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Unified Common Fixed Point Theorems in Complex Valued Metric Spaces Via an Implicit Relation With Applications

Waleed Mohd. Alfaqih [1], Mohammad Imdad and Fayyaz Rouzkard

ABSTRACT: The purpose of this paper is to prove some common fixed point theorems for two pairs of weakly compatible mappings in complex valued metric spaces satisfying an implicit relation. Several illustrative examples are given which demonstrate the usefulness of our utilized implicit relation. Beside generalizing and improving several well known core results of the existing literature we can deduce several new contractions which have not obtained before in complex valued metric spaces. As an application of our results, we prove the existence and uniqueness of common solution of Hammerstein as well as Urysohn integral equations.

Key Words: Complex valued metric spaces, Common fixed point, Weakly compatible mappings, Implicit relation, Hammerstein integral equations, Urysohn integral equations.

Contents

1. Introduction and Preliminaries

Due to its applications, fixed point theory has demonstrated to be a useful branch of nonlinear analysis. In 1922, Banach introduced the most powerful principle (Banach contraction principle) which has been extended and generalized to many directions with several applications to many branches.

In 1997, Popa [10] initiated the idea of an implicit relation which is designed to cover several well known contractions of the existing literature in one go besides admitting several new contractions. Thereafter, several authors proved a multitude fixed point theorems (see [9,13,11,12] and references cited therein). In fact, the strength of implicit relations lies in their unifying power besides being general enough to a multitude yield new contractions.

[1] The corresponding author
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Recently, Azam et al. [3] introduced the concept of complex valued metric spaces which is relatively more general than metric spaces and also proved common fixed point theorems for two mappings satisfying certain rational inequalities. Since then, several papers have dealt with fixed point theory in complex valued metric spaces (see [2,4,5,16,14,22,21,18,19,17,20,23,14] and references cited therein).

Though complex metric spaces form a special class of cone metric spaces, yet the definition of a cone metric space banks on the underlying Banach space which is not a division ring. Hence, rational expressions are not meaningful in cone metric spaces and henceforth many results involving rational contractions can not be generalized to cone metric spaces. So, with a view to prove results involving rational inequalities Azam et al. [3] propounded the idea of complex metric spaces. In cone metric spaces the underlying metric assumes values in linear spaces where the linear space may be even infinite dimensional, whereas in the case of complex metric spaces the metric values belong to the set of complex numbers which is one dimensional vector space over the complex field. This is an instance which paves the way to consider complex metric spaces independently.

The aim of this paper is to utilize the idea of implicit relation in complex valued metric spaces to prove unified common fixed point results for two pairs of weakly compatible mappings satisfying an implicit relation such that these results unify, improve and generalize many existence results of the literature. We furnish with some examples to clarify that our implicit relation covers many of the exitance results in the context of complex valued metric spaces and is also general enough to yield some new contraction conditions.

Let $\mathbb{C}$ be the set of all complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \Longleftrightarrow Re(z_1) \leq Re(z_2) \text{ and } Im(z_1) \leq Im(z_2).$$

It follows that $z_1 \precsim z_2$, if on of the following conditions is satisfied:

(i) $Re(z_1) = Re(z_2),\ Im(z_1) = Im(z_2)$,

(ii) $Re(z_1) < Re(z_2),\ Im(z_1) = Im(z_2)$,

(iii) $Re(z_1) = Re(z_2),\ Im(z_1) < Im(z_2)$,

(iv) $Re(z_1) < Re(z_2),\ Im(z_1) < Im(z_2)$.

In particular, we write $z_1 = z_2$ if (i) holds and we write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (ii), (iii) and (iv) is satisfied while $z_1 \prec z_2$ if only (iv) is satisfied.

Throughout this presentation, $\mathbb{N}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}_+$ respectively denote the set of natural numbers, the set of rational numbers, the set of real numbers and the set of all $z \in \mathbb{C}$ such that $0 \precsim z$. Also, $\succsim$ is the dual relation of $\precsim$ and I stands for the identity mapping.

Remark 1.1. *Note that the following assertions hold for all* $z_1, z_2, z_3 \in \mathbb{C}$*:*

1. $\alpha, \beta \in \mathbb{R}$ *with* $\alpha \leq \beta$ *and* $0 \precsim z_1 \Longrightarrow \alpha z_1 \precsim \beta z_1$*;*
2. $0 \precsim z_1 \precnsim z_2 \Longrightarrow |z_1| < |z_2|$*;*
3. $z_1 \precsim z_2, z_2 \prec z_3 \Longrightarrow z_1 \prec z_3$*;*

The following basic definitions and results are required in the sequel.

Definition 1.2. *[3] Let X be a nonempty set. Suppose that the mapping* $d : X \times X \longrightarrow \mathbb{C}_+$ *satisfies the following conditions:*

(i) $d(x, y) = 0$ *if and only if* $x = y$ *for all* $x, y \in X$*;*

(ii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$*;*

(iii) $d(x, y) \precsim d(x, z) + d(z, y)$ *for all* $x, y, z \in X$*.*

Then the mapping d is called a complex valued metric and the pair (X, d) is called a complex valued metric space.

Remark 1.3. *In Definition 1.2 we ignore stating the nonnegative property* $0 \precsim d(x, y)$ *for all* $x, y \in X$ *since it follows from* $(i), (ii)$ *and* (iii)*.*

Definition 1.4. *[3] Let (X, d) be a complex valued metric space. Then*

(i) a point x in X is said to be an interior point of a subset M of X, if there exists $0 \prec \varepsilon \in \mathbb{C}$ *such that*

$$N(x, \varepsilon) = \{y \in X : d(x, y) \prec \varepsilon\} \subseteq M,$$

(ii) a point x in X is called a limit point of a subset M of X, if for every $0 \prec \varepsilon \in \mathbb{C}$*,*

$$N(x, \varepsilon) \cap (M \setminus \{x\}) \neq \phi,$$

(iii) a subset M of X is called an open set, if every element of M is an interior point of M,

(iv) a subset M of X is called a closed set, if every limit point of M belongs to M,

(v) the family $\digamma = \{N(x, \varepsilon) : x \in X, 0 \prec \varepsilon \in \mathbb{C}\}$ *forms a subbasis of a Hausdorff topology τ on X.*

Example 1.5. *Let* $X = C\big([a, b], \mathbb{R}^n\big)$ *where* $a, b \in \mathbb{R}, 0 < a \leq b$*. Define a mapping* $d : X \times X \longrightarrow \mathbb{C}$ *as follows:*

$$d(x, y) = \max_{t \in [a,b]} \|x(t) - y(t)\|_\infty \sqrt{1 + a^2} e^{i \arctan a}.$$

Then (X, d) is a complex valued metric space.

Definition 1.6. *[8] The* max *function for complex numbers with partial order relation $\precsim$ is defined as follows for all $z_1, z_2 \in \mathbb{C}$:*

$$\max\{z_1, z_2\} = z_2 \Longleftrightarrow |z_1| \leq |z_2|.$$

Definition 1.7. *[3] Let $\{x_n\}$ be a sequence in a complex valued metric space (X, d) and $x \in X$. Then*

(i) $\{x_n\}$ converges to x, If for every $0 \prec \varepsilon \in \mathbb{C}$ there exists an $n_0 \in \mathbb{N}$ such that

$$d(x_n, x) \prec \varepsilon \quad \forall n > n_0,$$

and denote this symbiotically by $\lim_{n\to\infty} x_n = x$ or $x_n \longrightarrow x$, as $n \longrightarrow \infty$,

(ii) $\{x_n\}$ is said to be a Cauchy sequence if for every $0 \prec \varepsilon \in \mathbb{C}$ there exists an $n_0 \in \mathbb{N}$ such that

$$d(x_n, x_{n+m}) \prec \varepsilon \quad \forall n > n_0,$$

where $m \in \mathbb{N}$,

(iii) (X, d) is called a complete complex valued metric space if every Cauchy sequence in X is convergent in X.

