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Euler Type Integral Operator Involving k-Mittag-Leffler Function

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ABSTRACT: This paper deals with a Euler type integral operator involving k-Mittag-Leffler function defined by Gupta and Parihar [7]. Furthermore, some special cases are also taken into consideration.

Key Words: Euler type integrals, Extended k-beta function, Generalized k-Mittag-Leffler function, Generalized k-Wright function.

Contents

1. Introduction

Many authors namely, Diaz et al. [5], Kokologiannaki [12], Krasniqi [13], Mansour [18], Merovci [16], had introduced k-generalized gamma, k-Zeta, k-Beta functions. They had proven a number of their properties and inequalities for the above k-generalization functions. They had studied k-hypergeometric functions based on k-Pochhammer symbols for factorial functions. Propose our present research, we begin by mentioning the following definitions of some well known functions:

The integral representation of the k-gamma function as:

$$\Gamma_k(z) = k^{\frac{z}{k}-1}\Gamma(\frac{z}{k}) = \int_0^\infty e^{\frac{-t^k}{k}} t^{z-1} dt, k \in \mathbb{R}, z \in \mathbb{C}. \tag{1.1}$$

and k-beta function is defined as:

$$B_k(x,y) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1} dt = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}, x>0, y>0. \tag{1.2}$$

The generalized k-Wright function [6] represented as follows:

$${}_p\Psi_q^k \left[\begin{array}{ll} (\alpha_1, A_1), ..., (\alpha_p, A_p); & \\ & z \\ (\beta_1, B_1), ..., (\beta_p, B_p); & \end{array} \right] = {}_p\Psi_q^k \left((\alpha_j, A_j)_{1,p}; (\beta_j, B_j)_{1,q}; z \right)$$

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$$= \sum_{n=0}^{\infty} \frac{\Gamma_k(\alpha_1 + nA_1)..., \Gamma_k(\alpha_p + nA_p)}{\Gamma_k(\beta_1 + nB_1)..., \Gamma_k(\beta_p + nB_p)} \frac{z^n}{n!}. \quad (1.3)$$

In 1903, the Swedish Mathematician introduced the Mittag-Leffler function $E_\alpha(z)$ [17] as:

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad (1.4)$$

where $z \in \mathbb{C}$ and $\Gamma(s)$ is the Gamma function; $\alpha \geq 0$.

The Mittag-Leffler function is a direct generalization of $\exp(z)$ in which $\alpha = 1$. Mittag-Leffler function naturally occurs as the solution of fractional order differential equation or fractional order integral equations.

A generalization of $E_\alpha(z)$ was studied by Wiman [27,28] where he defined the function $E_{\alpha,\beta}(z)$ as:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad (1.5)$$

where $\alpha, \beta \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0$. Which is also known as Mittag-Leffler function or Wiman's function.

Prabhakar [19] introduced the function $E^{\gamma}_{\alpha,\beta}(z)$ in the form (see Killbas et al. [8]):

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}, \quad (1.6)$$

where $\alpha, \beta, \gamma \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0$.

Shukla and Prajapati [25] (see Srivastava and Tomovaski [26]) defined and investigated the function $E^{\gamma,q}_{\alpha,\beta}(z)$ as:

$$E^{\gamma,q}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}, \quad (1.7)$$

where $\alpha, \beta, \gamma \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, q \in (0,1) \bigcup \mathbb{N}$ and $(\gamma)_{qn} = \frac{\Gamma(\gamma + qn)}{\Gamma(\gamma)}$ denotes the generalized Pochhammer symbol.

Salim [23] introduced a new generalized Mittag-Leffler function and defined it as:

$$E^{\gamma,\delta}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{(\delta)_n} \quad (1.8)$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\delta) > 0$.

Afterward, Salim and Faraj [22] introduced the generalized Mittag-Leffler function $E^{\gamma,\delta,q}_{\alpha,\beta,p}(z)$ which is defined as:

$$E^{\gamma,\delta,q}_{\alpha,\beta,p}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\alpha n + \beta)} \frac{z^n}{(\delta)_{pn}}, \quad (1.9)$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}, \min\{\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\delta)\} > 0; p, q > 0$ and $q < \Re(\alpha) + p$.

Recently, Gupta and Parihar [7] introduced the generalized k-Mittag-Leffler function as follows:

$$E_{k,\alpha,\beta,p}^{\gamma,\delta,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn,k}}{\Gamma_k(\alpha n + \beta)} \frac{z^n}{(\delta)_{pn,k}}, \tag{1.10}$$

where $k \in \mathbb{R}; \alpha, \beta, \gamma, \delta \in \mathbb{C}, \min\{\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\delta)\} > 0; p, q > 0$ and $q < \Re\alpha + p$.

