hc_q^p}\eta_1^p<\|u_1\|^p<m_0\eta_2^p.$$

Now, we prove the existence of the second local minimum distinct from the first one. To this purpose, we verify the hypotheses of the mountain-pass theorem for the functional $\Phi-\lambda\Psi$. Clearly, the functional $\Phi-\lambda\Psi$ is of class C^1 and $(\Phi-\lambda\Psi)(0)=0$. The first part of proof guarantees that $u_1\in X$ is a local nontrivial local minimum for $\Phi-\lambda\Psi$ in X. Now, we can assume that u_1 is a strict local minimum of $\Phi-\lambda\Psi$ on X. Therefore, there is $s>0$ such that

$$\inf_{\|u-u_1\|=s}(\Phi-\lambda\Psi)(u)>(\Phi-\lambda\Psi)(u_1).$$

So the condition [31, (I_1), Theorem (2.2)] is verified. From $(A3)$ there is a positive constant C_1, C_2 such that

$$F(x,t) \geq C_1|t|^{\xi} + C_2.$$

Fixed $u \in X \setminus \{0\}$, for each $t > 1$ one has

$$\begin{aligned}(\Phi - \lambda\Psi)(tu) &= \frac{1}{p}\widehat{M}\left(\int_\Omega |\Delta tu|^p dx\right) - \frac{a}{p}\int_\Omega \frac{|tu|^p}{|x|^{2p}}dx - \lambda\int_\Omega F(x,tu)dx \\ &\leq \frac{m_1}{p}t^p\left(\int_\Omega |\Delta u|^p dx\right) - \lambda C t^{\xi}\int_\Omega |u|^\rho dx + \lambda C_2.\end{aligned}$$

Since $\xi > p$, this condition guarantees that I_λ is unbounded from below. So the condition [31, (I_2), Theorem (2.2)] is fulfilled. Now we prove that $I_\lambda := \Phi - \lambda\Psi$ satisfies (PS)-condition for every $\lambda > 0$. Namely, we will prove that any sequence $\{u_n\} \subset X$ satisfying

$$h := \sup_n I_\lambda(u_n) < +\infty, \qquad \lim_{n\to+\infty} \|I'_\lambda(u_n)\| = 0.$$

From above, we can actually assume that

$$|\frac{1}{\xi}\langle I'_\lambda(u_n), u_n\rangle| \leq \|u_n\|.$$

For n large enough, we have

$$h \geq I_\lambda(u_n) = \frac{1}{p}\widehat{M}\left(\int_\Omega |\Delta u_n(x)|^p dx\right) - \frac{a}{p}\int_\Omega \frac{|u_n(x)|^p}{|x|^{2p}}dx - \lambda\int_\Omega F(x,u_n(x))dx,$$

then

$$\begin{aligned}I_\lambda(u_n) - \frac{1}{\xi}\langle I'_\lambda(u_n), u_n\rangle &= \frac{1}{p}\widehat{M}(\int_\Omega |\Delta u_n(x)|^p dx) - \frac{a}{p}\int_\Omega \frac{|u_n(x)|^p}{|x|^{2p}}dx \\ &\quad -\lambda\int_\Omega F(x,u_n(x))dx \\ &\quad -\frac{1}{\xi}M(\int_\Omega |\Delta u_n(x)|^p dx)\int_\Omega |\Delta u_n(x)|^p dx \\ &\quad -\frac{a}{\xi}\int_\Omega \frac{|\Delta u_n(x)|^p}{|x|^{2p}}dx + \frac{\lambda}{\xi}\int_\Omega f(x,u_n(x))u_n(x)dx \\ &\geq m_0\left(\frac{1}{p} - \frac{1}{\xi}\right)\int_\Omega |\Delta u_n(x)|^p dx \\ &\quad -\frac{a}{H}\left(\frac{1}{p} - \frac{1}{\xi}\right)\int_\Omega |\Delta u_n(x)|^p dx \\ &= \left(\frac{1}{p} - \frac{1}{\xi}\right)\left(\frac{m_0 H - a}{H}\right)\|u_n\|^p.\end{aligned}$$