Lemma 1.8. *[22] Let (X, d) be a complex valued metric space, $\{x_n\}$ a sequence in X and $\lambda \in [0, 1)$. If $\alpha_n = |d(x_n, x_{n+1})|$ satisfies $\alpha_n \leq \lambda\alpha_{n-1}$, for all $n \in \mathbb{N}$, then $\{x_n\}$ is Cauchy sequence.*

Definition 1.9. *Let S, T, f and g be four self-mappings of a nonempty set X. Then*

(i) a point $u \in X$ is said to be a fixed point of S if $Su = u$,

(ii) a point $u \in X$ is said to be a common fixed point of S and T if $Su = Tu = u$,

(iii) a point $u \in X$ is said to be a coincidence point of S and f if $Su = fu$ and a point $t \in X$ such that $t = Su = fu$ is called a point of coincidence of S and f,

(iv) a pint $t \in X$ is said to be a common point of coincidence of the pairs (S, f) and (T, g) if there exist $u, v \in X$ such that $Su = fu = t$ and $Tv = gv = t$.

Now, we introduce the following definition involving four finite families of mappings.

Definition 1.10. *Four families of self mappings $\{S_i\}_1^l, \{f_i\}_1^m, \{T_i\}_1^n$ and $\{g_i\}_1^s$ defined over a nonempty set X,(where $l, m, n, s \in \mathbb{N}$), are said to be pairwise commuting if:*

(i) $S_iS_j = S_jS_i$ for $i, j \in \{1, 2, ..., l\}$,

(ii) $S_if_j = f_jS_i$ for $i \in \{1, 2, ..., l\}, j \in \{1, 2, ..., m\}$,

(iii) $S_iT_j = T_jS_i$ *for* $i \in \{1, 2, ..., l\}, j \in \{1, 2, ..., n\}$,

(iv) $S_ig_j = g_jS_i$ *for* $i \in \{1, 2, ..., l\}, j \in \{1, 2, ..., s\}$,

(v) $f_if_j = f_jf_i$ *for* $i, j \in \{1, 2, ..., m\}$,

(vi) $f_iT_j = T_jf_i$ *for* $i \in \{1, 2, ..., m\}, j \in \{1, 2, ..., n\}$,

(vii) $f_ig_j = g_jf_i$ *for* $i \in \{1, 2, ..., m\}, j \in \{1, 2, ..., s\}$,

(viii) $T_iT_j = T_jT_i$ *for* $i, j \in \{1, 2, ..., n\}$,

(ix) $T_ig_j = g_jT_i$ *for* $i \in \{1, 2, ..., n\}, j \in \{1, 2, ..., s\}$,

(x) $g_ig_j = g_jg_i$ *for* $i, j \in \{1, 2, ..., s\}$.

Remark 1.11. *On setting* $f_i = g_j = I, \forall\ i \in \{1, 2, ..., m\}$*and* $j \in \{1, 2, ..., s\}$ *in Definition 1.10 we deduce Definition 1.11 due to Imdad et al. [6].*

Definition 1.12. *[5] Let* (X, d) *be a complex valued metric space. A pair of self mappings* (S, T) *on* X *is said to be weakly compatible if they commute at their coincidence points. i.e.,* $STx = TSx$ *whenever* $Sx = Tx, x \in X$.

Definition 1.13. *[10] A function* $f : \mathbb{C} \longrightarrow \mathbb{C}$ *is said to be a lower semicontinuous at a point* z_0 *in* $\mathbb{C}$ *if for every* $0 \prec \varepsilon \in \mathbb{C}$ *there exists a neighborhood* N *of* z_0 *such that* $f(z) \succsim f(z_0) - \varepsilon$ *for all* z *in* N*. This can also be expressed as* $\liminf_{z \to z_0} f(z) \succsim f(z_0)$*. Also,* f *is said to be an upper semicontinuous at a point* z_0 *in* $\mathbb{C}$ *if for every* $0 \prec \varepsilon \in \mathbb{C}$ *there exists a neighborhood* N *of* z_0 *such that* $f(z) \precsim f(z_0) + \varepsilon$ *for all* z *in* N*. This can be expressed as* $\limsup_{z \to z_0} f(z) \precsim f(z_0)$.

Definition 1.14. *A mapping* $f : \mathbb{C} \longrightarrow \mathbb{C}$ *is said to be a non-increasing mapping with respect to* $\precsim$ *if for every* $z_1, z_2 \in \mathbb{C}$, $z_1 \precsim z_2$ *implies* $fz_1 \succsim fz_2$.

Definition 1.15. *[1] The required control functions are defined as follows:*

(i) $\psi : \mathbb{C}_+ \longrightarrow \mathbb{C}_+$ *is a continuous nondecreasing function with* $\psi(z) = 0$ *if and only if* $z = 0$,

(ii) $\phi : \mathbb{C}_+ \longrightarrow \mathbb{C}_+$ *is a lower semicontinuous function with* $\phi(z) = 0$ *if and only if* $z = 0$.

By Ψ *and* Φ*, we respectively denote the set of all* $\psi' s$ *and the set of all* $\phi' s$.

2. An Implicit Relation

In this section, we extend the idea of an implicit relation (due to Popa [10]) to complex valued metric spaces in order to prove unified complex metrical common fixed point theorems. We are not familiar with any article dealing with such implicit functions rigorously.

Definition 2.1. *Let* $\Im$ *be the set of all complex valued lower semi-continuous functions* $F : \mathbb{C}_+^6 \longrightarrow \mathbb{C}$ *satisfying the following conditions:*

F_1 : *F is non-increasing in the 5^{th} and 6^{th} variables,*

F_2 : *there exists $h \in [0,1)$ such that for $u, v \succsim 0$ we have*
$F(u,v,v,u,u+v,0) \precsim 0$ *or* $F(u,v,u,v,0,u+v) \precsim 0$ *implies* $|u| \precsim h|v|$,

F_3 : $F(u,u,0,0,u,u) \succ 0$ *for all* $u \succ 0$.

Now, we present multitude of illustrative examples to substantiate Definition 2.1.

Example 2.2. *Define a function $F : \mathbb{C}_+^6 \longrightarrow \mathbb{C}$ as follows:*
$F(z_1,z_2,z_3,z_4,z_5,z_6) = z_1 - \lambda_1(z_2)z_2 - \lambda_2(z_2)\frac{z_4 z_3}{1+z_2}$,
where $\lambda_1, \lambda_2 : \mathbb{C}_+ \longrightarrow [0,1)$ are given continuous mappings such that $2\lambda_1(z) + \lambda_2(z) \leq 1$ $\forall z \in \mathbb{C}_+$.
F_1 : *Obvious.*
F_2 : *Let $u \succ 0$ and $F(u,v,v,u,u+v,0) = F(u,v,u,v,0,u+v) = u - \lambda_1(v)v - \lambda_2(v)\frac{uv}{1+v} \precsim 0$. This implies that $|u| \leq \lambda_1(v)|v| + \lambda_2(v)\left|\frac{uv}{1+v}\right|$ which implies that $|u| \leq \frac{1}{2}|v|$. Hence, $|u| \leq h|v|$ with $h = \frac{1}{2}$. If $u = 0$ then it is clear.*
F_3 : *Let $u \succ 0$, then $F(u,u,0,0,u,u) = u - \lambda_1(u)u \succ 0$. Hence $F \in \Im$.*

Example 2.3. *Define a function $F : \mathbb{C}_+^6 \longrightarrow \mathbb{C}$ as follows:*
$F(z_1,z_2,z_3,z_4,z_5,z_6) = z_1 - \lambda_1(z_2)z_2 - \lambda_2(z_2)\frac{z_2 z_4}{1+z_2}$,
where $\lambda_1, \lambda_2 : \mathbb{C}_+ \longrightarrow [0,1)$ are given continuous mappings such that $2\lambda_1(z) + \lambda_2(z) \leq 1$ $\forall z \in \mathbb{C}_+$.

