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Existence of Entropy Solutions in Musielak-Orlicz Spaces Via a Sequence of Penalized Equations

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ABSTRACT: This paper, is devoted to an existence result of entropy unilateral solutions for the nonlinear parabolic problems with obstacle in Musielak- Orlicz–spaces:

$$\partial_t u + A(u) + H(x,t,u,\nabla u) = f + div(\Phi(x,t,u)),$$

and

$$u \geq \zeta \text{ a.e. in } Q_T.$$

Where A is a pseudomonotone operator of Leray-Lions type defined in the inhomogeneous Musielak-Orlicz space $W_0^{1,x}L_\varphi(Q_T)$, $H(x,t,s,\xi)$ and $\Phi(x,t,s)$ are only assumed to be Crathéodory's functions satisfying only the growth conditions prescribed by Musielak-Orlicz functions φ and ψ which inhomogeneous and does not satisfies Δ_2-condition. The data f and u_0 are still taken in $L^1(Q_T)$ and $L^1(\Omega)$.

Key Words: Musielak-Orlicz space, Nonlinear Obstacle-parabolic problems, Entropy solution.

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1. Introduction

One of the driving forces for the rapid development of the theory of variable exponents function spaces, and more generaly the Musielak–Orlicz–functions spaces has been the model of electro-rheological fluids introduced by Rajagopal and Rusicka [page 457]. The model leads naturally to a functional setting involving function spaces with variable exponents. Electrorheological fluids change their mechanical properties dramatically when an external electric field is applied. Also in the mathematical community such materials are intensively investigated in the recent

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years. the concept of weak solutions is not enough to give a formulation to all problems and does not provide uniqueness and stability properties. Hence, as an extension of distributional solutions, we can use the notion of entropy.
Statement of the problem: Let Ω be a bounded open set of $\mathbb{R}^N$ $(N \geq 2)$, T is a positive real number, and $Q_T = \Omega \times (0, T)$. Consider the following nonlinear Dirichlet equation:

$$\begin{cases} u \geq \zeta & \text{in} \quad Q_T, \\ \frac{\partial u}{\partial t} + A(u) + H(x, t, u, \nabla u) = f + div(\Phi(x, t, u)) & \text{in} \quad Q_T, \\ u(x, t) = 0 & \text{on} \quad \partial\Omega \times (0, T), \\ u(x, t = 0) = u_0(x) & \text{in} \quad \Omega. \end{cases} \tag{1.1}$$

Where $A(u) = -div(a(x, t, u, \nabla u))$ is a Leary-Lions operator defined on the inhomogeneous Musielak-Orlicz-Sobolev space $W_0^{1,x}L_\varphi(Q_T)$, φ is a Musielak-Orlicz-function related to the growths of the Carathéodory functions $a(x, t, u, \nabla u)$, $\Phi(x, t, u)$ and $H(x, t, u, \nabla u)$ (see assumptions (3.1), (3.4) and (3.5). The data f and u_0 in $L^1(Q_T)$ and $L^1(\Omega)$ respectively, and $u_0 \geq \zeta$.

The first prototype is taken from the Classical Sobolev spaces, having the following form:

$$\frac{\partial u}{\partial t} - \triangle_p(u) + div(c(.,t)|u|^{\gamma-1}u) + b|\nabla u|^\delta = f, \text{ in } Q_T.$$

Porzio et al. in [21] have proved the existence of weak solutions, with $c(.,.) \equiv 0$. For $c(.,.) \in L^2(Q_T)$ and $p = 2$, Boccardo et al. in [11] have proved the existence of entropy solutions, recently R. Di–Nardo et al. in [16] have proved an existence results of renormalized solutions in the case where $p \geq 2$ and $c(.,.) \in L^r(Q_T)$ with $r = \frac{N+p}{p-1}$, and by Aberqi et al. in [2] for more general parabolic term. For the elliptic version of the problem (1.1), more results are obtained see e.g. [12,13].

In the degenerate Sobolev-spaces an existence results is shown in [6] without sign condition in $H(x, t, u, \nabla u)$.

In the Orlicz-Sobolev spaces, Rhoudaf et al. in [19] proved the existence of entropy solutions of the problem (1.1) where $H(x, t, u, \nabla u) \equiv 0$ and the growth of the first lower order Φ prescribed by an anisotropic N-function φ defining space does not satisfy the $\triangle_2$-condition.
To our knowledge, differential equations in general Musielak–Sobolev spaces have been studied rarely see [3,10,17,20], then our aim in this paper is to overcome some difficulties encountered in these spaces and to generalize the result of [2,5,19,22], and we prove an existence result of entropy solutions for the obstacle parabolic problem (1.1), with less restrictive growth, and no coercivity condition in the first lower order term Φ, and without sign condition in the second lower order H, in the general framework of inhomogeneous Musielak-Orlicz-Sobolev spaces $W_0^{1,x}L_\varphi(Q_T)$, and the anisotropic N-function φ, defining space does not satisfy the $\triangle_2$-condition.

Remark 1.1. *The difficulties associated to the existence of entropy solutions of equations* (1.1) *lies in the fact that:*

1. *$\Phi(x, s)$ is non-coercive, and non-continuous with respect to x, we can't applied the Stoks formula.*
2. *The Musielak–Orlicz function φ not satisfy the Δ_2-condition which induce a loss of reflexivity of the framework space.*

Let us give an example of equations to which the present result can be applied:

$$\begin{cases} u \geq \zeta & \text{in} \quad Q_T, \\ \dfrac{\partial u}{\partial t} - \Delta_\varphi(u) + u\varphi(x, \nabla u) = f + c(x,t)\psi_x^{-1}\varphi(x, \alpha_0|u|) & \text{in} \quad Q_T, \\ u(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,T). \end{cases}$$

Where $-\Delta_\varphi(u) = -div(\frac{m(x,|\nabla u|)}{|\nabla u|}.\nabla u)$, $m(x,s)$ is the derivative of $\varphi(x,s)$ with respect to s, ζ is an admissible obstacle function.

Let us summarize the contents of this article. In section 2, we recall some definitions, properties and technical lemmas about Musielak-Orlicz-Sobolev. In section 3 is devoted to specify the assumptions on Φ, H, f, u_0 and giving the definition of a entropy solution of (1.1) . In section 4, we establish the existence result of such a solutions in theorem (4.1), this last section is divided in 6 steps.

2. Preliminaries

2.1. Musielak-Orlicz function

Let Ω be an open subset of $\mathbb{R}^N$ $(N \geq 2)$, and let φ be a real-valued function defined in $\Omega \times \mathbb{R}_+$ and satisfying conditions:
(Φ_1): $\varphi(x, .)$ is an N-function for all $x \in \Omega$ (i.e. convex, non-decreasing, continuous, $\varphi(x,0) = 0$, $\varphi(x,0) > 0$ for $t > 0$, $\lim_{t\to 0}\sup_{x\in\Omega}\frac{\varphi(x,t)}{t} = 0$ and $\lim_{t\to\infty}\inf_{x\in\Omega}\frac{\varphi(x,t)}{t} = \infty$).
(Φ_2): $\varphi(., t)$ is a measurable function for all $t \geq 0$.
A function φ which satisfies the conditions (Φ_1) and (Φ_2) is called a Musielak-Orlicz function.

For a Musielak-Orlicz function φ, we put $\varphi_x(t) = \varphi(x,t)$ and we associate its non-negative reciprocal function φ_x^{-1}, with respect to t, that is

$$\varphi_x^{-1}(\varphi(x,t)) = \varphi(x, \varphi_x^{-1}(t)) = t.$$

Let φ and γ be two Musielak-Orlicz functions, we say that φ dominate γ, and we write $\gamma \prec \varphi$, near infinity (resp. globally) if there exist two positive constants c and t_0 such that for a.e. $x \in \Omega$, $\gamma(x,t) \leq \varphi(x, ct)$ for all $t \geq t_0$ (resp. for all $t \geq 0$). We say that γ grows essentially less rapidly than φ at 0(resp. near infinity, and we write $\gamma \prec\prec \varphi$, for every positive constant c, we have

$$\lim_{t\to 0}\Big(\sup_{x\in\Omega}\frac{\gamma(x,ct)}{\varphi(x,t)}\Big) = 0 \quad \Big(\text{resp.} \quad \lim_{t\to\infty}\Big(\sup_{x\in\Omega}\frac{\gamma(x,ct)}{\varphi(x,t)}\Big) = 0\Big).$$

Remark 2.1. *[10] If $\gamma \prec\prec \varphi$ near infinity, then $\forall \epsilon > 0$ there exist $k(\epsilon) > 0$ such that for almost all $x \in \Omega$ we have*

$$\gamma(x,t) \leq k(\epsilon)\varphi(x,\epsilon t) \quad \forall t \geq 0.$$

Proposition 2.2. *Let $\gamma \prec\prec \varphi$ near infinity and $\forall t > 0$, $\sup_{x\in\Omega} \gamma(x,t) < \infty$, then for all$\epsilon > 0$, there exists $C_\epsilon > 0$ such that $\gamma(x,t) \leq \varphi(x,\epsilon t) + C_\epsilon$, for all $t > 0$.*

Example 2.3. *$\gamma(x,t) = \|x\|M(t)$ and M is an isotropic N-function.*
$\gamma(x,t) = \exp(\dfrac{1}{\|x\|+1})M(t)$ and M is an isotropic N-function.