Thus,

$$h + \|u_n\| \geq I_\lambda(u_n) - \frac{1}{\xi}\langle I'_\lambda(u_n), u_n\rangle \geq \left(\frac{1}{p} - \frac{1}{\xi}\right)\left(\frac{m_0 H - a}{H}\right)\|u_n\|^p.$$

Consequently, $\{\|u_n\|\}$ is bounded. By the Eberlian-Smulyan theorem, without loss of generality, we assume that $u_n \rightharpoonup u$. Then $\Psi'(u_n) \to \Psi'(u)$. Since $I'_\lambda(u_n) = \Phi'(u_n) - \lambda\Psi'(u_n) \to 0$, then $\Phi'(u_n) \to \lambda\Psi'(u)$. Since Φ' has a continuous inverse, $u_n \to u$ and so I_λ satisfies (PS)-condition. Hence, the classical theorem of Amberosetti and Rabinowitz gives a critical point u_2 of $\Phi - \lambda\Psi$ such that $(\Phi - \lambda\Psi)(u_2) > (\Phi - \lambda\Psi)(u_1)$. So u_1 and u_2 are distinct weak solutions of the problem (1.1). Hence, the proof is complete. □

Theorem 3.2. *Suppose that $f(x, 0) \neq 0$ for all $x \in \Omega$ and there exist two positive constants δ and η, with*

$$\delta < \left(\frac{m_0 H - a}{m_1 L H}\right)^{\frac{1}{p}} \frac{\eta}{c_q}$$

such that the assumptions (A1) and (A3) in Theorem 3.1 hold. Furthermore, assume that

$$\frac{\int_\Omega \sup_{\|t\|_{L^q(\Omega)} \leq \eta} F(x,t)dx}{(m_0 H - a)\eta^p} < \frac{\int_{B(0,\frac{s}{2})} F(x,\delta)dx}{m_1 H L c_q^p \delta^p}. \tag{3.10}$$

Then, for each

$$\lambda \in \left(\frac{1}{p}\frac{m_1 L \delta^p}{\int_{B(0,\frac{s}{2})} F(x,\delta)dx}, \frac{1}{pHc_q^p}\frac{(m_0 H - a)\eta^p}{\int_\Omega \sup_{\|t\|_{L^q(\Omega)} \leq \eta} F(x,t)dx}\right)$$

the problem (1.1) admits at least two nontrivial weak solutions u_1 and u_2 in X such that

$$\|u_1\|^p < m_0 \eta^p.$$

Proof: Our aim is to employ Theorem 3.1, by choosing $\eta_1 = 0$ and $\eta_2 = \eta$. Therefore, owing to the inequality (3.5) and (A1), we see that

$$\begin{aligned}
a_\eta(\delta) &= p\frac{\int_\Omega \sup_{\|t\|_{L^q(\Omega)} \leq \eta} F(x,t)dx - \int_{B(0,\frac{s}{2})} F(x,\delta)dx}{(m_0 H - a)\eta^p - m_1 H L c_q^p \delta^p} \\
&< p\frac{\left(1 - \frac{m_1 H L k^p \delta^p}{(m_0 H - a)\eta^p}\right)\int_\Omega \sup_{|t| \leq \eta} F(x,t)dx}{(m_0 H - a)\eta^p - m_1 H L c_q^p \delta^p} \\
&< p\frac{\int_\Omega \sup_{\|t\|_{L^q(\Omega)} \leq \eta} F(x,t)dx}{(m_0 H - a)\eta^p} \\
&< p\frac{\int_{B(0,\frac{s}{2})} F(x,\delta)dx}{m_1 H L c_q^p \delta^p} \\
&= a_0(\delta).
\end{aligned}$$

In particular, one has

$$a_\eta(\delta) < p\frac{\int_\Omega \sup_{\|t\|_{L^q(\Omega)}\leq\eta} F(x,t)dx}{(m_0H-a)\eta^p}$$

which follows

$$\frac{1}{pHc_q^p}\frac{(m_0H-a)\eta^p}{\int_\Omega \sup_{\|t\|_{L^q(\Omega)}\leq\eta} F(x,t)dx} < \frac{1}{Hc_q^p}\frac{1}{a_\eta(\delta)}.$$

Hence, Theorem 3.1 yields the desired conclusion. □

Now, we present an application of Theorem 2.2 which will be used later to obtain multiple solutions for the problem(1.1).