Example 2.4. *Define a function $F : \mathbb{C}_+^6 \longrightarrow \mathbb{C}$ as follows:*
$F(z_1,z_2,z_3,z_4,z_5,z_6) = \psi(z_1) - \psi\left(\frac{z_4 z_5 z_6}{1+z_2}\right)$, *where $\psi \in \Psi$ with $\psi(z_1) \precsim \psi(z_2) \Leftrightarrow z_1 \precsim z_2$.*

Example 2.5. *Define a function $F : \mathbb{C}_+^6 \longrightarrow \mathbb{C}$ as follows:*

$$F(z_1,z_2,z_3,z_4,z_5,z_6) = z_1 - \lambda z_2 - \mu\frac{z_3 z_4}{1+z_3+z_4},$$

where $\lambda, \mu \in \mathbb{R}_+$ such that $\lambda + \mu < 1$.
F_1 : *Obvious.*
F_2 : *Let $u \succ 0$, then $F(u,v,v,u,u+v,0) = u - \lambda v - \mu\frac{vu}{1+v+u} \precsim 0 \Rightarrow |u| \leq \lambda|v| + \mu\left|\frac{vu}{1+v+u}\right| \leq (\lambda+\mu)|v|$. Similarly, $F(u,v,u,v,0,u+v) \precsim 0$ implies that $|u| \leq (\lambda+\mu)|v|$. Hence, $|u| \leq h|v|$ for $h = \lambda + \mu$. If $u = 0$, then it is clear.*
F_3 : *Let $u \succ 0$, then $F(u,u,0,0,u,u) = u - \lambda u \succ 0$. Hence $F \in \Im$.*

Example 2.6. *Define a function $F : \mathbb{C}_+^6 \longrightarrow \mathbb{C}$ as follows:*

$$F(z_1,z_2,z_3,z_4,z_5,z_6) = \begin{cases} z_1 - \alpha z_3 - \beta z_3\frac{z_2+z_4}{z_3+z_4}, & if\, z_3+z_4 \neq 0; \\ z_1, & if\, z_3+z_4 = 0, \end{cases}$$

where $\alpha, \beta \in \mathbb{R}_+$ such that $\alpha + 2\beta < 1$.

Example 2.7. *Define a function $F : \mathbb{C}^6_+ \longrightarrow \mathbb{C}$ as follows:*

$$F(z_1, z_2, z_3, z_4, z_5, z_6) = \begin{cases} z_1 - \alpha_1 z_2 - \alpha_2 \frac{z_2 z_3}{z_2+z_3+z_4} - \alpha_3 \frac{z_2 z_4}{z_2+z_3+z_4} \\ \qquad - \alpha_4 \frac{z_3 z_4}{z_2+z_3+z_4}, & if\ \Delta \neq 0; \\ z_1, & if\ \Delta = 0, \end{cases}$$

where $\Delta = z_2 + z_3 + z_4$ and $\alpha_i \in \mathbb{R}_+, i = 1, 2, 3, 4$ such that $\sum_{i=1}^4 \alpha_i < 1$.

Example 2.8. *Define a function $F : \mathbb{C}^6_+ \longrightarrow \mathbb{C}$ as follows:*

$$F(z_1, z_2, z_3, z_4, z_5, z_6) = z_1 - \alpha_1 z_3 - \alpha_2(z_2 + z_5) - \alpha_3(z_4 + z_6),$$

where $\alpha_i \in \mathbb{R}_+, i = 1, 2, 3$ such that $\alpha_1 + 3\alpha_2 + 3\alpha_3 < 1$.

Example 2.9. *Define a function $F : \mathbb{C}^6_+ \longrightarrow \mathbb{C}$ as follows:*

$$F(z_1, z_2, z_3, z_4, z_5, z_6) = z_1 - \alpha z_3 - \beta \frac{z_2(1 + z_3)}{1 + z_3 + z_4},$$

where $\alpha, \beta \in [0, 1)$ such that $\alpha + \beta < 1$.

Example 2.10. *Define a function $F : \mathbb{C}^6_+ \longrightarrow \mathbb{C}$ as follows:*

$$\begin{aligned} F(z_1, z_2, z_3, z_4, z_5, z_6) &= z_1 - \alpha_1 z_2 - \alpha_2(z_3 + z_4) - \alpha_3(z_5 + z_6) \\ &\quad - \alpha_4 \frac{z_4(1 + z_3)}{1 + z_2 + z_3} - \alpha_5 \frac{z_3(1 + z_4)}{1 + z_2 + z_4} - \alpha_6 z_5, \end{aligned}$$

where $\alpha_i \in \mathbb{R}_+, i = 1, 2, ..., 6,$ such that $\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 + 2\alpha_6 < 1$.

Example 2.11. *Define a function $F : \mathbb{C}^6_+ \longrightarrow \mathbb{C}$ as follows:*

$$F(z_1, z_2, z_3, z_4, z_5, z_6) = \begin{cases} z_1 - \lambda \frac{z_2 z_3 + z_2 z_4 + z_3 z_4}{z_2+z_3+z_4}, & if\ z_2 + z_3 + z_4 \neq 0; \\ z_1, & if\ z_2 + z_3 + z_4 = 0, \end{cases}$$

where $\lambda \in \mathbb{R}_+$ such that $\lambda < \frac{1}{3}$.

Example 2.12. *Define a function $F : \mathbb{C}^6_+ \longrightarrow \mathbb{C}$ as follows:*
$F(z_1, z_2, z_3, z_4, z_5, z_6) = z_1 - \alpha z_3 - \beta \max\{z_2, z_3, z_6\} - \gamma \max\{z_3, z_5\}$, where $\alpha, \beta, \gamma \in \mathbb{R}_+$ such that $\alpha + 2\beta + 2\gamma < 1$.

Example 2.13. *Define a function $F : \mathbb{C}^6_+ \longrightarrow \mathbb{C}$ as follows:*
$F(z_1, z_2, z_3, z_4, z_5, z_6) = z_1 - \alpha \max\{z_2, z_3, z_4, z_5, z_6\}, \alpha \in [0, 1/2)$.

Example 2.14. *Define a function $F : \mathbb{C}^6_+ \longrightarrow \mathbb{C}$ as follows:*
$F(z_1, z_2, z_3, z_4, z_5, z_6) = \alpha_1 z_1 - \alpha_2 z_2 - \alpha_3 z_3 - \alpha_4 z_4 - \alpha_5 z_5 - \alpha_6 z_6$,
where $\alpha_i \in \mathbb{R}_+, i = 1, 2, ..., 6$ such that $\alpha_2 + \alpha_3 + \alpha_4 + 2\alpha_5 + 2\alpha_6 < \alpha_1$ and $\alpha_1 > 0$.

Example 2.15. *Define a function $F : \mathbb{C}^6_+ \longrightarrow \mathbb{C}$ as follows:*
$F(z_1, z_2, z_3, z_4, z_5, z_6) = z_1 - \alpha \max\{z_3 + z_4, z_5 + z_6\}, \alpha \in [0, 1/2)$.

Example 2.16. *Define a function* $F : \mathbb{C}_+^6 \longrightarrow \mathbb{C}$ *as follows:*
$F(z_1, z_2, z_3, z_4, z_5, z_6) = z_1 - \alpha \max\{z_2, z_3, z_4, z_5 + z_6\}, \alpha \in [0, 1/2)$.

Example 2.17. *Define a function* $F : \mathbb{C}_+^6 \longrightarrow \mathbb{C}$ *as follows:*
$F(z_1, z_2, z_3, z_4, z_5, z_6) = z_1 - \alpha \max\{z_2, z_3, \ z_5, z_4 + z_6\}, \alpha \in [0, 1/3)$.