Proof: We have by definition, $\forall \epsilon > 0, \exists t_0 > 0,$ such that $\forall x \in \Omega, \forall t > t_0, \gamma(x,t) \leq \varphi(x,\epsilon t)$, for $0 < t < t_0$, since γ is increasing in t, we have

$$\gamma(x,t) \leq \gamma(x,t_0) \leq \sup_{x\in\Omega} \gamma(x,t_0) = C_\epsilon,$$

then $\gamma(x,t) \leq \varphi(x,\epsilon t) + \gamma(x,t_0) \leq \varphi(x,\epsilon t) + C_\epsilon, \forall t > 0.$ □

2.2. Musielak-Orlicz space

For a Musielak-Orlicz function φ and a measurable function $u : \Omega \to \mathbb{R}$, we define the functional

$$\varrho_{\varphi,\Omega}(u) = \int_\Omega \varphi(x, |u(x)|)dx.$$

The set $K_\varphi(\Omega) = \{u : \Omega \to \mathbb{R} \quad \text{mesurable} : \quad \varrho_{\varphi,\Omega}(u) < \infty\}$ is called the Musielak-Orlicz class. The Musielak-Orlicz space $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$; that is, $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$. Equivalently

$$L_\varphi(\Omega) = \{u : \Omega \to \mathbb{R} \quad \text{mesurable} : \quad \varrho_{\varphi,\Omega}(\frac{u}{\lambda}) < \infty, \quad \text{for some} \quad \lambda > 0\}.$$

For any Musielak-Orlicz function φ, we put $\psi(x,s) = \sup_{t\geq 0}(st - \varphi(x,s))$.
ψ is called the Musielak-Orlicz function complementary to φ (or conjugate of φ) in the sense of Young with respect to s. We say that a sequence of function $u_n \in L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant $\lambda > 0$ such that $\lim_{n\to\infty} \varrho_{\varphi,\Omega}(\frac{u_n - u}{\lambda}) = 0$, this implies convergence for $\sigma(\Pi L_\varphi, \Pi L_\psi)$ (see [9]).
In the space $L_\varphi(\Omega)$, we define the following two norms

$$\|u\|_\varphi = \inf\{\lambda > 0 : \int_\Omega \varphi(x, \frac{|u(x)|}{\lambda})dx \leq 1\},$$

which is called the Luxemburg norm, and the so-called Orlicz norm by

$$|||u|||_{\varphi,\Omega} = \sup_{\|v\|_\psi \leq 1} \int_\Omega |u(x)v(x)|dx,$$

where ψ is the Musielak-Orlicz function complementary to φ. These two norms are equivalent [9]. $K_\varphi(\Omega)$ is a convex subset of $L_\varphi(\Omega)$. We define $E_\varphi(\Omega)$ as the subset of $L_\varphi(\Omega)$ of all measurable functions $u : \Omega \mapsto \mathbb{R}$ such that $\int_\Omega \varphi(x, \frac{|u(x)|}{\lambda})dx < \infty$ for all $\lambda > 0$. It is a separable space and $(E_\varphi(\Omega))^* = L_\varphi(\Omega)$. We have $E_\varphi(\Omega) = K_\varphi(\Omega)$, if and only if φ satisfies the Δ_2-condition for large values of t or for all values of t, according to whether Ω has finite measure or not.
We define

$$W^1L_\varphi(\Omega) = \{u \in L_\varphi(\Omega) : D^\alpha u \in L_\varphi(\Omega), \quad \forall \alpha \leq 1\},$$
$$W^1E_\varphi(\Omega) = \{u \in E_\varphi(\Omega) : D^\alpha u \in E_\varphi(\Omega), \quad \forall \alpha \leq 1\},$$

where $\alpha = (\alpha_1, ..., \alpha_N)$, $|\alpha| = |\alpha_1| + ... + |\alpha_N|$ and $D^\alpha u$ denote the distributional derivatives. The space $W^1L_\varphi(\Omega)$ is called the Musielak-Orlicz-Sobolev space.
Let
$\overline{\varrho}_{\varphi,\Omega}(u) = \sum_{|\alpha|\leq 1} \varrho_{\varphi,\Omega}(D^\alpha u)$ and

$$\|u\|^1_{\varphi,\Omega} = \inf\{\lambda > 0 : \overline{\varrho}_{\varphi,\Omega}(\frac{u}{\lambda}) \leq 1\}$$

for $u \in W^1L_\varphi(\Omega)$.
These functionals are convex modular and a norm on $W^1L_\varphi(\Omega)$, respectively. Then the pair $(W^1L_\varphi(\Omega), \|u\|^1_{\varphi,\Omega})$ is a Banach space if φ satisfies the following condition (see [20]): There exists a constant $c > 0$ such that $\inf_{x\in\Omega} \varphi(x, 1) > c$.
The space $W^1L_\varphi(\Omega)$ is identified to a subspace of the product $\Pi_{\alpha\leq 1} L_\varphi(\Omega) = \Pi L_\varphi$
We denote by $\mathcal{D}(\Omega)$ the Schwartz space of infinitely smooth functions with compact support in Ω and by $\mathcal{D}(\overline{\Omega})$ the restriction of $\mathcal{D}(\mathbb{R})$ on Ω. The space $W^1_0L_\varphi(\Omega)$ is defined as the $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closure of $\mathcal{D}(\Omega)$ in $W^1L_\varphi(\Omega)$ and the space $W^1_0E_\varphi(\Omega)$ as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1L_\varphi(\Omega)$.
For two complementary Musielak-Orlicz functions φ and ψ, we have [9].

- The Young inequality:

$$st \leq \varphi(x, s) + \psi(x, t) \text{ for all } s, t \geq 0 \text{ , } x \in \Omega.$$

- The Hölder inequality:

$$\Big|\int_\Omega u(x)v(x)dx\Big| \leq \|u\|_{\varphi,\Omega}\||v|\|_{\psi,\Omega} \text{ for all } u \in L_\varphi(\Omega), v \in L_\psi(\Omega).$$

We say that a sequence of functions u_n converges to u for the modular convergence in $W^1L_\varphi(\Omega)$ (resp. in $W^1_0L_\varphi(\Omega)$) if, for some $\lambda > 0$, $\lim_{n\to\infty} \overline{\varrho}_{\varphi,\Omega}\big(\frac{u_n-u}{\lambda}\big) = 0$.
The following spaces of distributions will also be used

$$W^{-1}L_\psi(\Omega) = \big\{f \in \mathcal{D}^{'}(\Omega) : f = \sum_{\alpha\leq 1}(-1)^\alpha D^\alpha f_\alpha \quad \text{where} \quad f_\alpha \in L_\psi(\Omega)\big\},$$

and

$$W^{-1}E_{\psi}(\Omega)=\big\{f\in\mathcal{D}^{'}(\Omega):f=\sum_{\alpha\leq 1}(-1)^{\alpha}D^{\alpha}f_{\alpha}\quad\text{where}\quad f_{\alpha}\in E_{\psi}(\Omega)\big\}.$$

Lemma 2.4. *[9] Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and let φ and ψ be two complementary Musielak-Orlicz functions which satisfy the following conditions:*

1. *There exists a constant $c>0$ such that*
$$\inf_{x\in\Omega}\varphi(x,1)>c,$$
2. *There exists a constant $A>0$ such that for all $x,y\in\Omega$ with $|x-y|\leq\frac{1}{2}$, we have*
$$\frac{\varphi(x,t)}{\varphi(y,t)}\leq|t|^{\left(\frac{A}{\log(\frac{1}{|x-y|})}\right)}\quad \textit{for all}\quad t\geq 1,$$
3. $$\int_K\varphi(y,\lambda)dx<\infty,\ \textit{for any constant}\lambda>0\ \textit{and every compact}\ K\subset\Omega,$$
4. *There exists a constant $C>0$ such that $\psi(y,t)\leq C$ a.e. in Ω.*

Under this assumptions $\mathcal{D}(\Omega)$ is dense in $L_{\varphi}(\Omega)$ with respect to the modular topology, $\mathcal{D}(\Omega)$ is dense in $W_0^1L_{\varphi}(\Omega)$ for the modular convergence and $\mathcal{D}(\overline{\Omega})$ is dense in $W_0^1L_{\varphi}(\Omega)$ for the modular convergence. Consequently, the action of a distribution S in in $W^{-1}L_{\psi}(\Omega)$ on an element u of $W_0^1L_{\varphi}(\Omega)$ is well defined. It will be denoted by $<S,u>$.