Theorem 3.3. *Suppose that there exist two positive constants $\overline{\eta}$ and $\overline{\delta}$ with*

$$\overline{\delta} > \left(\frac{m_0H-a}{m_1LH}\right)^{\frac{1}{p}}\frac{\overline{\eta}}{c_q}$$

such that assumption (A1) in Theorem 3.1 holds. Moreover, assume that

$$\int_\Omega \sup_{\|t\|_{L^q(\Omega)}\leq\overline{\eta}} F(x,t)dx < \int_{B(0,\frac{s}{2})} F(x,\overline{\delta})dx$$

and

$$\limsup_{|\xi|\to+\infty}\frac{F(x,\xi)}{|\xi|^p} \leq 0 \quad \text{uniformly in } \mathbb{R}. \tag{3.11}$$

Then, for each $\lambda > \widehat{\lambda}$, where

$$\widehat{\lambda} := \frac{m_1HLc_q^p\overline{\delta}^p - (m_0H-a)\overline{\eta}^p}{pHc_q^p(\int_{B(0,\frac{s}{2})} F(x,\overline{\delta})dx - \int_\Omega \sup_{\|t\|_{L^q(\Omega)}\leq\overline{\eta}} F(x,t)dx)}$$

the problem (1.1) admits at least one nontrivial weak solution $\overline{u_1} \in X$ such that

$$\|\overline{u_1}\|^p > \frac{m_0H-a}{m_1Hc_q^p}\overline{\eta}^p.$$

Proof: Our goal is apply Theorem 2.2 to the functional $I_\lambda = \Phi - \lambda\Psi$ where Φ and Ψ are given as in (3.3) and (3.4), respectively. We observe that the all assumptions of Theorem 2.2 on Φ and Ψ are satisfied. Moreover, for $\lambda > 0$, the functional I_λ is coercive. Indeed, fix $0 < \epsilon < \frac{m_0H-a}{p\lambda Hc_q^p}$. From (3.11) there is a function $\varrho_\varepsilon \in L^1(\Omega)$ such that

$$F(x,t) \leq \varepsilon t^p + \varrho_\varepsilon(x),$$

for every $x \in \Omega$ and $t \in \mathbb{R}$. Therefore, for each $u \in X$ with $\|u\| \geq 1$, we see that

$$\begin{aligned}\Phi(u) - \lambda\Psi(u) &\geq \frac{m_oH-a}{pH}\|u\|^p - \lambda\varepsilon\int_\Omega u^p(x)dx - \lambda\|\varrho_\varepsilon\|_{L^1} \\ &\geq \left(\frac{m_0H-a}{pH} - \lambda c_q^p\varepsilon\right)\|u\|^p - \lambda\|\varrho_\varepsilon\|_{L1}\end{aligned}$$

and thus

$$\lim_{\|u\|\to+\infty}(\Phi(u)-\lambda\Psi(u))=+\infty$$

which means the functional $I_\lambda=\Phi-\lambda\Psi$ is coercive. Put

$$\overline{r}=\frac{m_0H-a}{pHc_q^p}\overline{\eta}^p,$$

and choose

$$\overline{w}(x):=\begin{cases}0, & x\in\overline{\Omega}\setminus B(0,s),\\ \overline{\delta}(\frac{4}{s^3}l^3-\frac{12}{s^2}l^2+\frac{9}{s}l-1), & x\in B(0,s)\setminus B(0,\frac{s}{2}),\\ \overline{\delta}, & x\in B(0,\frac{s}{2}).\end{cases}$$

Using the condition (A1) and arguing as in the proof of Theorem 3.1, we obtain that

$$\rho_2(\overline{r})\geq pHc_q^p\frac{\int_{B(0,\frac{s}{2})}F(x,\overline{\delta})dx-\int_\Omega\sup_{\|t\|_{L^q(\Omega)}\leq\overline{\eta}}F(x,t)dx}{m_1HLc_q^p\overline{\delta}^p-(m_0H-a)\overline{\eta}^p}.$$