Example 2.18. *Define a function* $F : \mathbb{C}_+^6 \longrightarrow \mathbb{C}$ *as follows:*
$F(z_1, z_2, z_3, z_4, z_5, z_6) = z_1 - \alpha(z_4 + z_6) - \alpha \max\{z_2, z_5\}, \alpha \in [0, 1/4)$.

Example 2.19. *Define a function* $F : \mathbb{C}_+^6 \longrightarrow \mathbb{C}$ *as follows:*
$F(z_1, z_2, z_3, z_4, z_5, z_6) = z_1 - \alpha \max\ \{\frac{2z_2+z_5}{2}, \frac{2z_2+z_4}{2}, \frac{2z_2+z_6}{2}\}, \alpha \in [0, \frac{1}{2})$.

Example 2.20. *Define a function* $F : \mathbb{C}_+^6 \longrightarrow \mathbb{C}$ *as follows:*
$F(z_1, z_2, z_3, z_4, z_5, z_6) = z_1 - \alpha \max\ \{\frac{z_2+z_5}{2}, \frac{z_4+z_6}{2}\}, \alpha \in [0, 2/3)$.

Example 2.21. *Define a function* $F : \mathbb{C}_+^6 \longrightarrow \mathbb{C}$ *as follows:*
$F(z_1, z_2, z_3, z_4, z_5, z_6) = z_1 - \alpha \max\{z_2, z_3, z_4, z_5, z_6\} - \beta[z_5 + z_6]$,
where $\alpha, \beta \in \mathbb{R}_+$ *such that* $\alpha + \beta < \frac{1}{2}$.

Example 2.22. *Define a function* $F : \mathbb{C}_+^6 \longrightarrow \mathbb{C}$ *as follows:*
$F(z_1, z_2, z_3, z_4, z_5, z_6) = z_1 - \alpha \max\{z_2, \frac{z_3+z_4}{2}, \frac{z_5+z_6}{2}\}, \alpha \in [0, 1)$.

3. Main Results

In this section, we present the main results of this paper.

Theorem 3.1. *Let S, T, f and g be four self mappings on a complex valued metric space (X, d) such that $SX \subseteq gX$ and $TX \subseteq fX$. Assume that there exists $F \in \Im$ such that for all $x, y \in X$,*

$$F(d(Sx, Ty), d(fx, gy), d(Sx, fx), d(Ty, gy), d(Ty, fx), d(Sx, gy)) \precsim 0. \tag{3.1}$$

If $fX \cup gX$ is complete subspace of X, then the pairs (S, f) and (T, g) have a unique common point of coincidence.

Moreover, if the pairs (S, f) and (T, g) are weakly compatible, then S, T, f and g have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. Since $SX \subseteq gX$, we can find a point x_1 in X such that $Sx_0 = gx_1$. Also, since $TX \subseteq fX$, we can choose a point x_2 in X with $Tx_1 = fx_2$. Thus, in general for the point x_{2n} one can find a point x_{2n+1} such that $Sx_{2n} = gx_{2n+1}$ and also a point x_{2n+2} with $Tx_{2n+1} = fx_{2n+2}$ for $n = 0, 1, 2, \ldots$. Hence, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ by the rule

$$Sx_{2n-2} = gx_{2n-1} = y_{2n-1} \ \ \textit{and} \ \ Tx_{2n-1} = fx_{2n} = y_{2n} \quad , n \in \mathbb{N}. \tag{3.2}$$

Clearly $\{y_n\} \subseteq fX \cup gX$. Now, we prove that $\{y_n\}$ is a Cauchy sequence. Taking $x = x_{2n}$ and $y = x_{2n+1}$ in (3.1), we have

$$F(d(Sx_{2n}, Tx_{2n+1}), d(fx_{2n}, gx_{2n+1}), d(Sx_{2n}, fx_{2n}), d(Tx_{2n+1}, gx_{2n+1}) \\ , d(Tx_{2n+1}, fx_{2n}), d(Sx_{2n}, gx_{2n+1})) \precsim 0. \tag{3.3}$$

On using (3.2) and (3.3), we have

$$F(d(y_{2n+1}, y_{2n+2}), d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n}),$$
$$d(y_{2n+2}, y_{2n+1}), d(y_{2n+2}, y_{2n}), 0) \precsim 0.$$

Now, due to F_1 and triangular inequality, we have

$$F(d(y_{2n+1}, y_{2n+2}), d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n}), d(y_{2n+2}, y_{2n+1}),$$
$$d(y_{2n+2}, y_{2n+1}) + d(y_{2n+1}, y_{2n}), 0) \precsim 0,$$

implying thereby $|d(y_{2n+1}, y_{2n+2})| \leq h|d(y_{2n}, y_{2n+1})|$ (due to F_2). Similarly, by taking $x = x_{2n+2}$ and $y = x_{2n+1}$ in (3.1), one can prove that $|d(y_{2n}, y_{2n+1})| \leq h|d(y_{2n-1}, y_{2n})|$. Thus, $|d(y_n, y_{n+1})| \leq h|d(y_{n-1}, y_n)| \quad \forall n \in \mathbb{N} - \{1\}$. Hence, by Lemma 1.8, $\{y_n\}$ is a Cauchy sequence in $fX \cup gX$. Since $fX \cup gX$ is complete it follows that $\{y_n\}$ converges to some $t \in fX \cup gX$. Therefore, in the light of (3.2), one can have

$$\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} gx_{2n+1} = t. \tag{3.4}$$

Now, if $t \in gX$, then there exists $u \in X$ such that $gu = t$. We assert that $Tu = t$. On contrary, assume that $d(Tu, t) \succ 0$. Putting $x = x_{2n}$ and $y = u$ in (3.1), we have

$$F(d(Sx_{2n}, Tu), d(fx_{2n}, gu), d(Sx_{2n}, fx_{2n}),$$
$$d(Tu, gu), d(Tu, fx_{2n}), d(Sx_{2n}, gu)) \precsim 0.$$

Taking $n \longrightarrow \infty$ and using (3.4), we obtain

$$F(d(t, Tu), d(t, gu), 0, d(Tu, gu), d(Tu, t), d(t, gu)) \precsim 0.$$

Since $gu = t$, we have

$$F(d(t, Tu), 0, 0, d(Tu, t), d(Tu, t), 0) \precsim 0,$$

yielding thereby $|d(t, Tu)| = 0$ (due to F_2). Hence, $Tu = t$. Therefore, we have

$$Tu = gu = t, \tag{3.5}$$

proving that t is a point of coincidence of the pair (T, g).

Since $TX \subseteq fX$, there exists $v \in X$ such that $fv = t$. Setting $x = v$ and $y = x_{2n+1}$ in (3.1), and using similar arguments one can prove that

$$Sv = fv = t. \tag{3.6}$$

That is, t is a point of coincidence of the pair (S, f). Hence, t is a common point of coincidence of (S, f) and (T, g).

Now, we prove that t is unique. Let t' be a point of coincidence of both (S,f) and (T,g) such that $d(t,t') \succ 0$. Then there exist $u',v' \in X$ such that $Su' = fu' = t'$ and $Tv' = gv' = t'$. Setting $x = u'$ and $y = v$ in (3.1), we have

$$F(d(Su',Tv),d(fu',gv),d(Su',fu'),d(Tv,gv),d(Tv,fu'),d(Su',gv) \quad \precsim \quad 0,$$

so that $F(d(t',t),d(t',t),0,0,d(t,t'),d(t',t) \precsim 0$, which is a contradiction to F_3. Therefore, (S,f) and (T,g) have a unique point of coincidence.

Now, on using (3.5), (3.6) and the weak compatibility of the pairs (S,f) and (T,g), we have

$$St = Sfv = fSv = ft, \tag{3.7}$$

$$Tt = Tgu = gTu = gt. \tag{3.8}$$

That is, t is a coincidence point of the pairs (S,f) and (T,g).