Now, we state the classical beta function denoted by $B(a, b)$ which is defined (see [15], see also [20]) by Euler's integral as:

$$B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, (\Re(a) > 0, \Re(b) > 0). \tag{1.11}$$

In 1997, Chaudhary et al. [4] presented the following extension of Euler's Beta function as follows:

$$B_\sigma(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1} \exp\left[-\frac{p}{t(1-t)}\right] dt. \tag{1.12}$$

In continuation of his work, Lee et al. [14] introduced the generalizations of Euler beta functions and defined it as:

$$B(x, y; p; m) = \int_0^1 t^{x-1}(1-t)^{y-1} \exp\left[-\frac{p}{t^m(1-t)^m}\right] dt. \tag{1.13}$$

where $\Re(p) > \Re(m) > 0$.

In this paper, we consider the new generalizations of Euler type k-Beta functions as follows:

$$B_k(x, y; p; m) = \int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1} \exp\left[-\frac{p}{t^{\frac{m}{k}}(1-t)^{\frac{m}{k}}}\right] dt, \tag{1.14}$$

where $k \in \mathbb{R}; \Re(p) > \Re(m) > 0$.

Clearly, when $m = k = 1$, equation (1.14) reduces to (1.12) and further, by taking $p = 0$ in (1.14), we get (1.11).

In this paper, we have obtained some theorems on Euler type integral operator involving generalized k-Mittag-Leffler function and have discussed some special cases.

2. Basic properties of Euler type integral operator involving generalized k-Mittag-Leffler function

Theorem 2.1. *If* $k \in \mathbb{R}, \alpha, \beta, \gamma, \delta \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\delta) > 0, \Re(A) > 0,\ p, q > 0$ *and* $q < \Re(\alpha) + p$, *then,*

$$\int_0^1 t^{\frac{a}{k}-1}(1-t)^{\frac{b}{k}-1}\exp\left(\frac{-A}{t^{\frac{m}{k}}(1-t)^{\frac{m}{k}}}\right)E^{\gamma,\delta,q}_{k,\alpha,\beta,p}(zt^{\frac{\alpha}{k}})dt$$

$$= \sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}z^n}{\Gamma_k(\alpha n+\beta)(\delta)_{pn,k}}B_k(a+n\alpha, b; A; m). \tag{2.1}$$

Proof: In order to derive (2.1), we denote L.H.S. of (2.1) by I_1 and then expanding $E^{\gamma,\delta,q}_{k,\alpha,\beta,p}(zt^{\alpha})$ by using (1.10), to get:

$$I_1 = \int_0^1 t^{\frac{a}{k}-1}(1-t)^{\frac{b}{k}-1}\exp\left(\frac{-A}{t^{\frac{m}{k}}(1-t)^{\frac{m}{k}}}\right)\sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}z^n t^{\frac{n\alpha}{k}}}{\Gamma_k(\alpha n+\beta)(\delta)_{pn,k}}dt$$

Now changing the order of summation and integration (which is guaranteed under the given conditions), we get:

$$I_1 = \sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}z^n}{\Gamma_k(\alpha n+\beta)(\delta)_{pn,k}}\int_0^1 t^{\frac{a+n\alpha}{k}-1}(1-t)^{\frac{b}{k}-1}\exp\left(\frac{-A}{t^{\frac{m}{k}}(1-t)^{\frac{m}{k}}}\right)dt$$

By using (1.14) as in the above equation, we attain the required result. □

Corollary 2.1

For $A = 0, a = \beta$ in Theorem 2.1, we deduce the following result:

$$\frac{1}{k}\frac{1}{\Gamma_k(b)}\int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{b}{k}-1}E^{\gamma,\delta,q}_{k,\alpha,\beta,p}(zt^{\frac{\alpha}{k}})dt$$

$$= \sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}z^n}{(\delta)_{pn,k}\Gamma_k(\beta+b+\alpha n)}. \tag{2.2}$$

Theorem 2.2. *If* $k \in \mathbb{R}, \alpha, \beta, \gamma, \delta, \rho, \mu \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\delta) > 0, \Re(\rho), \Re(\mu), \Re(A) > 0,\ p, q > 0$ *and* $q < \Re(\alpha) + p$; $\mid arg(\frac{b^{'}c+d}{a^{'}c+d}) \mid < \pi$, *then*

$$\int_t^x (x-s)^{\frac{\rho}{k}-1}(s-t)^{\frac{\mu}{k}-1}\exp\left(\frac{-A}{(x-s)^{\frac{m}{k}}(s-t)^{\frac{m}{k}}}\right)E^{\gamma,\delta,q}_{k,\alpha,\beta,p}(z(s-t)^{\frac{\alpha}{k}})dt$$

$$= \sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \frac{(x-t)^{\frac{\rho+\mu+n\alpha-2mr}{k}-2}(-A)^r(\gamma)_{qn,k}z^n}{\Gamma_k(\alpha n+\beta)(\delta)_{pn,k}r!} B_k(\mu+n\alpha-mr, \rho-mr). \quad (2.3)$$