Thus, it follows that $\rho(\overline{r})>0$. Hence, from Theorem 2.2 for each $\lambda>\widehat{\lambda}$, the functional I_λ admits at least one local minimum $\overline{u_1}$ such that

$$\|\overline{u_1}\|^p>\frac{m_0H-a}{m_1Hc_q^p}\overline{\eta}^p$$

the desired conclusion is achieved. □

Now, we point out a special situation of our main result when the function f has separated variables. To be precise, let $\alpha:\Omega\to\mathbb{R}$ be a nonnegative and nonzero function such that $\alpha\in L^1(\Omega)$ and let $g:\mathbb{R}\to\mathbb{R}$ be a nonnegative continuous function. Consider the following problem

$$\begin{cases}M\left(\int_\Omega|\Delta u|^pdx\right)\Delta_p^2u-\frac{a}{|x|^{2p}}|u|^{p-2}u=\lambda\alpha(x)g(u) & \text{in }\Omega,\\ u=\Delta u=0, & \text{on }\partial\Omega.\end{cases}\tag{3.12}$$

Put $G(t)=\int_0^tg(\xi)d\xi$ for all $t\in\mathbb{R}$ and set $f(x,t)=\alpha(x)g(t)$ for every $(x,t)\in\Omega\times\mathbb{R}$. The following existence results are consequences of Theorems (3.1)-(3.3), respectively.

Theorem 3.4. *Suppose that $g(0)\neq0$ and there exist a nonnegative constant η_1 and two positive constants η_2 and δ, with*

$$\frac{1}{c_qL^{\frac{1}{p}}}\eta_1<\delta<\left(\frac{m_0H-a}{m_1LH}\right)^{\frac{1}{p}}\frac{\eta_2}{c_q}$$

such that

$$\frac{\|\alpha\|_{L^1(\Omega)}G(\eta_2)-\|\alpha\|_{L^1(B(0,\frac{s}{2}))}G(\delta)}{(m_0H-a)\eta_2^p-m_1HLc_q^p\delta^p}<\frac{\|\alpha\|_{L^1(\Omega)}G(\eta_1)-\|\alpha\|_{L^1(B(0,\frac{s}{2}))}G(\delta)}{(m_0H-a)\eta_1^p-m_1HLc_q^p\delta^p}.$$

Moreover, assume that there exist constants $v > p$ and $R > 0$ such that for all $|\xi| \geq R$ and for all $x \in \Omega$

$$0 < vG(\xi) \leq \xi g(\xi). \tag{3.13}$$

Then, for each $\lambda \in]\lambda_1, \lambda_2[$, where

$$\lambda_1 := \frac{1}{pHc_q^p} \frac{(m_0H - a)\eta_1^p - m_1HLc_q^p\delta^p}{\|\alpha\|_{L^1(\Omega)}G(\eta_1) - \|\alpha\|_{L^1(B(0,\frac{s}{2}))}G(\delta)}$$

and

$$\lambda_2 := \frac{1}{pHc_q^p} \frac{(m_0H - a)\eta_2^p - m_1HLc_q^p\delta^p}{\|\alpha\|_{L^1(\Omega)}G(\eta_2) - \|\alpha\|_{L^1(B(0,\frac{s}{2}))}G(\delta)}$$

the problem (3.12) admits at least two nontrivial weak solutions u_1 and u_2 such that

$$\|u_1\|^p < m_0\eta^p.$$

Theorem 3.5. *Suppose that $g(0) \neq 0$ and there exist two positive constant δ and η, with*

$$\delta < \left(\frac{m_0H - a}{m_1LH}\right)^{\frac{1}{p}} \frac{\eta}{c_q}$$

such that

$$\frac{\|\alpha\|_{L^1(\Omega)}G(\eta)}{\eta^p} < \frac{m_0H - a}{m_1HLc_q^p\delta^p}\|\alpha\|_{L^1(B(0,\frac{s}{2}))}G(\delta). \tag{3.14}$$