Next, we show that t is a common fixed point of S,T,f and g. First we show that $St = t$. If not, then $d(St,t) \succ 0$. Setting $x = t$ and $y = u$ in (3.1), we have

$$F(d(St,Tu),d(ft,gu),d(St,ft),d(Tu,gu),d(Tu,ft),d(St,gu) \precsim 0.$$

Using (3.5) and (3.7),we obtain

$$F(d(St,t),d(St,t),0,0,d(t,St),d(St,t) \precsim 0,$$

which is a contradiction to F_3. Thus, $St = ft = t$. Similarly, one can prove that $Tt = gt = t$. Hence, we have $St = Tt = ft = gt = t$. That is, t is a common fixed point of S,T,f and g.

The uniqueness of the common fixed point of S,T,f and g is an easy consequence of the uniqueness of the common point of coincidence of the pairs (S,f) and (T,g). The proof is similar in case $t \in fX$, hence, it is omitted. This completes the proof.

Remark 3.2. *Theorem 3.1 generalizes and improves Theorem 2 of Popa [10].*

Theorem 3.3. *The conclusions of Theorem 3.1 remain true if the completeness of $gX \cup fX$ is replaced by the completeness of one of the subspaces SX, TX, fX or gX.*

Remark 3.4. *Theorem 3.3 generalizes Theorem 2.1 of Imdad et al. [7].*

As a consequence of Theorems 3.1 and 3.3, we have the following theorem for four finite families of self mappings defined on a complex valued metric space which can be viewed as a generalization to Theorem 2.2 of Imdad et al. [7].

Theorem 3.5. *Let $\{S_i\}_1^l, \{T_j\}_1^n, \{f_k\}_1^m$ and $\{g_r\}_1^s$ be four finite pairwise commuting families of self mappings defined on a complex valued metric space (X,d). Let $S = S_1S_2...S_l, T = T_1T_2...T_n, f = f_1f_2...f_m$ and $g = g_1g_2...g_s$ satisfying inequality (3.1), $SX \subseteq gX$, $TX \subseteq fX$ and one of SX, TX, fX, gX and $gX \cup fX$ is complete subspace of X. Then* *(a) the pairs (S, f) and (T, g) have a unique common point of coincidence,*

(b) S, T, f and g have a unique common fixed point,

(c) the component maps of the families $\{S_i\}_1^l, \{T_j\}_1^m, \{f_k\}_1^n$ and $\{g_r\}_1^s$ have a unique common fixed point.

Proof. By the componentwise commutativity of the pairs $\left(\{S_i\}_1^l, \{f_k\}_1^m\right)$ and $\left(\{T_j\}_1^n, \{g_r\}_1^s\right)$, one can prove that $Sf = fS$ and $Tg = gT$ and hence, the pairs (S, f) and (T, g) are weak compatible. Consequently, Theorems 3.1 and 3.3 are applicable for S, T, f and g which establish (a) and (b).

Now, we show that t is also a common fixed point of the component maps of the families $\{S_i\}_1^l, \{T_j\}_1^m, \{f_k\}_1^n$ and $\{g_r\}_1^s$. To do this, consider

$$\begin{aligned} S(S_it) &= (SS_i)t = (S_1S_2...S_lS_i)t = (S_1S_2...S_iS_l)t = (S_1S_i...S_l)t \\ &= (S_iS_1...S_l)t = (S_iS)t = S_i(St) = S_it \ , i \in \{1, 2, ..., l\}. \end{aligned}$$

Similarly, one can also show that $S(T_jt) = T_jt, S(f_kt) = f_kt, S(g_rt) = g_rt, T(T_jt) = T_jt, T(S_it) = S_it, T(f_kt) = f_kt, T(g_rt) = g_rt, f(f_kt) = f_kt, f(S_it) = S_it, f(T_jt) = T_jt, f(g_rt) = g_rt, g(g_rt) = g_rt, g(S_it) = S_it, g(T_jt) = T_jt, g(f_kt) = f_kt$, for all $i \in \{1, 2, ..., l\}, j \in \{1, 2, ..., m\}, k \in \{1, 2, ..., n\}$ and $r \in \{1, 2, ..., s\}$. Therefore, S_it, T_jt, f_kt and g_rt are also fixed points of S, T, f and g. But in view of (b) the common fixed point of S, T, f and g is unique and hence (for all i, j, k and r) one gets

$$S_it = T_jt = f_kt = g_rt = t,$$

proving that t is a common fixed point of S_it, T_jt, f_kt and g_rt for all i, j, k and r. Finally, we observe that t is unique common fixed point of S_it, T_jt, f_kt and g_rt for all i, j, k and r. Otherwise, let t^* another common fixed point of S_it, T_jt, f_kt and g_rt for all i, j, k and r. Then one can prove that t^* is also a common fixed point of S, T, f and g which is a contradiction. This completes the proof.

By setting $S_i = S, T_j = T, f_k = f, g_r = g$, for all i, j, k and r, in Theorem 3.5 one can deduce the following theorem which can be viewed as a partial generalization of Theorems 3.1 and 3.3.

Theorem 3.6. *Let (X, d) be a complex valued metric space and S, T, f and g be four self mappings on X. Assume that there exists $F \in \Im$ such that for all $x, y \in X$,*

$$\begin{aligned} F(d(S^lx, T^my), d(f^nx, g^sy), d(S^lx, f^nx), d(T^my, g^sy), d(T^my, f^nx), \\ d(S^lx, g^sy)) \precsim 0, \quad (3.9) \end{aligned}$$

where $l, m, n, s \in \mathbb{N}$.

If $S^lX \subseteq g^sX$, $T^mX \subseteq f^nX$ and one of S^lX, T^mX, f^nX, g^sX and $g^sX \cup f^nX$ is complete, then

(a) the pairs (S, f) and (T, g) have a unique common point of coincidence,

(b) S, T, f and g have a unique common fixed point.

The following example shows that Theorem 3.6 is genuine but partial extension of Theorem 3.1.

Example 3.7. *Consider $X = [0,1]$ equipped with the complex metric $d(x,y) = i|x-y|$. Let S, T, f and g be four self mappings defined on X as follows:*
$Sx = \begin{cases} 1, & if \quad x \in [0,1] \cap \mathbb{Q}; \\ 0, & if \quad x \in [0,1] \cap \mathbb{Q}^c, \end{cases}$ $Tx = \begin{cases} 1, & if \quad x \in [0,1] \cap \mathbb{Q}; \\ \frac{1}{2}, & if \quad x \in [0,1] \cap \mathbb{Q}^c, \end{cases}$
$fx = \begin{cases} \frac{1}{2}, & if \quad 0 \leq x < 1; \\ 1, & if \quad x = 1, \end{cases}$ *and* $gx = \begin{cases} \frac{1}{4}, & if \quad 0 \leq x < 1; \\ 1, & if \quad x = 1. \end{cases}$
Then, $S^2X = \{1\} \subseteq \{\frac{1}{4}, 1\} = gX$ and $T^2X = \{1\} \subseteq \{\frac{1}{2}, 1\} = fX$. Also

$$\begin{aligned} 0 = d(S^2x, T^2y) &\precsim \frac{1}{2} \begin{cases} i|\frac{1}{2} - \frac{1}{4}|, & if \quad 0 \leq x, y < 1; \\ i|\frac{1}{2} - 1|, & if \quad 0 \leq x < 1, y = 1; \\ i|1 - \frac{1}{4}|, & if \quad x = 1, 0 \leq y < 1; \\ 0, & if \quad x = y = 1; \end{cases} \\ &= \frac{1}{2} d(fx, gy) \end{aligned}$$

Define $F : \mathbb{C}_+^6 \longrightarrow \mathbb{C}_+$ as follows:

$$F(z_1, z_2, z_3, z_4, z_5, z_6) = z_1 - \frac{1}{2} z_2, \quad \textit{for all } z_1, z_2 \in \mathbb{C}_+.$$

Then $F \in \Im$. Hence, the conclusions of Theorem 3.6 remain true if (for all $x, y \in X$) implicit relation (3.9) is replaced by $d(S^2x, T^2y) \precsim \frac{1}{2} d(fx, gy)$. Thus all the conditions of Theorem 3.6 are satisfied and 1 is the unique common fixed point of S, T, f and g. Here, it is worth noting that Theorems 3.1 and 3.3 can not be used in this example since $SX = \{1, 0\} \nsubseteq \{\frac{1}{4}, 1\} = gX$.