Proof: On taking L.H.S. of (2.3) and then by changing the variable s to $u = \frac{s-t}{x-t}$, we get:

$$\int_t^x (x-s)^{\frac{\rho}{k}-1}(s-t)^{\frac{\mu}{k}-1} \exp\left(\frac{-A}{(x-s)^{\frac{m}{k}}(s-t)^{\frac{m}{k}}}\right) E_{k,\alpha,\beta,p}^{\gamma,\delta,q}(z(s-t)^{\frac{\alpha}{k}})dt,$$

$$= \int_0^1 (1-u)^{\frac{\rho}{k}-1}(x-t)^{\frac{\rho}{k}-1}u^{\frac{\mu}{k}-1}(x-t)^{\frac{\mu}{k}-1} \exp\left(\frac{-A}{((x-t)(1-u))^{\frac{m}{k}}(u(x-t))^{\frac{m}{k}}}\right)$$

$$\times E_{k,\alpha,\beta,p}^{\gamma,\delta,q}(zu^{\frac{\alpha}{k}}(x-t)^{\frac{\alpha}{k}})dt.$$

Expanding the exponential function and k-Mittag-Leffler function in their respective series, we attain:

$$\int_0^1 (1-u)^{\frac{\rho}{k}-1}(x-t)^{\frac{\rho}{k}-1}u^{\frac{\mu}{k}-1}(x-t)^{\frac{\mu}{k}-1} \sum_{r=0}^{\infty} \frac{(-A)^r}{(x-t)^{\frac{2mr}{k}}(1-u)^{\frac{mr}{k}}u^{\frac{mr}{k}}r!}$$

$$\times \sum_{n=0}^{\infty} \frac{(\gamma)_{qn,k}z^n u^{\frac{n\alpha}{k}}(x-t)^{\frac{\alpha n}{k}}}{\Gamma_k(\alpha n+\beta)(\delta)_{pn,k}}.$$

By changing the order of summation and integration (which is guaranteed under the given conditions), we get:

$$= \sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \frac{(x-t)^{\frac{\rho+\mu+n\alpha-2mr}{k}-2}(-A)^r(\gamma)_{qn,k}z^n}{\Gamma_k(\alpha n+\beta)(\delta)_{pn,k}r!} \int_0^1 (1-u)^{\frac{\rho-mr}{k}-1}u^{\frac{\mu+n\alpha-mr}{k}-1}du,$$

which further on using the integral(1.11) yields the required result. □

Corollary 2.2

For $A = 0, \mu = \beta$ in Theorem 2.2, we get:

$$\frac{1}{k}\frac{1}{\Gamma_k(\rho)} \int_t^x (x-t)^{\frac{\rho}{k}-1}(s-t)^{\frac{\beta}{k}-1} E_{k,\alpha,\beta,p}^{\gamma,\delta,q}(z(s-t)^{\frac{\alpha}{k}})dt = \sum_{n=0}^{\infty} \frac{(x-t)^{\frac{\rho+\beta+n\alpha}{k}-2}(\gamma)_{qn,k}z^n}{\Gamma_k(\alpha n+\beta+\rho)(\delta)_{pn,k}}. \quad (2.4)$$

Theorem 2.3. *If* $k \in \mathbb{R}, \alpha, \beta, \gamma, \delta, \rho, \mu, \lambda, \sigma \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\sigma) > 0, \Re(\delta) > 0, \Re(\rho), \Re(\mu), \Re(\lambda) > 0, \Re(A) > 0$*;* $p, q > 0$ *and* $q < \Re(\alpha) + p$*, then*

$$\int_0^1 t^{\frac{\lambda}{k}-1}(1-t)^{\frac{\mu-\lambda}{k}-1}\left(1-ut^{\frac{\rho}{k}}(1-t)^{\frac{\sigma}{k}}\right)^{-a}\exp\left(\frac{-A}{t^{\frac{m}{k}}(1-t)^{\frac{m}{k}}}\right)E_{k,\alpha,\beta,p}^{\gamma,\delta,q}(zt^{\frac{\alpha}{k}})dt$$

$$=\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_r u^r(\gamma)_{qn,k}z^n}{\Gamma_k(\alpha n+\beta)(\delta)_{pn,k}r!}B_k(\lambda+n\alpha+\rho r, \mu-\lambda+r\sigma; A; m). \tag{2.5}$$