Moreover, assume that the assumption (3.11) holds. Then, for every

$$\lambda \in \left(\frac{m_1L\delta^p}{p\|\alpha\|_{L^1(B(0,\frac{s}{2}))}G(\delta)}, \frac{(m_0H - a)\eta^p}{pHc_q^p\|\alpha\|_{L^1(\Omega)}G(\eta)}\right)$$

the problem (3.12) admits at least two nontrivial weak solutions u_1 and u_2 in X such that

$$\|u_1\|^p < m_0\eta^p.$$

Theorem 3.6. *Suppose that there exist two positive constant $\overline{\eta}$ and $\overline{\delta}$ with*

$$\overline{\delta} > \left(\frac{m_0H - a}{m_1LH}\right)^{\frac{1}{p}} \frac{\overline{\eta}}{c_q}$$

such that

$$G(\overline{\eta}) < \frac{\|\alpha\|_{L^1(B(0,\frac{s}{2}))}}{\|\alpha\|_{L^1(\Omega)}}G(\overline{\delta}) \tag{3.15}$$

and

$$\limsup_{\xi \to +\infty} \frac{g(\xi)}{|\xi|^{p-1}} \leq 0.$$

Then, for each $\lambda > \overline{\lambda}$*, where*

$$\overline{\lambda} := \frac{1}{pHc_q^p} \frac{(m_0H - a)\overline{\eta}^p - m_1HLc_q^p\overline{\delta}^p}{\|\alpha\|_{L^1(\Omega)}G(\overline{\eta}) - \|\alpha\|_{L^1(B(0,\frac{s}{2}))}G(\overline{\delta})}$$

the problem (3.12) *admits at least one nontrivial weak solution* $\overline{u_1}$ *such that*

$$\|\overline{u_1}\|^p > \frac{m_0H - a}{m_1Hc_q^p}\overline{\eta}^p.$$

A further consequence of Theorem 3.1 is the following existence result.

Theorem 3.7. *Suppose that* $g(0) \neq 0$ *and*

$$\lim_{\xi \to 0^+} \frac{g(\xi)}{\xi^{p-1}} = +\infty. \tag{3.16}$$

Moreover, assume that the assumption (3.13) *holds. Then, for every* $\lambda \in (0, \lambda_\eta^*)$*, where*

$$\lambda_\eta^* := \frac{m_0H - a}{pHc_q^p\|\alpha\|_{L^1(\Omega)}} \sup_{\eta > 0} \frac{\eta^p}{G(\eta)}$$

the problem (3.12) *admits at least two nontrivial weak solutions in* X*.*

Proof: Fix $\lambda \in]0, \lambda_\eta^*[$. then there is $\eta > 0$ such that $\lambda < \frac{m_0H-a}{pHc_q^p\|\alpha\|_{L^1(\Omega)}} \frac{\eta^p}{G(\eta)}$. From (3.16) there exists a positive constant δ with

$$\delta < \left(\frac{m_0H - a}{m_1LH}\right)^{\frac{1}{p}} \frac{\eta}{c_q}$$

such that

$$\lambda > \frac{m_1L\delta^p}{p\|\alpha\|_{L^1(B(0,\frac{s}{2}))}G(\delta)}.$$

Therefore, the conclusion follows from Theorem 3.2. □

Now, we present the following example to illustrate Theorem 3.7.

Example 3.8. *Let* $N = 3$, $p = \frac{5}{4}$, $\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3; x_1^2 + x_2^2 + x_3^2 < 1\}$, $M(t) = 3 + \cos x$ *for all* $t \in [0, +\infty]$, $\alpha(x_1, x_2, x_3) = \frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}$ *for all* $(x_1, x_2, x_3) \in \mathbb{R}^3$ *and* $g(t) = \frac{1}{6} + t^2|t|$*. thus*

$$\|\alpha\|_{L^1(\Omega)} = \int_0^{2\pi}\int_0^{\pi}\int_0^1 r\sin\phi dr d\phi d\theta = 2\pi$$

and

$$\lim_{\xi \to 0^+} \frac{g(\xi)}{\xi^{p-1}} = \lim_{\xi \to 0^+} \left(\frac{1}{6\xi^{\frac{1}{4}}} + \xi^{\frac{7}{4}}\right) = +\infty$$