In view of Examples 2.2-2.22, we have the following corollaries which cover, generalize and improve several known results beside yielding new contraction conditions in the context of complex valued metric spaces (e.g. $(a_2), (a_3)-(a_7), (a_{15}), (a_{16})$ and (a_{19})).

Corollary 3.8. *The conclusions of Theorems 3.1 and 3.3 remain true if (for all $x, y \in X$) implicit relation (3.1) is replaced by any one of the following:*

(a_1)

$$d(Sx, Ty) \precsim \lambda_1(d(fx, gy)) d(fx, gy) + \lambda_2(d(fx, gy)) \frac{d(Sx, fx) d(Ty, gy)}{1 + d(fx, gy)},$$

where $\lambda_1, \lambda_2 : \mathbb{C}_+ \longrightarrow [0, 1)$ are given upper semi-continuous mappings such that $2\lambda_1(z) + \lambda_2(z) \leq 1$ for all $z \in \mathbb{C}_+$.

(a_2)

$$d(Sx,Ty) \precsim \lambda_1(d(fx,gy))d(fx,gy) + \lambda_2(d(fx,gy))\frac{d(fx,gy)d(Ty,gy)}{1+d(fx,gy)},$$

where $\lambda_1, \lambda_2 : \mathbb{C}_+ \longrightarrow [0,1)$ *are given upper semi-continuous mappings such that* $2\lambda_1(z) + \lambda_2(z) \leq 1$ *for all* $z \in \mathbb{C}_+$.

(a_3)

$$\psi(d(Sx,Ty)) \precsim \psi\left(\frac{d(Ty,gy)d(Ty,fx)d(Sx,gy)}{1+d(fx,gy)}\right),$$

where $\psi \in \Psi$ *with* $\psi(z_1) \precsim \psi(z_2) \Leftrightarrow z_1 \precsim z_2$.

(a_4)

$$d(Sx,Ty) \precsim \lambda d(fx,gy) + \mu\frac{d(Sx,fx)d(Ty,gy)}{1+d(Sx,fx)+d(Ty,gy)},$$

where $\lambda, \mu \in \mathbb{R}_+$ *such that* $\lambda + \mu < 1$.

(a_5)

$$d(Sx,Ty) \precsim \begin{cases} \alpha d(Sx,fx) + \beta d(Sx,fx)\frac{d(fx,gy)+d(Ty,gy)}{d(Sx,fx)+d(Ty,gy)}, & if \Delta \neq 0; \\ 0, & if \Delta = 0, \end{cases}$$

where $\Delta = d(Sx,fx) + d(Ty,gy)$ *and* $\alpha, \beta \in \mathbb{R}_+$ *such that* $\alpha + 2\beta < 1$.

(a_6)

$$d(Sx,Ty) \precsim \begin{cases} \alpha_1 d(fx,gy) + \alpha_2\frac{d(fx,gy)d(Sx,fx)}{d(fx,gy)+d(Sx,fx)+d(Ty,gy)} & \\ \qquad +\alpha_3\frac{d(fx,gy)d(Ty,gy)}{d(fx,gy)+d(Sx,fx)+d(Ty,gy)} & \\ \qquad +\alpha_4\frac{d(Sx,fx)d(Ty,gy)}{d(fx,gy)+d(Sx,fx)+d(Ty,gy)}, & if \Delta \neq 0; \\ 0, & if \Delta = 0, \end{cases}$$

where $\Delta = d(fx,gy) + d(Sx,fx) + d(Ty,gy)$ *and* $\alpha_i \in \mathbb{R}_+, i = 1,2,3,4$ *such that* $\sum_{i=1}^4 \alpha_i < 1$.

(a_7)

$$d(Sx,Ty) \precsim \alpha_1 d(Sx,fx) + \alpha_2[d(fx,gy)+d(Ty,fx)] + \alpha_3[d(Ty,gy)+d(Sx,gy)],$$

where $\alpha_i \in \mathbb{R}_+, i = 1,2,3$ *such that* $\alpha_1 + 3\alpha_2 + 3\alpha_3 < 1$.

(a_8)

$$d(Sx,Ty) \precsim \alpha d(Sx,fx) + \beta\frac{d(fx,gy)[1+d(Sx,fx)]}{1+d(Sx,fx)+d(Ty,gy)},$$

where $\alpha, \beta \in [0,1)$ *such that* $\alpha + \beta < 1$.

(a_9)

$$d(Sx,Ty) \precsim \alpha_1 d(fx,gy) + \alpha_2[d(Sx,fx) + d(Ty,gy)] + \alpha_3[d(Ty,fx) + d(Sx,gy)] + \alpha_4 \frac{d(Ty,gy)[1 + d(Sx,fx)]}{1 + d(fx,gy) + d(Sx,fx)} + \alpha_5 \frac{d(Sx,fx)[1 + d(Ty,gy)]}{1 + d(fx,gy) + d(Ty,gy)} + \alpha_6 d(Ty,fx),$$

where $\alpha_i \in \mathbb{R}_+, i = 1,2,...,7,$ *such that* $\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 + 2\alpha_6 < 1.$

(a_{10})

$$d(Sx,Ty) \precsim \begin{cases} \lambda \frac{d(fx,gy)d(Sx,fx)+d(fx,gy)d(Ty,gy)+d(Sx,fx)d(Ty,gy)}{d(fx,gy)+d(Sx,fx)+d(Ty,gy)}, & if \Delta \neq 0; \\ 0, & if \Delta = 0, \end{cases}$$

where $\Delta = d(fx,gy) + d(Sx,fx) + d(Ty,gy)$ *and* $\lambda \in \mathbb{R}_+$ *such that* $\lambda < \frac{1}{3}$.

(a_{11})

$$d(Sx,Ty) \precsim \alpha d(Sx,fx) + \beta \max\{d(fx,gy), d(Sx,fx), d(Sx,gy)\} + \gamma \max\{d(Sx,fx), d(Ty,fx)\},$$

where $\alpha, \beta, \gamma \in \mathbb{R}_+$ *such that* $\alpha + 2\beta + 2\gamma < 1.$

(a_{12})

$$d(Sx,Ty) \precsim \alpha \max\{d(fx,gy), d(Sx,fx), d(Ty,gy), d(Ty,fx), d(Sx,gy)\},$$

where $\alpha \in \mathbb{R}_+$ *such that* $\alpha < \frac{1}{2}$.

(a_{13})

$$\alpha_1 d(Sx,Ty) \precsim \alpha_2 d(fx,gy) + \alpha_3 d(Sx,fx) + \alpha_4 d(Ty,gy) + \alpha_5 d(Ty,fx) + \alpha_6 d(Sx,gy),$$

where $\alpha_i \in \mathbb{R}_+, i = 1,2,...,6$ *such that* $\alpha_2 + \alpha_3 + \alpha_4 + 2\alpha_5 + 2\alpha_6 < \alpha_1$ *and* $\alpha_1 > 0$.

(a_{14})

$$d(Sx,Ty) \precsim \alpha \max\{d(Sx,fx) + d(Ty,gy), d(Ty,fx) + d(Sx,gy)\},$$

where $\alpha \in \mathbb{R}_+$ *such that* $\alpha < \frac{1}{2}$.

(a_{15})

$$d(Sx,Ty) \precsim \alpha \max\ \{d(fx,gy), d(Sx,fx), d(Ty,gy), d(Ty,fx) + d(Sx,gy)\},$$

where $\alpha \in \mathbb{R}_+$ *such that* $\alpha < \frac{1}{2}$.