2.3. Truncation Operator

T_k, $k>0$, denotes the truncation function at level k defined on $\mathbb{R}$ by $T_k(r)=\max(-k,\min(k,r))$. The following abstract lemmas will be applied to the truncation operators.

Lemma 2.5. *[10] Let $F:\mathbb{R}\to\mathbb{R}$ be uniformly lipschitzian,with $F(0)=0$. Let φ be an Musielak-Orlicz function and let $u\in W_0^1L_{\varphi}(\Omega)$(resp.$u\in W^1E_{\varphi}(\Omega)$). Then $F(u)\in W^1L_{\varphi}(\Omega)$(resp.$u\in W_0^1E_{\varphi}(\Omega)$). Moreover, if the set of discontinuity points D of F' is finite, then*

$$\frac{\partial}{\partial x_i}F(u)=\begin{cases}F'(x)\frac{\partial u}{\partial x_i} & a.e.\ in\ \{x\in\Omega;\ u(x)\notin D\},\\ 0 & a.e.\ in\ \{x\in\Omega;\ u(x)\in D\}.\end{cases}$$

Lemma 2.6. *[9] Suppose that Ω satisfies the segment property and let*

$$u \in W_0^1 L_\varphi(\Omega).$$

Then, there exists a sequence $u_n \in \mathcal{D}(\Omega)$ such that

$$u_n \to u \text{ for modular convergence in } W_0^1 L_\varphi(\Omega).$$

Furthermore, if $u \in W_0^1 L_\varphi(\Omega) \cap L^\infty(\Omega)$ then $\|u_n\|_\infty \leq (N+1)\|u\|_\infty$.

Let Ω be an open subset of $\mathbb{R}^N$ and let φ be a Musielak-Orlicz function satisfying:

$$\int_0^1 \frac{\varphi_x^{-1}(t)}{t^{\frac{N+1}{N}}} dt = \infty \quad \text{a.e.} \quad x \in \Omega, \tag{2.1}$$

and the conditions of Lemma (2.4). We may assume without loss of generality that

$$\int_0^1 \frac{\varphi_x^{-1}(t)}{t^{\frac{N+1}{N}}} dt < \infty \quad \text{a.e.} \quad x \in \Omega. \tag{2.2}$$

Define a function $\varphi^* : \Omega \times [0, \infty)$ by $\varphi^*(x, s) = \displaystyle\int_0^s \frac{\varphi_x^{-1}(t)}{t^{\frac{N+1}{N}}} dt$ $x \in \Omega$ and $s \in [0, \infty)$. φ^* its called the Sobolev conjugate function of φ (see [1] for the case of Orlicz function).

Theorem 2.7. *[17] Let Ω be a bounded Lipschitz domain and let φ be a Musielak-Orlicz function satisfying (2.1)-(2.2) and the conditions of lemma (2.4). Then*

$$W_0^1 L_\varphi(\Omega) \hookrightarrow L_{\varphi^*}(\Omega),$$

where φ^* is the Sobolev conjugate function of φ. Moreover, if Φ is any Musielak-Orlicz function increasing essentially more slowly than φ^* near infinity, then the imbedding

$$W_0^1 L_\varphi(\Omega) \hookrightarrow L_\Phi(\Omega),$$

is compact.

Corollary 2.8. *[17] Under the same assumptions of theorem (2.7), we have*

$$W_0^1 L_\varphi(\Omega) \hookrightarrow\hookrightarrow L_\varphi(\Omega).$$

Lemma 2.9. *[10] If a sequence un $u_n \in L_\varphi(\Omega)$ converges a.e. to u and if u_n remains bounded in $L_\varphi(\Omega)$, then $u \in L_\varphi(\Omega)$ and $u_n \rightharpoonup u$ for $\sigma(L_\varphi(\Omega), E_\psi(\Omega))$.*

Lemma 2.10. *Let $u_n, u \in L_\varphi(\Omega)$. If $u_n \to u$ with respect to the modular convergence, then $u_n \rightharpoonup u$ for $\sigma(L_\varphi(\Omega), L_\psi(\Omega))$.*

Proof: Let $\lambda > 0$ such that $\displaystyle\int_\Omega \varphi(x, \frac{u_n - u}{\lambda}) dx \to 0$. Thus, for a subsequence, $u_n \to u$ a.e. in Ω. Take $v \in L_\psi(\Omega)$ and multiplying v by a suitable constant, we can assume $\lambda v \in \mathcal{L}_\psi(\Omega)$.
By Young's inequality, we have $|(u_n - u)v| \leq \varphi(x, \frac{u_n - u}{\lambda}) + \psi(x, \lambda v)$ which implies, by Vitali's theorem, that $\displaystyle\int_\Omega |(u_n - u)v| dx \to 0$. □

2.4. Inhomogeneous Musielak-Orlicz-Sobolev spaces

Let Ω an bounded open subset $\mathbb{R}^N$ and let $Q_T = \Omega\times]0,T[$ with some given $T>0$. Let φ be an Musielak-Orlicz function, for each $\alpha\in\mathbb{N}^N$, denote by ∇_x^α the distributional derivative on Q_T of order α with respect to the variable $x\in\mathbb{N}^N$. The inhomogeneous Musielak-Orlicz-Sobolev spaces are defined as follows,

$$W^{1,x}L_\varphi(Q_T) = \{u\in L_\varphi(Q_T) : \nabla_x^\alpha u \in L_\varphi(Q_T), \forall\alpha\in\mathbb{N}^N, |\alpha|\le 1\}, \tag{2.3}$$
$$W^{1,x}E_\varphi(Q_T) = \{u\in E_\varphi(Q_T) : \nabla_x^\alpha u \in E_\varphi(Q_T), \forall\alpha\in\mathbb{N}^N, |\alpha|\le 1\}.$$

The last space is a subspace of the first one, and both are Banach spaces under the norm

$$\|u\| = \sum_{|\alpha|\le m} \|\nabla_x^\alpha u\|_{\varphi,Q_T}.$$

We can easily show that they form a complementary system when Ω satisfies the Lipschitz domain [9]. These spaces are considered as subspaces of the product space $\Pi L_\varphi(Q_T)$ which have as many copies as there is α-order derivatives,$|\alpha|\le 1$. We shall also consider the weak topologies $\sigma(\Pi L_\varphi, \Pi E_\psi)$ and $\sigma(\Pi L_\varphi, \Pi L_\psi)$. If $u\in W^{1,x}L_\varphi(Q_T)$ then the function : $t\mapsto u(t) = u(t,.)$ is defined on $(0,T)$ with values $W^1L_\varphi(\Omega)$. If, further, $u\in W^{1,x}E_\varphi(Q_T)$ then the concerned function is a $W^{1,x}E_\varphi(\Omega)$-valued and is strongly measurable. Furthermore the following imbedding holds $W^{1,x}E_\varphi(\Omega)\subset L^1(0,T,W^{1,x}E_\varphi(\Omega))$.
The space $W^{1,x}L_\varphi(Q_T)$ is not in general separable, if $W^{1,x}L_\varphi(Q_T)$, we can not conclude that the function $u(t)$ is measurable on $(0,T)$. However, the scalar function $t\mapsto \|u(t)\|_{\varphi,\Omega}$, is in $L^1(0,T)$. The space $W_0^{1,x}E_\varphi(Q_T)$ is defined as the (norm) closure $W^{1,x}E_\varphi(Q_T)$ of $\mathcal{D}(Q_T)$. We can easily show as in [9], that when Ω has the segment property, then each element u of the closure of $\mathcal{D}(Q_T)$ with respect of the weak* topology $\sigma(\Pi L_\varphi, \Pi E_\psi)$ is a limit, in $W_0^{1,x}E_\varphi(Q_T)$, of some subsequence $(u_i)\subset\mathcal{D}(Q_T)$ for the modular convergence, i.e. there exists $\lambda>0$ such that for all $|\alpha|\le 1$

$$\int_{Q_T} \varphi(x, \frac{\nabla_x^\alpha u_i - \nabla_x^\alpha u}{\lambda})dx\,dt \to 0 \quad \text{as} \quad i\to\infty.$$