Now, by choosing $\nu = 3$ and $R = 2$, we see that the assumption (3.13) is satisfied. Hence, by applying theorem 3.7, for every $\lambda \in (0, \lambda^*)$ where

$$\begin{aligned}
\lambda^* &= \frac{m_0H - a}{pHc_q^p\|\alpha\|_{L^1(\Omega)}} \sup_{\eta>0} \frac{\eta^p}{G(\eta)} \\
&= \left(\frac{4\sqrt[4]{6} - \sqrt{5}}{5\sqrt[4]{6}c_q^{\frac{5}{4}}\pi}\right) \sup_{\eta>0} \frac{\eta^{\frac{1}{4}}}{2 + 3\eta^3} \\
&\geq \left(\frac{4\sqrt[4]{6} - \sqrt{5}}{5\sqrt[4]{6}c_q^{\frac{5}{4}}\pi}\right) \frac{\eta^{\frac{1}{4}}}{2 + 3\eta^3} |_{\eta=1} \\
&= \frac{48\sqrt[4]{6} - 12\sqrt{5}}{25\sqrt[4]{6}c_q^{\frac{5}{4}}\pi}
\end{aligned}$$

the problem

$$\begin{cases} \left(3 + \cos(\int_\Omega |\Delta u|^p dx)\right)\Delta_p^2 u - \dfrac{6}{50|x|^{2p}}|u|^{p-2}u = \dfrac{\lambda}{\sqrt{x_1^2 + x_2^2 + x_3^2}} g(u(x_1, x_2, x_3)) & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases}$$

has at least two nontrivial weak solutions.

Next, by applying Theorems 3.5 and 3.6, we obtain the following theorem of existence of three solutions for the problem (3.12).

Theorem 3.9. *Suppose that $g(0) \neq 0$ and*

$$\limsup_{|\xi|\to+\infty} \frac{G(\xi)}{|\xi|^p} \leq 0. \tag{3.17}$$

Furthermore, assume that there exist four positive constants $\eta, \delta, \overline{\eta}$ and $\overline{\delta}$ with

$$\delta < \left(\frac{m_0H - a}{m_1LH}\right)^{\frac{1}{p}} \frac{\eta}{c_q} \leq \left(\frac{m_0H - a}{m_1LH}\right)^{\frac{1}{p}} \frac{\overline{\eta}}{c_q} < \overline{\delta}$$

such that (3.14) and (3.15) hold, and

$$\frac{\|\alpha\|_{L^1(\Omega)}G(\eta)}{(m_0H - a)\eta^p} < \frac{\|\alpha\|_{L^1(\Omega)}G(\overline{\eta}) - \|\alpha\|_{L^1(B(0,\frac{s}{2}))}G(\overline{\delta})}{(m_0H - a)\overline{\eta}^p - m_1HLc_q^p\overline{\delta}^p} \tag{3.18}$$

is satisfied. Then, for each

$$\lambda \in \Lambda = \left(\max\{\overline{\lambda}, \frac{m_1L\delta^p}{p\|\alpha\|_{L^1(B(0,\frac{s}{2}))}G(\delta)}\}, \frac{(m_0H - a)\eta^p}{pHc_q^p\|\alpha\|_{L^1(\Omega)}G(\eta)}\right)$$

the problem (3.12) admits at least three nontrivial weak solutions u_1, $\overline{u_1}$ and u_3 such that

$$\|u_1\|^p < m_0\eta^p, \quad \|\overline{u_1}\|^p > \frac{m_0H - a}{m_1Hc_q^p}\overline{\eta}^p.$$

Proof: It is easy to see that $\Lambda \neq \emptyset$ from (3.18). Fix $\lambda \in \Lambda$. Using Theorem 3.5, there is a nontrivial weak solution u_1 such that

$$\|u_1\|^p < m_0\eta^p$$

which is a local minimum for the associated functional I_λ, and Theorem 3.6 guarantees the second a nontrivial weak solution $\overline{u_1}$ such that

$$\|\overline{u_1}\|^p > \frac{m_0H - a}{m_1Hc_q^p}\overline{\eta}^p$$

which is local minimum for I_λ. Now, by employing the proof of Theorem 3.3, from the condition (3.17) we see that the functional I_λ satisfies the (PS) condition. Hence, the conclusion follows from the mountain pass theorem as given by Pucci and Serrin (see [30]). □

We now point out the following consequence of Theorem 3.9.