(a_{16})

$$d(Sx,Ty) \precsim \alpha \max\{d(fx,gy), d(Sx,fx), d(Ty,fx), d(Ty,gy)+d(Sx,gy)\},$$

where $\alpha \in \mathbb{R}_+$ *such that* $\alpha < \frac{1}{3}$.

(a_{17})

$$d(Sx,Ty) \precsim \alpha[d(Ty,gy)+d(Sx,gy)] + \alpha \max\{d(fx,gy), d(Ty,fx)\},$$

where $\alpha \in \mathbb{R}_+$ *such that* $\alpha < \frac{1}{4}$.

(a_{18})

$$d(Sx,Ty) \precsim \alpha \max\left\{\frac{2d(fx,gy)+d(Ty,fx)}{2}, \frac{2d(fx,gy)+d(Ty,gy)}{2}, \frac{2d(fx,gy)+d(Sx,gy)}{2}\right\},$$

where $\alpha \in \mathbb{R}_+$ *such that* $\alpha < \frac{1}{2}$.

(a_{19})

$$d(Sx,Ty) \precsim \alpha \max\left\{\frac{d(fx,gy)+d(Ty,fx)}{2}, \frac{d(Ty,gy)+d(Sx,gy)}{2}\right\},$$

where $\alpha \in \mathbb{R}_+$ *such that* $\alpha < \frac{2}{3}$.

(a_{20})

$$d(Sx,Ty) \precsim \alpha \max\{d(fx,gy), d(Sx,fx), d(Ty,gy), d(Ty,fx), d(Sx,gy)\} + \beta[d(Ty,fx)+d(Sx,gy)],$$

where $\alpha, \beta \in \mathbb{R}_+$ *such that* $\alpha+\beta < \frac{1}{2}$.

(a_{21})

$$d(Sx,Ty) \precsim \alpha \max\left\{d(fx,gy), \frac{d(Sx,fx)+d(Ty,gy)}{2}, \frac{d(Ty,fx)+d(Sx,gy)}{2}\right\},$$

where $\alpha \in [0,1)$.

Proof. The proof of each contraction condition in this corollary follows easily from Theorems 3.1 and 3.3 in view of Examples 2.2-2.22.

Remark 3.9. *Here, we point out the following fallacies.*

1. *We have noticed some fallacy in the proof of Theorem 8 of Abbas et al. [1]. Observe that in equation (6) in [1] authors used* $\frac{z_1 z_2}{1+z_2} \precsim z_1$ *which is not true in general (e.g. take* $z_1 = 1$ *and* $z_2 = 1 + i$ *then* $\frac{1+i}{2+i} \not\precsim 1$*).*

2. *We have also noticed yet another fallacy in the proof of Theorem 8 of Abbas et al. [1]. Notice that on page 4 second line authors used* $\psi(z_1) \precsim \psi(z_2) \Longrightarrow z_1 \precsim z_2$ *where* $\psi \in \Psi$ *which is not true in general (e.g. define* $\psi : \mathbb{C}_+ \longrightarrow \mathbb{C}_+$ *by* $\psi(z) = Re(z) + Im(z) \quad \forall z \in \mathbb{C}_+$*, then* $\psi \in \Psi$ *and* $\psi(i) = 1 \precsim 1 = \psi(1)$ *,* $i \not\precsim 1$ *).*

Remark 3.10. *The majority of results corresponding to contraction conditions in Corollary 3.8 generalize and improve a multitude results of the existing literature. Corollary 3.8 corresponding to contraction condition*

1. (a_1) *improves Theorem 3.2 of Sintunavarat et al [20].*

2. (a_9) *remains a generalized form of Theorem 2.1 of [4]. Particularly, substituting* $a_4 = a_5 = a_6 = 0$ *in* (a_9)*, we get Theorem 2.1 of Bhatt et al. [4].*

3. (a_{12}) *generalizes Theorem 2.1 of [23]. Especially, substituting* $f = g = I$ *in* (a_{12})*, we get Theorem 2.1 of Verma et al. [23].*

4. (a_{13}) *present Theorem 2.1 of Rouzkard [15].*

5. *Corollary 3.8 corresponding to contraction condition* (a_{20}) *generalizes Corollary 2.2 of Verma et al. [23].*

6. (a_{21}) *represents Theorem 2.2 of Sastry et al. [17].*

Next, we indicate that a corollary analogous to Corollary 3.8 can be outlined in respect of Theorem 3.6 involving various iterates of mappings namely: S^l, T^m, f^n and g^s.

Corollary 3.11. *The conclusions of Theorem 3.6 remain true if (for all* $x, y \in X$*) implicit relation (3.9) is replaced by any one of the contraction conditions in Corollary 3.8 with* $S = S^l, T = T^m, f = f^n$ *and* $g = g^s$*.*

4. Applications to Integral Equations

Let $X = C\big([a,b], \mathbb{R}^n\big)$, $a > 0$. In this section, we show that Theorem 3.1 (corresponding to contraction condition (a_1)) can be applied to prove the existence and uniqueness of common solution for the system of Hammerstein integral equations:

$$x(t) = \psi_j(t) + \int_a^b k_j(t,s) f_j(s, x(s)) ds, \tag{4.1}$$

where $t \in [a,b] \subseteq \mathbb{R}, x, \psi_j \in X, k_j : [a,b] \times [a,b] \longrightarrow \mathbb{R}^n$ and $f_j : [a,b] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n, j = 1, 2.$

Also, Theorem 3.1 (corresponding to contraction condition (a_3)) can be applied to prove the existence and uniqueness of common solution for the system of Urysohn integral equations:

$$x(t) = \varphi_j(t) + \int_a^b k_j(t,s,x(s))ds, \tag{4.2}$$

where $t \in [a,b] \subseteq \mathbb{R}, x, \varphi_j \in X, k_j : [a,b] \times [a,b] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $j = 1,2$.

Throughout this section, we use the following symbols:

$$\begin{aligned}
H_j\big(x(t)\big) &= \int_a^b k_j(t,s) f_j(s,x(s))ds, \ \ j = 1,2, \\
U_j\big(x(t)\big) &= \int_a^b k_j(t,s,x(s))ds, \ \ j = 1,2, \\
\Omega_{xy}(t) &= \|H_1x(t) + \psi_1(t) - H_2y(t) - \psi_2(t)\|_\infty \sqrt{1+a^2}\ e^{i \arctan a}, \\
\Lambda_{xy}(t) &= \|x(t) - y(t)\|_\infty \sqrt{1+a^2}\ e^{i \arctan a}, \\
\Omega^*_{xy}(t) &= \|U_1x(t) + \varphi_1(t) - U_2y(t) - \varphi_2(t)\|_\infty \sqrt{1+a^2}\ e^{i \arctan a}, \\
\mathbf{A}_{xy}(t) &= \|U_2y(t) + \varphi_2(t) - y(t)\|_\infty \sqrt{1+a^2}\ e^{i \arctan a}, \\
\mathbf{B}_{xy}(t) &= \|U_2y(t) + \varphi_2(t) - x(t)\|_\infty \sqrt{1+a^2}\ e^{i \arctan a}, \\
\mathbf{C}_{xy}(t) &= \|U_1x(t) + \varphi_1(t) - y(t)\|_\infty \sqrt{1+a^2}\ e^{i \arctan a}.
\end{aligned}$$

$$\Xi_{xy}(t) = \left(\frac{\|H_2y(t) + \psi_2(t) - y(t)\|_\infty \sqrt{1+a^2}\ e^{i \arctan a}}{1 + \max_{t \in [a,b]} \Lambda_{xy}(t)} \right) \times \|H_1x(t) + \psi_1(t) - x(t)\|_\infty \sqrt{1+a^2}\ e^{i \arctan a},$$

$$\Lambda^*_{xy}(t) = \left(\frac{\mathbf{A}_{xy}(t)\mathbf{B}_{xy}(t)\mathbf{C}_{xy}(t)}{1 + \max_{t \in [a,b]} \Lambda_{xy}(t)} \right).$$