This implies that (u_i) converge to u in $W^{1,x}L_\varphi(Q_T)$ for the weak topology

$$\sigma(\Pi L_\varphi, \Pi L_\psi).$$

Consequently,

$$\overline{\mathcal{D}(Q_T)}^{\sigma(\Pi L_\varphi,\Pi E_\psi)} = \overline{\mathcal{D}(Q_T)}^{\sigma(\Pi L_\varphi,\Pi L_\psi)}.$$

This space will be denoted by $W_0^{1,x}L_\varphi(Q_T)$. Furthermore,

$$W_0^{1,x}E_\varphi(Q_T) = W_0^{1,x}L_\varphi(Q_T)\cap \Pi E_\varphi.$$

We have the following complementary system $\begin{pmatrix} W_0^{1,x}L_\varphi(Q_T) & F \\ W_0^{1,x}E_\varphi(Q_T) & F_0 \end{pmatrix}$

F being the dual space of $W_0^{1,x}E_\varphi(Q_T)$. It is also, except for an isomorphism, the quotient of ΠL_ψ by the polar set $W_0^{1,x}E_\varphi(Q_T)^\perp$, and will be denoted by $F = W^{-1,x}L_\psi(Q_T)$ and it is show that,

$$W^{-1,x}L_\psi(Q_T) = \left\{ f = \sum_{|\alpha|\le 1} \nabla_x^\alpha f_\alpha : \quad f_\alpha \in L_\psi(Q_T) \right\}. \tag{2.4}$$

This space will be equipped with the usual quotient norm

$$\|f\| = \inf \sum_{|\alpha|\le 1} \|f_\alpha\|_{\psi,Q_T}, \tag{2.5}$$

where the infimum is taken on all possible decompositions

$$f = \sum_{|\alpha|\le 1} \nabla_x^\alpha f_\alpha, \quad f_\alpha \in L_\psi(Q_T). \tag{2.6}$$

The space F_0 is then given by,

$$F_0 = \left\{ f = \sum_{|\alpha|\le 1} \nabla_x^\alpha f_\alpha : \quad f_\alpha \in E_\psi(Q_T) \right\}, \tag{2.7}$$

and is denoted by $F_0 = W^{-1,x}E_\psi(Q_T)$.

Lemma 2.11. *[3] Under the assumptions of lemma (2.4), and by assuming that $\varphi(x,.)$ decreases with respect to one of coordinate of x, there exists a constant $\delta > 0$ which depends only on Ω such that*

$$\int_{Q_T} \varphi(x, |u|)dxdt \le \int_{Q_T} \varphi(x, \delta|\nabla u|)dxdt. \tag{2.8}$$

Definition 2.12. *We say that $u_n \to u$ in $W^{-1,x}L_\psi(Q_T)+L^1(Q_T)$ for the modular convergence if we can write $u_n = \sum_{|\alpha|\le 1} D_x^\alpha u_n^\alpha + u_n^0$ and $u = \sum_{|\alpha|\le 1} D_x^\alpha u^\alpha + u^0$ with $u_n^\alpha \to u^\alpha$ in $L_\psi(Q_T)$ for modular convergence for all $|\alpha| \le 1$, and $u_n^0 \to u^0$ strongly in $L^1(Q_T)$.*

Lemma 2.13. *Let $\{u_n\}$ be a bounded sequence in $W^{1,x}L_\varphi(Q_T)$ such that*

$$\frac{\partial u_n}{\partial t} = \alpha_n + \beta_n \ \text{in } \mathcal{D}'(Q_T),$$

$$u_n \rightharpoonup u, \quad \text{weakly in} \quad W^{1,x}L_\varphi(Q_T), \quad \text{for} \quad \sigma(\Pi L_\varphi, \Pi E_\psi),$$

with $\{\alpha_n\}$ and $\{\beta_n\}$ two bounded sequences respectively in $W^{-1,x}L_\psi(Q_T)$ and in $\mathcal{M}(Q_T)$. Then $u_n \to u$ in $L^1_{loc}(Q_T)$. Furthermore, if $u_n \in W_0^{1,x}L_\varphi(Q_T)$, then $u_n \to u$ strongly in $L^1(Q_T)$.

Proof: It is easily adapted from that given in [14] by using Theorem 4.4 and Remark 4.3 instead of Lemma 8 of [23]. □

Theorem 2.14. *if $u \in W^{1,x}L_\varphi(Q_T) \cap L^1(Q_T)$ (resp. $W_0^{1,x}L_\varphi(Q_T) \cap L^1(Q_T)$) and $\frac{\partial u}{\partial t} \in W^{-1,x}L_\psi(Q_T) + L^1(Q_T)$, then there exists a sequence (v_j) in $\mathcal{D}(\overline{Q}_T)$ (resp. $\mathcal{D}(\overline{I}, \mathcal{D}(Q_T))$) such that $v_j \to u$ in $W^{1,x}L_\varphi(Q_T)$ and*

$$\frac{\partial v_j}{\partial t} \to \frac{\partial u}{\partial t} \text{ in } W^{-1,x}L_\psi(Q_T) + L^1(Q_T),$$

for the modular convergence.

Proof: Let $u \in W^{1,x}L_\varphi(Q_T) \cap L^1(Q_T)$ and $\frac{\partial u}{\partial t} \in W^{-1,x}L_\psi(Q_T) + L^1(Q_T)$, then for any $\epsilon > 0$. Writing $\frac{\partial u}{\partial t} = \sum_{|\alpha| \le 1} D_x^\alpha u^\alpha + u^0$, where $u^\alpha \in L_\psi(Q_T)$ for all $|\alpha| \le 1$ and $u^0 \in L^1(Q_T)$, we will show that there exits $\lambda > 0$ (depending Only on u and N) and there exists $v \in \mathcal{D}(\overline{Q}_T)$ for which we can write $\frac{\partial u}{\partial t} = \sum_{|\alpha| \le 1} D_x^\alpha v^\alpha + v^0$ with $v^\alpha, v^0 \in \mathcal{D}(\overline{Q}_T)$ such that

$$\int_{Q_T} \varphi(x, \frac{D_x^\alpha v - D_x^\alpha u}{\lambda}) dxdt \le \epsilon, \forall |\alpha| \le 1, \tag{2.9}$$

$$\|v - u\|_{L^1(Q_T)} \le \epsilon, \tag{2.10}$$

$$\|v^0 - u^0\|_{L^1(Q_T)} \le \epsilon, \tag{2.11}$$

$$\int_{Q_T} \psi(x, \frac{v^\alpha - u^\alpha}{\lambda}) dxdt \le \epsilon, \quad \forall |\alpha| \le 1. \tag{2.12}$$

The equation (2.9) flows from a slight adaptation of the arguments [9], the equations (2.10)-(2.11) flows also from classical approximation results.
For The equation (2.12) we know that $\mathcal{D}(\overline{Q}_T)$ is dense in $L_\psi(Q_T)$ for the modular convergence.
The case where $u \in W_0^{1,x}L_\varphi(Q_T) \cap L^1(Q_T))$ can be handled similarly without essential difficulty as it mentioned [9]. □

Remark 2.15. *The assumption $u \in L^1(Q_T)$ in theorem (2.14) is needed only when Q_T has infinite measure, since else, we have $L_\varphi(Q_T) \subset L^1(Q_T)$ and so $W^{1,x}L_\varphi(Q_T) \cap L^1(Q_T) = W^{1,x}L_\varphi(Q_T)$.*

Remark 2.16. *If in the statement of theorem (2.14) above, one takes $I = \mathbb{R}$, we have that $\mathcal{D}(\Omega \times \mathbb{R})$ is dense in $\{u \in W_0^{1,x}L_\varphi(\Omega \times \mathbb{R}) \cap L^1(\Omega \times \mathbb{R}) : \frac{\partial u}{\partial t} \in W^{-1,x}L_\psi(\Omega \times \mathbb{R}) + L^1(\Omega \times \mathbb{R})\}$ for the modular convergence. This trivially follows from the fact that $\mathcal{D}(\mathbb{R}, \mathcal{D}(\Omega)) = \mathcal{D}(\Omega \times \mathbb{R})$.*

Lemma 2.17. *Let $a < b \in \mathbb{R}$ and Ω be a bounded open subset of $\mathbb{R}^N$ with the segment property, then $\{u \in W_0^{1,x}L_\varphi(\Omega \times (a,b)) \cap L^1(\Omega \times (a,b)) : \dfrac{\partial u}{\partial t} \in W^{-1,x}L_\psi(\Omega \times (a,b)) + L^1(\Omega \times (a,b))\} \subset \mathcal{C}([a,b], L^1(\Omega))$.*