Proof: On taking L.H.S. of Theorem 2.3, using the definition of generalized k-Mittag-Leffler function (1.10), and then by changing the order of summation and integration, we get:

$$\int_0^1 t^{\frac{\lambda}{k}-1}(1-t)^{\frac{\mu-\lambda}{k}-1}\left(1-ut^{\frac{\rho}{k}}(1-t)^{\frac{\sigma}{k}}\right)^{-a}\exp\left(\frac{-A}{t^{\frac{m}{k}}(1-t)^{\frac{m}{k}}}\right)E_{k,\alpha,\beta,p}^{\gamma,\delta,q}(zt^{\frac{\alpha}{k}})dt$$

$$=\int_0^1 t^{\frac{\lambda}{k}-1}(1-t)^{\frac{\mu-\lambda}{k}-1}\sum_{r=0}^{\infty}\frac{(a)_r u^r t^{\frac{\rho r}{k}}(1-t)^{\frac{r\sigma}{k}}}{r!}\exp\left(\frac{-A}{t^{\frac{m}{k}}(1-t)^{\frac{m}{k}}}\right)$$

$$\times\sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}z^n t^{\frac{n\alpha}{k}}}{\Gamma_k(\alpha n+\beta)(\delta)_{pn,k}}dt,$$

$$=\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_r u^r(\gamma)_{qn,k}z^n}{\Gamma_k(\alpha n+\beta)(\delta)_{pn,k}r!}\int_0^1 t^{\frac{\lambda+n\alpha+\rho r}{k}-1}(1-t)^{\frac{\mu-\lambda+\sigma r}{k}-1}\exp\left(\frac{-A}{t^{\frac{m}{k}}(1-t)^{\frac{m}{k}}}\right)dt.$$

By using (1.14) as in the above equation, we derive the required result. □

Corollary 2.3

For $a = 0$ in Theorem 2.3 reduces to the following result:

$$\int_0^1 t^{\frac{\lambda}{k}-1}(1-t)^{\frac{\mu-\lambda}{k}-1}\exp\left(\frac{-A}{t^{\frac{m}{k}}(1-t)^{\frac{m}{k}}}\right)E_{k,\alpha,\beta,p}^{\gamma,\delta,q}(zt^{\frac{\alpha}{k}})dt$$

$$=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}z^n}{\Gamma_k(\alpha n+\beta)(\delta)_{pn,k}}B_k(\lambda+n\alpha, \mu-\lambda; A; m). \tag{2.6}$$

Corollary 2.4
Setting $a = 0$, $A = 0$ in Theorem 2.3, we deduces the following result:

$$\int_0^1 t^{\frac{\lambda}{k}-1}(1-t)^{\frac{\mu-\lambda}{k}-1} E^{\gamma,\delta,q}_{k,\alpha,\beta,p}(zt^{\frac{\alpha}{k}})dt = \sum_{n=0}^{\infty} \frac{k(\gamma)_{qn,k} z^n}{\Gamma_k(\alpha n+\beta)(\delta)_{pn,k}} B_k(\lambda + n\alpha, \mu - \lambda). \tag{2.7}$$

Remark 2.4. *If we consider $p = q = k = m = 1$ in Theorem (2.1), (2.2) and (2.3), we get a new class of Beta type integral operators involving the generalized Mittag-Leffler function defined by Salim [23] and the case $\delta = p = k = 1$ of (2.1), (2.3) and (2.5) is seen to yield the known results of Ahmed and Khan [1].*

3. Special Cases

In this section, we establish the following potentially useful integral operators involving generalized k-Beta type functions as special cases of our main results:
1. On setting $\gamma = q = 1$ in Theorem 2.1, we get:

$$\int_0^1 t^{\frac{a}{k}-1}(1-t)^{\frac{b}{k}-1} \exp\left(\frac{-A}{t^{\frac{m}{k}}(1-t)^{\frac{m}{k}}}\right) E^{\delta}_{k,\alpha,\beta,p}(zt^{\frac{\alpha}{k}})dt$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_k(\alpha n+\beta)(\delta)_{pn,k}} B_k(n\alpha + a, b; A; m). \tag{3.1}$$