Theorem 3.10. *Suppose that $g(0) \neq 0$,*

$$\limsup_{\xi \to 0^+} \frac{G(\xi)}{\xi^p} = +\infty, \tag{3.19}$$

and

$$\limsup_{\xi \to +\infty} \frac{G(\xi)}{\xi^p} = 0. \tag{3.20}$$

Moreover, assume that there exist two positive constants $\overline{\eta}$ and $\overline{\delta}$ with

$$\left(\frac{m_0H - a}{m_1LH}\right)^{\frac{1}{p}} \frac{\overline{\eta}}{c_q} < \overline{\delta} \tag{3.21}$$

such that

$$\frac{\|\alpha\|_{L^1(\Omega)}G(\overline{\eta})}{(m_0H - a)\overline{\eta}^p} < \frac{\|\alpha\|_{L^1(B(0,\frac{s}{2}))}G(\overline{\delta})}{m_1HLc_q^p\overline{\delta}^p} \tag{3.22}$$

Then, for each

$$\lambda \in \left(\frac{m_1L\overline{\delta}^p}{p\|\alpha\|_{L^1(B(0,\frac{s}{2}))}G(\overline{\delta})}, \frac{(m_0H - a)\overline{\eta}^p}{pHc_q^p\|\alpha\|_{L^1(\Omega)}G(\overline{\eta})}\right)$$

the problem (3.12) admits at least three nontrivial weak solutions.

Proof: We easily observe form (3.20) that the condition (3.17) is satisfied. Moreover, by choosing δ small enough and $\eta = \overline{\eta}$, one can drive the condition (3.14) from (3.19) as well as the conditions (3.15) and (3.18) from (3.22). Hence, the conclusion follows from Theorem 3.9. □

Finally, we illustrate Theorem 3.10 by presenting the following example.

Example 3.11. *Consider the problem*

$$\begin{cases} (3+\sin(\int_\Omega |\Delta u|^p dx))\Delta_p^2 u - \dfrac{1}{4\sqrt[5]{4}|x|^{2p}}|u|^{p-2}u = \lambda u(x) & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.23}$$

where $N = 3, p = \frac{6}{5}, \Omega = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3; |x_1| + |x_2| + |x_3| < 2\}, M(t) = 3 + \sin t$ *for all* $t \in [0, +\infty)$, $\alpha(x_1, x_2, x_3) = x_1^2+x_2^2+x_3^2$ *for all* $(x_1, x_2, x_3) \in \mathbb{R}^3$, $s = 2$. *Thus*

$$\|\alpha\|_{L^1(\Omega)} = \frac{64}{5}, \qquad \|\alpha\|_{L^1(B(0,1))} = \frac{4\pi}{5},$$

Let $g(t) = 1 + e^{-t}(1-t)$ *for all* $t \in \mathbb{R}$. *Thus* g *is nonnegative continuous,* $g(0) \neq 0$ *and*

$$G(t) = t(1 + e^{-t}).$$

Since

$$\lim_{t\to 0^+} \frac{t(1+e^{-t})}{t^{\frac{6}{5}}} = +\infty, \qquad \lim_{t\to +\infty} \frac{t(1+e^{-t})}{t^{\frac{6}{5}}} = 0,$$

we see that conditions (3.19) *and* (3.20) *hold true.Moreover, by* $\overline{\eta} = 4480c_q$ *and* $\overline{\delta} = \frac{1}{c_q}$, *we see that the conditions* (3.21) *and* (3.22) *hold true. Then, by applying Theorem 3.10, for every*

$$\lambda \in \left(\frac{755}{3c_q^{\frac{1}{5}}(1 + e^{-c_q^{-1}})}, \frac{875}{3c_q^{\frac{1}{5}}(1 + e^{-4480c_q})} \right)$$

the problem (3.23) *has at least three nontrivial weak solutions.*

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