Proof: Let $u \in W_0^{1,x}L_\varphi(\Omega\times(a,b))$ and $\frac{\partial u}{\partial t} \in W^{-1,x}L_\psi(\Omega\times(a,b))+L^1(\Omega\times(a,b))$. After two consecutive reflections first with respect to $t=b$ and then with respect to $t=a$,

$$\hat{u}(x,t) = u(x,t)\chi_{(a,b)} + u(x,2b-t)\chi_{(b,2b-a)} \quad \text{in} \quad \Omega\times(b,2b-a),$$

and

$$\tilde{u}(x,t) = \hat{u}(x,t)\chi_{(a,2b-a)} + \hat{u}(x,2a-t)\chi_{(3a-2b,a)} \quad \text{in} \quad \Omega\times(3a-2b,2b-a).$$

We get function $\tilde{u} \in W_0^{1,x}L_\varphi(\Omega\times(3a-2b,2b-a))$ with $\frac{\partial \tilde{u}}{\partial t} \in W^{-1,x}L_\psi(\Omega\times(3a-2b,2b-a))+L^1(\Omega\times(3a-2b,2b-a))$. Now by letting a function $\eta\in\mathcal{D}(\mathbb{R})$ with $\eta=1$ on $[a,b]$ and *supp* $(\eta)\subset(3a-2b,2b-a)$, we set $\overline{u}=\eta\tilde{u}$, therefore, by standard arguments (see [15]), we have $\overline{u}=u$ on $(\Omega\times(a,b))$,
$\overline{u}\in W_0^{1,x}L_\varphi(\Omega\times\mathbb{R})\cap L^1(\Omega\times\mathbb{R})$ and $\frac{\partial\overline{u}}{\partial t}\in W_0^{-1,x}L_\psi(\Omega\times\mathbb{R})+L^1(\Omega\times\mathbb{R})$.
Let now v_j the sequence given by theorem (2.14) corresponding to $\overline{u}$, that is,

$$v_j \to \overline{u} \quad \text{in} \quad W_0^{1,x}L_\varphi(\Omega\times\mathbb{R}),$$

and

$$\frac{\partial v_j}{\partial t} \to \frac{\partial \overline{u}}{\partial t} \quad \text{in} \quad W_0^{-1,x}L_\psi(\Omega\times\mathbb{R})+L^1(\Omega\times\mathbb{R}),$$

for the modular convergence.
If we denote $S_k(s)=\int_0^s T_k(t)dt$ the primitive of T_k. We have,

$$\int_\Omega S_1(v_i-v_j)(\tau)dx = \int_\Omega\int_{-\infty}^{r} T_1(v_i-v_j)(\frac{\partial v_i}{\partial t}-\frac{\partial v_j}{\partial t})dxdt \to 0 \quad \text{as} \quad i,j\to 0,$$

from which, one deduces that v_j is a Cauchy sequence in $C(\mathbb{R};L^1(\Omega))$ and hence $\overline{u}\in C(\mathbb{R},L^1(\Omega))$. Consequently, $u\in C([a;b];L^1(\Omega))$. □

3. Essential assumptions

Let Ω be an open subset of $\mathbb{R}^N$ $(N\geq 2)$ satisfying the segment property, and let φ and γ be two Musielak-Orlicz functions such that φ and its complementary ψ satisfies conditions of Lemma 2.4 and Lemma 2.11 and $\gamma\prec\prec\varphi$.
$A: D(A)\subset W_0^1L_\varphi(Q_T)\to W^{-1}L_\psi(Q_T)$ defined by $A(u)=-diva(x,u,\nabla u)$, where $a: Q_T\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N$ is Carathéodory function such that for a.e. $x\in\Omega$ and for all $s\in\mathbb{R}$,$\xi,\xi^*\in\mathbb{R}^N$,$\xi\neq\xi^*$,

$$|a(x,t,s,\xi)| \leq \beta(a_0(x,t)+\psi_x^{-1}\gamma(x,k_1|s|)+\psi_x^{-1}\varphi(x,k_1|\xi|)), \tag{3.1}$$

with $a_0(.)\in E_\psi(Q_T)$, and $\beta>0$,

$$(a(x,t,s,\xi)-a(x,t,s,\xi^*))(\xi-\xi^*)>0, \tag{3.2}$$

$$a(x,t,s,\xi).\xi \geq \alpha\varphi(x,|\xi|) + \varphi(x,|s|). \tag{3.3}$$

$\Phi : Q_T \times \mathbb{R} \to \mathbb{R}^N$ is a Carathéodory function such that

$$|\Phi(x,t,s)| \leq c(x,t)\psi_x^{-1}\varphi(x,\alpha_0|s|), \tag{3.4}$$

$$\text{where} \quad \|c(.,.)\|_{L^\infty(Q_T)} \leq \alpha \quad \text{and} \quad 0 < \alpha_0 < \min(1;\frac{1}{\alpha}).$$

$H : Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function such that

$$|H(x,t,s,\xi)| \leq h(x,t) + \rho(s)\varphi(x,|\xi|), \tag{3.5}$$

$\rho : \mathbb{R} \to \mathbb{R}^+$ is continuous positive function which belong $L^1(IR)$ and $h(.,.)$ belong $L^1(Q_T)$.

$$f \in L^1(Q_T), \tag{3.6}$$

$$u_0 \in L^1(\Omega). \tag{3.7}$$

Let ζ a measurable function with values in $\mathbb{R}$ such that

$$\zeta \in W_0^1E_\varphi(Q_T) \cap L^\infty(Q_T), \quad \frac{\partial\zeta}{\partial t} \in L^1(Q_T) \quad \text{such that} \quad u_0 \geq \zeta,$$

and let $K_\zeta = \Big\{u \in W_0^{1,x}L_\varphi(Q_T) : u \geq \zeta \text{ a.e. in } Q_T\Big\}$.

Note that $<,>$ means for either the pairing between $W_0^{1,x}L_\varphi(Q_T) \cap L^\infty(Q_T)$ and $W^{-1,x}L_\psi(Q_T) + L^1(Q_T)$ or between $W_0^{1,x}L_\varphi(Q_T)$ and $W^{-1,x}L_\psi(Q_T)$.

The definition of a entropy solution of Problem (1.1) can be stated as follows.

Definition 3.1. *A measurable function u defined on Q_T is a entropy solution of Problem (1.1), if it satisfies the following conditions:*

$$\begin{cases} \displaystyle\int_\Omega S_k(u(T)-v(T))dx + \int_{Q_T}\frac{\partial v}{\partial t}T_k(s-v(x,0))dxdt \\ \displaystyle+ \int_{Q_T} a(x,t,u,\nabla u)\nabla T_k(u-v)dxdt \\ \displaystyle+ \int_{Q_T} \Phi(x,t,u)\nabla T_k(u-v)dxdt + \int_{Q_T} H(x,t,u,\nabla u)T_k(u-v)dxdt \\ \displaystyle\leq \int_{Q_T} fT_k(u-v)dxdt + \int_\Omega S_k(u_0-v(x,0))dx, \\ \forall k>0, \quad \textit{and} \quad \forall v \in K_\zeta \cap L^\infty(Q_T) \quad \textit{such that} \quad \frac{\partial v}{\partial t} \in L_\psi(0,T;W^{-1}L_\psi(\Omega)). \end{cases} \tag{3.8}$$

4. Main result

Theorem 4.1. *Assume that (3.1) − (3.7) hold true . Then there exists at least one entropy solution u of the problem (1.1) in the sense of definition (3.1).*

Remark 4.2. *The results obtained in Theorem (4.1), remains true if we replace (3.4) by the growth condition $|\Phi(x,t,s)| \leq c(x)\overline{\gamma}_x^{-1}\gamma(x,|s|)$, where $c(.) \in E_\gamma(\Omega)$ and $\gamma \prec\prec \varphi$.*

Remark 4.3. *Condition* (3.3) *can be replaced by the weaker one*

$$a(x, s, \xi)\xi \geq \alpha\varphi(x, |\xi|) - b(x),$$

where $b(x)$ is in L^1-function.