2. On setting $\alpha = \beta = q = \gamma = 1$ in Theorem 2.1, we find:

$$\int_0^1 t^{\frac{a}{k}-1}(1-t)^{\frac{b}{k}-1} \exp\left(\frac{-A}{t^{\frac{m}{k}}(1-t)^{\frac{m}{k}}}\right) E^{\delta}_{k,p}(zt^{\frac{1}{k}})dt$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{(\delta)_{pn,k}} B_k(n + a, b; A; m). \tag{3.2}$$

3. On setting $\delta = p = 1$ in Theorem 2.1, we attain:

$$\int_0^1 t^{\frac{a}{k}-1}(1-t)^{\frac{b}{k}-1} \exp\left(\frac{-A}{t^{\frac{m}{k}}(1-t)^{\frac{m}{k}}}\right) E^{\gamma,q}_{k,\alpha,\beta}(zt^{\frac{\alpha}{k}})dt$$

$$= \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} z^n}{\Gamma_k(\alpha n+\beta)} B_k(n\alpha + a, b; A; m). \tag{3.3}$$

4. On setting $\gamma = q = 1$ in Theorem 2.2, we achieve:

$$\int_t^x (x-s)^{\frac{\rho}{k}-1}(s-t)^{\frac{\mu}{k}-1} \exp\left(\frac{-A}{(x-s)^{\frac{m}{k}}(s-t)^{\frac{m}{k}}}\right) E^{\delta}_{k,\alpha,\beta,p}(z(x-s)^{\frac{\alpha}{k}})dt$$

$$= \sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \frac{(x-t)^{\frac{\rho+\mu+n\alpha-2mr}{k}-2}(-A)^r z^n}{\Gamma_k(\alpha n+\beta)(\delta)_{pn,k} r!} B_k(\mu+n\alpha-mr, \rho-mr). \tag{3.4}$$

5. On setting $\alpha=\beta=\gamma=q=1$ in Theorem 2.2, we acquire:

$$\int_t^x (x-s)^{\frac{\rho}{k}-1}(s-t)^{\frac{\mu}{k}-1} \exp\left(\frac{-A}{(x-s)^{\frac{m}{k}}(s-t)^{\frac{m}{k}}}\right) E_{k,p}^{\delta}(z(x-s)^{\frac{1}{k}})dt$$

$$= \sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \frac{(x-t)^{\frac{\rho+\mu+n-2mr}{k}-2}(-A)^r z^n}{(\delta)_{pn,k} r!} B_k(\mu+n-mr, \rho-mr). \tag{3.5}$$

6. On setting $\delta=p=1$ in Theorem 2.2, we found:

$$\int_t^x (x-s)^{\frac{\rho}{k}-1}(s-t)^{\frac{\mu}{k}-1} \exp\left(\frac{-A}{(x-s)^{\frac{m}{k}}(s-t)^{\frac{m}{k}}}\right) E_{k,\alpha,\beta}^{\gamma,q}(z(x-s)^{\frac{\alpha}{k}})dt$$

$$= \sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \frac{(x-t)^{\frac{\rho+\mu+n\alpha-2mr}{k}-2}(\gamma)_{qn,k}(-A)^r z^n}{\Gamma(\alpha n+\beta) r!} B_k(\mu+n\alpha-mr, \rho-mr). \tag{3.6}$$

7. On setting $\gamma=q=1$ in Theorem 2.3, we find:

$$\int_0^1 t^{\frac{\lambda}{k}-1}(1-t)^{\frac{\mu-\lambda}{k}-1}\left(1-ut^{\frac{\rho}{k}}(1-t)^{\frac{\sigma}{k}}\right)^{-a} \exp\left(\frac{-A}{t^{\frac{m}{k}}(1-t)^{\frac{m}{k}}}\right) E_{k,\alpha,\beta,p}^{\delta}(zt^{\frac{\alpha}{k}})dt$$

$$= \sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_r u^r z^n}{\Gamma_k(\alpha n+\beta)(\delta)_{pn,k} r!} B_k(\lambda+n\alpha+\rho r, \mu-\lambda+r\sigma; A; m). \tag{3.7}$$

8. On setting $\alpha=\beta=\gamma=q=1$ in Theorem 2.3, we get:

$$\int_0^1 t^{\frac{\lambda}{k}-1}(1-t)^{\frac{\mu-\lambda}{k}-1}\left(1-ut^{\frac{\rho}{k}}(1-t)^{\frac{\sigma}{k}}\right)^{-a} \exp\left(\frac{-A}{t^{\frac{m}{k}}(1-t)^{\frac{m}{k}}}\right) E_{k,p}^{\delta}(zt^{\frac{1}{k}})dt$$

$$= \sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_r u^r z^n}{(\delta)_{pn} r!} B_k(\lambda+n+\rho r, \mu-\lambda+r\sigma; m; A). \tag{3.8}$$

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