Theorem 4.1. *Let $X = C\big([a,b],\mathbb{R}^n\big)$, $a > 0$. Consider the system of Hammerstein integral equations described by (4.1). Suppose that k_1, k_2, f_1 and f_2 are such that $H_1(x), H_2(x) \in X$ for all $x \in X$. If there exist mappings $\lambda_1, \lambda_2 : \mathbb{C}_+ \longrightarrow [0,1)$ such that for each $z \in \mathbb{C}$, $x, y \in X$ and $t \in [0,1)$, we have*

(i) $2\lambda_1(z) + \lambda_2(z) \leq 1$,

(ii) $\Omega_{xy}(t) \precsim \lambda_1\big(\max_{t \in [a,b]} \Lambda_{xy}(t)\big)\Lambda_{xy}(t) + \lambda_2\big(\max_{t \in [a,b]} \Lambda_{xy}(t)\big)\Xi_{xy}(t)$.

Then the system of integral equations (4.1) have a unique common solution.

Proof. Define a metric $d : X \times X \longrightarrow \mathbb{C}_+$ by

$$d(x,y) = \max_{t\in[a,b]} \|x(t) - y(t)\|_\infty \sqrt{1+a^2}\ e^{i\arctan a}.$$

Then (X, d) is a complete complex valued metric space. Define two mappings $S, T : X \longrightarrow X$ as follows:

$$S\big(x(t)\big) = \psi_1(t) + H_1\big(x(t)\big) = \psi_1(t) + \int_a^b k_1(t,s) f_1(s, x(s))ds,$$

$$T\big(x(t)\big) = \psi_2(t) + H_2\big(x(t)\big) = \psi_2(t) + \int_a^b k_2(t,s) f_2(s, x(s))ds.$$

Let $x, y \in X$. Then we have
$d(Sx,Ty) = \max_{t\in[a,b]} \|H_1x(t) + \psi_1(t) - H_2y(t) - \psi_2(t)\|_\infty \sqrt{1+a^2}\ e^{i\arctan a}$,
$d(Sx,x) = \max_{t\in[a,b]} \|H_1x(t) + \psi_1(t) - x(t)\|_\infty \sqrt{1+a^2}\ e^{i\arctan a}$.
$d(Ty,y) = \max_{t\in[a,b]} \|H_2y(t) + \psi_2(t) - y(t)\|_\infty \sqrt{1+a^2}\ e^{i\arctan a}$.

From assumption (ii), for each $t \in [a,b]$ we have

$$\begin{aligned}\Omega_{xy}(t) &\precsim \lambda_1\left(\max_{t\in[a,b]} \Lambda_{xy}(t)\right)\Lambda_{xy}(t) + \lambda_2\left(\max_{t\in[a,b]} \Lambda_{xy}(t)\right)\Xi_{xy}(t)\\ &\precsim \lambda_1\left(\max_{t\in[a,b]} \Lambda_{xy}(t)\right)\max_{t\in[a,b]}\Lambda_{xy}(t) + \lambda_2\left(\max_{t\in[a,b]} \Lambda_{xy}(t)\right)\max_{t\in[a,b]}\Xi_{xy}(t),\end{aligned}$$

which implies that

$$\max_{t\in[a,b]}\Omega_{xy}(t) \precsim \lambda_1\left(\max_{t\in[a,b]} \Lambda_{xy}(t)\right)\max_{t\in[a,b]}\Lambda_{xy}(t) + \lambda_2\left(\max_{t\in[a,b]} \Lambda_{xy}(t)\right)\max_{t\in[a,b]}\Xi_{xy}(t),$$

yielding thereby

$$d(Sx,Ty) \precsim \lambda_1(d(x,y))d(x,y) + \lambda_2(d(x,y))\frac{d(Sx,x)d(Ty,y)}{1+d(x,y)}.$$

Thus, all the conditions of Theorem 3.1 corresponding to contraction condition (a_1) with $f = g = I$ are satisfied . Therefore, the Hammerstein integral equations described by system (4.1) have a unique solution.

Theorem 4.2. *Let $X = C\big([a,b],\mathbb{R}^n\big)$, $a > 0$. Consider the system of Urysohn integral equations described by (4.2). Suppose that k_1 and k_2 are such that $U_1(x), U_2(x) \in X$ for all $x \in X$. If for each $x, y \in X$ and $t \in [a,b]$, we have*

$$\Omega^*_{xy}(t) \precsim \Lambda^*_{xy}(t). \tag{4.3}$$

Then the system of integral equations (4.2) have a unique common solution.

Proof. Define a metric $d : X \times X \longrightarrow \mathbb{C}_+$ by

$$d(x, y) = \max_{t\in[a,b]} \|x(t) - y(t)\|_\infty \sqrt{1 + a^2}\ e^{i \arctan a}.$$

Then (X, d) is a complete complex valued metric space. Define two mappings $S, T : X \longrightarrow X$ as follows:

$$S\big(x(t)\big) = \varphi_1(t) + U_1\big(x(t)\big) = \varphi_1(t) + \int_a^b k_1(t, s, x(s))ds,$$

$$T\big(x(t)\big) = \varphi_2(t) + U_2\big(x(t)\big) = \varphi_2(t) + \int_a^b k_2(t, s, x(s))ds.$$

Let $x, y \in X$. Then we have
$d(Sx, Ty) = \max_{t\in[a,b]} \|U_1x(t) + \varphi_1(t) - U_2y(t) - \psi_2(t)\|_\infty \sqrt{1 + a^2}\ e^{i \arctan a}$,
$d(Sx, x) = \max_{t\in[a,b]} \|U_1x(t) + \varphi_1(t) - x(t)\|_\infty \sqrt{1 + a^2}\ e^{i \arctan a}$,
$d(Ty, y) = \max_{t\in[a,b]} \|U_2y(t) + \varphi_2(t) - y(t)\|_\infty \sqrt{1 + a^2}\ e^{i \arctan a}$,
$d(Sx, y) = \max_{t\in[a,b]} \|U_1x(t) + \varphi_1(t) - y(t)\|_\infty \sqrt{1 + a^2}\ e^{i \arctan a}$.

On using assumption (4.3), for each $t \in [a, b]$ we have

$$\begin{aligned}\Omega^*_{xy}(t) &\precsim \Lambda^*_{xy}(t)\\ &\precsim \max_{t\in[a,b]} \Lambda^*_{xy}(t),\end{aligned}$$

which implies that

$$\max_{t\in[a,b]} \Omega^*_{xy}(t) \precsim \max_{t\in[a,b]} \Lambda^*_{xy}(t),$$

or

$$\psi\big(\max_{t\in[a,b]} \Omega^*_{xy}(t)\big) \precsim \psi\big(\max_{t\in[a,b]} \Lambda^*_{xy}(t)\big),$$

so that

$$\psi(d(Sx, Ty)) \precsim \psi\left(\frac{d(Ty, y)d(Ty, x)d(Sx, y)}{1 + d(x, y)}\right).$$

Thus, all the conditions of Theorem 3.1 corresponding to contraction condition (a_3) with $f = g = I$ are satisfied. Therefore, the Urysohn integral equations described by system (4.2) have a unique solution.

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Waleed Mohd. Alfaqih,
Department of Mathematics,
Aligarh Muslim University,
India.
E-mail address: waleedmohd2016@gmail.com

and

Mohammad Imdad,
Department of Mathematics,
Aligarh Muslim University,
India.
E-mail address: mhimdad@yahoo.co.in

and

Fayyaz Rouzkard,
Department of Mathematics,
Farhangian University,
Iran.
E-mail address: fayyazrouzkard@gmail.com