Remark 4.4. *We obtain the existence result without assuming the coercivity condition. However one can overcome this difficulty by introduced an appropriate test function.*

Remark 4.5. *We will denote by C_i with $i = 1, 2, ...$ any constant which depends on the various quantities of the problem but not on n.*

Proof:
Step 1: Approximate problem.
For each $n > 0$, we define the following approximations

$$a_n(x, t, s, \xi) = a(x, t, T_n(s), \xi) \quad \text{a.e. } (x, t) \in Q_T, \ \forall\ s \in \mathbb{R}, \ \forall\ \xi \in \mathbb{R}^N, \tag{4.1}$$

$$\Phi_n(x, t, s) = \Phi(x, t, T_n(s)) \quad \text{a.e. } (x, t) \in Q_T, \ \forall\ s \in \mathbb{R}, \tag{4.2}$$

$$H_n(x, t, s, \xi) = \frac{H(x, t, s, \xi)}{1 + \frac{1}{n}|H(x, t, s, \xi)|}, \tag{4.3}$$

$$f_n \in L^1(Q_T) \text{ such that } f_n \to f \text{ strongly in } L^1(Q_T), \text{and } \|f_n\|_{L^1(Q_T)} \leq \|f\|_{L^1(Q_T)}, \tag{4.4}$$

and

$$u_{0n} \in \mathcal{C}_0^\infty(\Omega) \text{ such that } u_{0n} \to u_0 \text{ strongly in } L^1(\Omega). \tag{4.5}$$

We define $sg_n(s) = \frac{T_n(s)}{n}$.
Let us now consider the approximate problem :

$$\begin{cases} \dfrac{\partial u_n}{\partial t} - div(a_n(x, t, u_n, \nabla u_n)) + H_n(x, t, u_n, \nabla u_n) & \\ +nT_n(u_n - \zeta)^- sg_{\frac{1}{n}}(u_n) = f_n + div(\Phi_n(x, t, u_n)) & \text{in} \quad Q_T, \\ u_n(x, t) = 0 & \text{on} \quad \partial\Omega \times (0, T), \\ u_n(x, 0) = u_{0n} & \text{in} \quad \Omega. \end{cases} \tag{4.6}$$

Since H_n is bounded for any fixed $n > 0$, there exists at last one solution $u_n \in W_0^{1,x}L_\varphi(Q_T)$ of (4.6)(see [18]).
Step 2: A priori estimates.

Lemma 4.6. *Let $\{u_n\}_n$ be a solution of the approximate problem* (4.6)*, then for all $k > 0$, there exists a constants C_1 and C_2 such that*

$$\int_{Q_T} a(x, t, T_k(u_n), \nabla T_k(u_n))\nabla T_k(u_n)dx \leq kC_1, \tag{4.7}$$

and

$$\int_{Q_T} \varphi(x, |\nabla T_k(u_n)|)dx \leq kC_2, \tag{4.8}$$

where C_1 and C_2 does not depend on the n and k.

Proof: Fixed $k > 0$,
Let $\tau \in (0,T)$ and using $\exp(G(u_n))T_k(u_n)^+\chi_{(0,\tau)}$ as a test function in problem (4.6), where $G(s) = \int_0^s \frac{\rho(r)}{\alpha'}dr$, and $\alpha' > 0$ is a parameter to be specified later. We get

$$\int_{Q_\tau} \frac{\partial u_n}{\partial t} \exp(G(u_n))T_k(u_n)^+ dxdt \tag{4.9}$$

$$+\int_{Q_\tau} a_n(x,t,u_n,\nabla u_n)\nabla\big(\exp(G(u_n))T_k(u_n)^+\big)dxdt \tag{4.10}$$

$$+\int_{Q_\tau} \Phi_n(x,t,u_n)\nabla\big(\exp(G(u_n))T_k(u_n)^+\big)dxdt \tag{4.11}$$

$$+\int_{Q_\tau} H(x,t,u_n,\nabla u_n)\exp(G(u_n))T_k(u_n)^+ dxdt \tag{4.12}$$

$$+\int_{Q_\tau} nT_n(u_n-\zeta)^- sg_{\frac{1}{n}}(u_n)\exp(G(u_n))T_k(u_n)^+ dxdt \tag{4.13}$$

$$\leq k\exp(\frac{\|\rho\|_{L^1}}{\alpha'})\|f_n\|_{L^1(Q_T)}. \tag{4.14}$$

* For the (4.9), we take

$$\widetilde{T}_k(r) = \int_0^r \exp(G(s))T_k(s)^+ ds,$$

then

$$\int_{Q_\tau} \frac{\partial u_n}{\partial t}\exp(G(u_n))T_k(u_n)^+ dxdt = \int_\Omega \widetilde{T}_k(u_n(\tau))dx - \int_\Omega \widetilde{T}_k(u_n(0))dx. \tag{4.15}$$

By definition we have

$$\int_\Omega \widetilde{T}_k(u_n(\tau))dx \geq 0$$

and

$$\int_\Omega \widetilde{T}_k(u_n(0))dx \leq k\exp(\frac{\|\rho\|_{L^1}}{\alpha'})\|u_0\|_{L^1(\Omega)}.$$

* For the (4.11) we use (3.4) and Young inequality, we get

$$\int_{Q_\tau} \Phi_n(x,t,u_n)\nabla(\exp(G(u_n))T_k(u_n)^+)dxdt$$

$$\leq \frac{\|c(.,.)\|_{L^\infty(Q_T)}}{\alpha'}\Big[\alpha_0\int_{Q_\tau}\varphi(x,u_n)\rho(u_n)\exp(G(u_n))T_k(u_n)^+ dxdt$$

$$+\int_{Q_\tau}\varphi(x,\nabla u_n)\rho(u_n)\exp(G(u_n))T_k(u_n)^+ dxdt\Big]$$

$$+\|c(.,.)\|_{L^\infty(Q_T)}\alpha_0 \int_{Q_\tau} \varphi(x,u_n)\exp(G(u_n))dxdt$$

$$+\|c(.,.)\|_{L^\infty(Q_T)} \int_{Q_\tau} \varphi(x,|\nabla T_k(u_n)^+|)\exp(G(u_n))dxdt.$$

* For the (4.12) we have,

$$\int_{Q_\tau} H_n(x,t,u_n,\nabla u_n)\exp(G(u_n))T_k(u_n)^+dxdt \le k\exp(\frac{\|\rho\|_{L^1}}{\alpha'})\int_{Q_T}|h(x,t)|dxdt$$

$$+\int_{Q_\tau}\rho(u_n)\exp(G(u_n))\varphi(x,\nabla u_n)T_k(u_n)^+dxdt.$$

Finally using the previous inequalities and (3.3), we obtain

$$\begin{cases} \frac{1}{\alpha'}\int_{Q_\tau}\varphi(x,u_n)\rho(u_n)\exp(G(u_n))T_k(u_n)^+dxdt \\ +\frac{\alpha}{\alpha'}\int_{Q_T}\varphi(x,\nabla u_n)\rho(u_n)\exp(G(u_n))T_k(u_n)^+dxdt \\ +\int_{Q_\tau}a(x,t,u_n,\nabla u_n)\exp(G(u_n))\nabla T_k(u_n)^+dxdt \\ +\int_{Q_t}nT_n(u_n-\zeta)^-sg_{\frac{1}{n}}(u_n)\exp(G(u_n))T_k(u_n)^+dxdt \\ \le \frac{\|c(.,.)\|_{L^\infty(Q_T)}}{\alpha'}\Big[\alpha_0\int_{Q_\tau}\varphi(x,u_n)\rho(u_n)\exp(G(u_n))T_k(u_n)^+dxdt \\ +\int_{Q_\tau}\varphi(x,\nabla u_n)\rho(u_n)\exp(G(u_n))T_k(u_n)^+dxdt\Big] \\ +\alpha_0\|c(.,.)\|_{L^\infty(Q_T)}\int_{\{0\le u_n\le k\}}\varphi(x,u_n)\exp(G(u_n))dxdt \\ +\|c(.,.)\|_{L^\infty(Q_T)}\int_{Q_\tau}\varphi(x,\nabla T_k(u_n)^+)\exp(G(u_n))dxdt \\ +\int_{Q_\tau}\varphi(x,\nabla u_n)\rho(u_n)\exp(G(u_n))T_k(u_n)^+dxdt \\ +k\Big[\exp(\frac{\|\rho\|_{L^1}}{\alpha'}(\|f\|_{L^1(Q_T)}+\|u_0\|_{L^1(\Omega)}+\int_{Q_T}|h(x,t)|dxdt\Big], \end{cases} \tag{4.16}$$

Using again (3.3) in (4.16) we get

$$\Big[\frac{1-\alpha_0\|c(.,.)\|_{L^\infty(Q_T)}}{\alpha'}\Big]\int_{Q_\tau}\varphi(x,u_n)\rho(u_n)\exp(G(u_n))T_k(u_n)^+dxdt$$

$$+\Big[\frac{\alpha-\|c(.,.)\|_{L^\infty(Q_T)}-\alpha'}{\alpha'}\Big]\int_{Q_\tau}\varphi(x,\nabla u_n)\rho(u_n)\exp(G(u_n))T_k(u_n)^+dxdt$$

$$+\int_{Q_\tau}a(x,t,u_n,\nabla u_n)\exp(G(u_n))\nabla T_k(u_n)^+dxdt$$

$$+\int_{Q_t}nT_n(u_n-\zeta)^-sg_{\frac{1}{n}}(u_n)\exp(G(u_n))T_k(u_n)^+dxdt$$

$$\leq \frac{\|c(.,.)\|_{L^\infty(Q_T)}}{\alpha}\Big[\alpha_0\alpha\int_{\{0\leq u_n\leq k\}}\varphi(x,u_n)\exp(G(u_n))dxdt$$
$$+\alpha\varphi(x,\nabla T_k(u_n)^+)\exp(G(u_n))dxdt\Big]+kc_1.$$

If we choose α' such that $\alpha'<\alpha-\|c(.,.)\|_{L^\infty(Q_T)}$ and using again (3.3) we get

$$\Big[1-\frac{\|c(.,.)\|_{L^\infty(Q_T)}}{\alpha}\Big]\int_{Q_\tau}a(x,t,u_n,\nabla u_n)\exp(G(u_n))\nabla T_k(u_n)^+dxdt$$
$$+\int_{Q_t}nT_n(u_n-\zeta)^-sg_{\frac{1}{n}}(u_n)\exp(G(u_n))T_k(u_n)^+dxdt\leq kc_1. \tag{4.17}$$

Taking $\dfrac{1}{c_2}=\Big[1-\dfrac{\|c(.,.)\|_{L^\infty(Q_T)}}{\alpha}\Big]$. Thus,

$$\int_{Q_\tau}a(x,t,u_n,\nabla u_n)\exp(G(u_n))\nabla T_k(u_n)^+dxdt$$
$$+c_2\int_{Q_\tau}nT_n(u_n-\zeta)^-sg_{\frac{1}{n}}(u_n)\exp(G(u_n))T_k(u_n)^+dxdt\leq kc_1c_2.$$

It follow that

$$0\leq\int_{Q_\tau}nT_n(u-\zeta)^-sg_{\frac{1}{n}}(u_n)\exp(G(u_n))\frac{T_k(u_n)^+}{k}dxdt\leq c_1,$$

by Fatou's lemma as $k\to 0$ we have

$$0\leq\int_{\{u_n\geq 0\}}nT_n(u_n-\zeta)^-sg_{\frac{1}{n}}(u_n)\exp(G(u_n))dxdt\leq c_1.$$

Return to (4.17), we deduce easily

$$\int_{\{0\leq u_n\leq k\}}a(x,t,u_n,\nabla u_n)\exp(G(u_n))\nabla T_k(u_n)dxdt\leq kc_1c_2.$$

And as one has $\exp(G(u_n))\geq 1$ for $0\leq u_n\leq k$, then

$$\int_{\{0\leq u_n\leq k\}}a(x,t,u_n,\nabla u_n)\nabla T_k(u_n)dxdt\leq kc_1c_2, \tag{4.18}$$

by (3.3)

$$\int_{Q_\tau}\varphi(x,|\nabla T_k(u_n)^+|)dxdt\leq\frac{kc_1c_2}{\alpha}, \tag{4.19}$$

and

$$0\leq\int_{\{u_n\geq 0\}}nT_n(u_n-\zeta)^-sg_{\frac{1}{n}}(u_n)dxdt\leq c_1. \tag{4.20}$$

Similarly, taking $\exp(-G(u_n))T_k(u_n)^-\chi_{(0,\tau)}$ as a test function in problem (4.6), we get

$$\int_{Q_\tau} \frac{\partial u_n}{\partial t} \exp(-G(u_n))T_k(u_n)^- dxdt \tag{4.21}$$

$$+ \int_{Q_\tau} a_n(x,t,u_n,\nabla u_n)\nabla(\exp(-G(u_n))\nabla T_k(u_n)^-)dxdt \tag{4.22}$$

$$+ \int_{Q_\tau} \Phi_n(x,t,u_n)\nabla(\exp(-G(u_n))\nabla T_k(u_n)^-)dxdt \tag{4.23}$$

$$+ \int_{Q_\tau} H(x,t,u_n,\nabla u_n)\exp(-G(u_n))T_k(u_n)^- dxdt \tag{4.24}$$

$$+ \int_{Q_\tau} nT_n(u_n-\zeta)^- sg_{\frac{1}{n}}(u_n)\exp(-G(u_n))T_k(u_n)^- dxdt \tag{4.25}$$

$$\geq \int_{Q_\tau} f_n \exp(-G(u_n))T_k(u_n)^- dxdt, \tag{4.26}$$

we take

$$\widetilde{T}_k(r) = \int_0^r \exp(-G(s))T_k(s)^- ds,$$

then

$$\int_{Q_\tau} \frac{\partial u_n}{\partial s} \exp(-G(u_n))T_k(u_n)^- dxdt = \int_\Omega \widetilde{T}_k(u_n(\tau))dx - \int_\Omega \widetilde{T}_k(u_n(0))dx, \tag{4.27}$$

and using same techniques, we obtain also

$$\int_{Q_\tau} a(x,t,u_n,\nabla u_n)\exp(-G(u_n))\nabla T_k(u_n)dxdt$$

$$+ c_2 \int_{Q_\tau} nT_n(u-\zeta)^- sg_{\frac{1}{n}}(u_n)\exp(-G(u_n))T_k(u_n)^- dxdt \leq kc_1c_2. \tag{4.28}$$

It follow that

$$0 \leq \int_{Q_\tau} nT_n(u_n-\zeta)^- sg_{\frac{1}{n}}(u_n)\exp(-G(u_n))\frac{T_k(u_n)^-}{k}dxdt \leq c_1,$$

we deduce by Fatou's lemma as $k \to 0$ that

$$0 \leq \int_{\{u_n \leq 0\}} nT_n(u_n-\zeta)^- sg_{\frac{1}{n}}(u_n)\exp(-G(u_n))dxdt \leq c_1,$$

And as one has $\exp(-G(u_n)) \geq 1$ since $-k \leq u_n \leq 0$, then

$$\int_{\{-k \leq u_n \leq 0\}} a(x,t,u_n,\nabla u_n)\nabla T_k(u_n)dxdt \leq kc_1c_2, \tag{4.29}$$

$$\int_{Q_\tau} \varphi(x, |\nabla T_k(u_n)^-|)dxdt \leq \frac{kc_1c_2}{\alpha}, \tag{4.30}$$

and

$$0 \leq \int_{\{u_n \leq 0\}} nT_n(u_n - \zeta)^- sg_{\frac{1}{n}}(u_n)dxdt \leq c_1. \tag{4.31}$$

Combining now (4.18) and (4.29) we get,

$$\int_{Q_T} a(x, t, u_n, \nabla u_n)\nabla T_k(u_n)dxdt \leq kC_1. \tag{4.32}$$

Of the same with (4.19) and (4.30) we get,

$$\int_{Q_T} \varphi(x, |\nabla T_k(u_n)|)dxdt \leq kC_2. \tag{4.33}$$

we conclude that $T_k((u_n))$ is bounded in $W_0^{1,x}L_\varphi(Q_T)$ independently of n and for any $k > 0$, so there exists a subsequence still denoted by u_n such that

$$T_k(u_n) \rightharpoonup \xi_k \quad \text{weakly in} \quad W_0^{1,x}L_\varphi(Q_T). \tag{4.34}$$

On the other hand, using (4.33), we have

$$\inf_{x\in\Omega} \varphi(x, \frac{k}{\delta}) meas\{|u_n| > k\} \leq \int_{|u_n|>k} \varphi(x, \frac{|T_k(u_n)|}{\delta})dxdt$$
$$\leq \int_{Q_T} \varphi(x, |\nabla T_k(u_n)|)dxdt \leq kC_2.$$

Then

$$meas\{|u_n| > k\} \leq \frac{kC_2}{\inf_{x\in\Omega} \varphi(x, \frac{k}{\delta})},$$

for all n and for all k.
Assuming that there exists a positive function M such that $\lim_{t\to\infty} \frac{M(t)}{t} = +\infty$ and $M(t) \leq ess \inf_{x\in\Omega} \varphi(x, t)$, $\forall t \geq 0$. Thus, we get

$$\lim_{k\to\infty} meas\{|u_n| > k\} = 0. \tag{4.35}$$

□

Step 3:
Now we turn to prove the almost every convergence of u_n and convergence of $a_n(x, t, T_k(u_n), \nabla T_k(u_n))$.

Proposition 4.7. *Let u_n be a solution of the approximate problem, then*

$$u_n \to u \quad a.e \ in \quad Q_T, \tag{4.36}$$

$$a_n(x, t, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup \varpi_k \quad in \quad (L_\psi(Q))^N, \quad for \quad \sigma(\Pi L_\psi, \Pi E_\varphi), \tag{4.37}$$

for some $\varpi_k \in (L_\psi(Q))^N$.

Proof of (4.36) : Let $\lambda > 0$ then

$$meas\{|u_m - u_n| > \lambda\} \leq meas\{|u_m| > k\}$$
$$+meas\{|u_n| > k\} + meas\{|T_k(u_m) - T_k(u_n)| > \lambda\}.$$

By (4.34), we can assume that $T_k(u_n)$ is a Cauchy sequence in measure in Q_T and using (4.35) we deduce that for any $\epsilon > 0$ there exists some $k(\epsilon) > 0$ such that

$$meas\{|u_m - u_n| > \lambda\} \leq \epsilon \quad \text{for all} \quad n, m > N_{k(\epsilon),\lambda}.$$

Which means that u_n is a Cauchy sequence in measure in Q_T, thus converge almost every where to some measurable function u.

Proof of (4.37) : We shall prove that $\{a(x,t,T_k(u_n),\nabla T_k(u_n))\}_n$ is bounded in $\big(L_\psi(\Omega)\big)^N$ for all $k > 0$.

Let $w \in (E_\varphi(\Omega))^N$ be arbitrary. By condition (3.2) we have,

$$(a(x,t,u_n,\nabla u_n) - a(x,t,u_n,w))(\nabla u_n - w) > 0,$$

then

$$\begin{aligned}\int_{\{|u_n|\leq k\}} a(x,t,u_n,\nabla u_n) w\,dxdt \quad &\leq \quad \int_{\{|u_n|\leq k\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dxdt \\ &\quad + \int_{\{|u_n|\leq k\}} a(x,t,u_n,w)(w - \nabla u_n)\,dxdt,\end{aligned}$$

by (3.1) we have for $\nu > \beta$

$$\begin{aligned}\int_{\{|u_n|\leq k\}} \psi_x\Big(x, \frac{a(x,t,u_n,\frac{w}{k_2})}{3\nu}\Big)dxdt \quad &\leq \quad \frac{\beta}{3\nu}\int_{Q_T}\Big[\psi(x,a_0(x,t)) + \gamma(x,k_1|T_k(u_n)|)\Big]dxdt \\ &\quad + \frac{\beta}{3\nu}\int_{Q_T}\Big[\varphi(x,|w|)\Big]dxdt \\ &\leq \quad \frac{\beta}{3\nu}\Big[\int_{Q_T}\psi(x,a_0(x,t)) + \gamma(x,k_1 k)dxdt\Big] \\ &\quad + \frac{\beta}{3\nu}\Big[\int_Q \varphi(x,|w|)dxdt\Big]. \end{aligned} \tag{4.38}$$

Thus $\{a(x,t,T_k(u_n),\dfrac{w}{k_2})\}$ is bounded in $(L_\psi(\Omega))^N$.

By (4.38),(4.7) and by the theorem of Banach–Steinhaus, the sequence

$$\{a(x,t,T_k(u_n),\nabla T_k(u_n))\}$$

remains bounded in $(L_\psi(\Omega))^N$ and we conclude (4.37).

Lemma 4.8. *If the subsequence u_n satisfies (4.6), then*

$$\lim_{m\to+\infty}\limsup_{n\to+\infty}\int_{\{m\leq|u_n|\leq m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dxdt = 0. \tag{4.39}$$

Proof: Taking the function $Z_m(u_n) = T_1(u_n - T_m(u_n))^-$ and multiplying the approximating equation (4.6) by the test function $\exp(-G(u_n))Z_m(u_n)$ we get

$$\begin{cases} \int_\Omega \widetilde{T}_m(x, u_n(T))dx + \int_{Q_T} a_n(x,t,u_n,\nabla u_n)\nabla(\exp(-G(u_n))Z_m(u_n))dx\,dt \\ + \int_{Q_T} \Phi_n(x,t,u_n)\nabla(\exp(-G(u_n))Z_m(u_n))dx\,dt \\ + \int_{Q_T} H_n(x,t,u_n,\nabla u_n)\exp(-G(u_n))Z_m(u_n)dx\,dt \\ + \int_{Q_T} nT_n(u_n-\zeta)^- sg_{\frac{1}{n}}(u_n)\exp(-G(u_n))Z_m(u_n)\,dx\,dt \\ = \int_{Q_T} f_n \exp(-G(u_n))Z_m(u_n)\,dx\,dt + \int_\Omega \widetilde{T}_m(u_{0n})dx, \end{cases} \tag{4.40}$$

where $\widetilde{T}_m(r) = \int_0^r \exp(-G(s))Z_m(s)ds$.
we know that $\int_\Omega \widetilde{T}_m(x, u_n(T))dx \geq 0$,
and $\int_{Q_T} nT_n(u_n-\zeta)^- sg_{\frac{1}{n}}(u_n)\exp(-G(u_n))Z_m(u_n)\,dx\,dt \geq 0$.
Then, using the same argument in step 2 to remove the term

$$\int_{Q_T} \rho(u_n)\exp(-G(u_n))\nabla u_n Z_m(u_n)dx\,dt,$$

we obtain,

$$\begin{cases} \int_{\{-(m+1)\leq u_n \leq -m\}} a_n(x,t,u_n,\nabla u_n)\nabla u_n dx\,dt \\ + \int_{Q_T} \Phi_n(x,t,u_n)\exp(-G(u_n))\nabla Z_m(u_n)dx\,dt \\ \leq \exp(\frac{\|\rho\|_{L^1}}{\alpha'})\Big[\int_{Q_T} |f_n|Z_m(u_n)\,dx\,dt \int_{Q_T} |h(x,t)|Z_m(u_n)dx\,dt + \int_{|u_{0n}|>m} |u_{0n}|dx\Big]. \end{cases} \tag{4.41}$$

Thanks to the (3.4) and (4.41) we obtain,

$$\int_{Q_T} \varphi(x, |\nabla Z_m(u_n)|)\exp(-G(u_n))dxdt \leq \frac{c_2}{\alpha}\exp(\frac{\|\rho\|_{L^1}}{\alpha'})\Big(\int_{Q_T} f_n Z_m(u_n)dxdt$$
$$+ \int_{Q_T} |h(x,t)|Z_m(u_n)dxdt + \int_{\{|u_{0n}|>m\}} |u_{0n}|dx\Big),$$

Passing to limit as $n \to +\infty$, since the pointwise convergence of u_n and strongly convergence in $L^1(Q_T)$ of f_n and u_{0n} we get

$$\lim_{n\to+\infty} \int_{Q_T} \varphi(x, |\nabla Z_m(u_n)|)\exp(-G(u_n))dxdt \leq C\Big(\int_{Q_T} fZ_m(u)dxdt$$
$$+ \int_{Q_T} |h(x,t)|Z_m(u)dxdt + \int_{\{|u_0|>m\}} |u_0|dx\Big).$$

By using Lebesgue's theorem and passing to limit as $m \to +\infty$, in the all term of the right-hand side, we get

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{Q_T}\varphi(x,|\nabla Z_m(u_n)|)\exp(-G(u_n))dxdt = 0. \tag{4.42}$$

On the other hand, we have

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{Q_T}\Phi_n(x,t,u_n)\exp(-G(u_n))\nabla Z_m(u_n)dxdt$$
$$\leq \lim_{m\to+\infty}\lim_{n\to+\infty}\int_{Q_T}\varphi(x,|\nabla Z_m(u_n)|)\exp(-G(u_n))dxdt$$
$$+\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{m\leq|u_n|\leq m+1}\psi(x,|\Phi_n(x,t,u_n)|)\exp(-G(u_n))dxdt.$$

Using the pointwise convergence of u_n and Lebegue's theorem in the second term of the right side, we get

$$\lim_{n\to+\infty}\int_{m\leq|u_n|\leq m+1}\psi(x,|\Phi_n(x,t,u_n)|)\exp(-G(u_n))dxdt$$
$$=\int_{m\leq|u|\leq m+1}\psi(x,|\Phi(x,t,u)|)\exp(-G(u))dxdt,$$

and also, by Lebesgue's theorem

$$\lim_{m\to+\infty}\int_{m\leq|u|\leq m+1}\psi(x,|\Phi(x,t,u)|)dxdt = 0. \tag{4.43}$$

Then, we deduce

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{Q_T}\Phi_n(x,t,u_n)\exp(-G(u_n))\nabla Z_m(u_n)dx\,dt = 0.$$

Finally passing to the limit in (4.41), we get

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{\{-(m+1)\leq u_n\leq -m\}}a_n(x,t,u_n,\nabla u_n)\nabla u_n dx\,dt = 0.$$

In the same way we take $Z_m(u_n) = T_1(u_n - T_m(u_n))^+$ and multiplying the approximating equation (4.6) by the test function $\exp(G(u_n))Z_m(u_n)$ and we also obtain

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{\{m\leq u_n\leq m+1\}}a_n(x,t,u_n,\nabla u_n)\nabla u_n dx\,dt = 0.$$

On the above we get (4.39). □

Step 4: Almost everywhere convergence of the gradients.

This step is devoted to introduce a time regularization of the $T_k(u)$ for $k > 0$ in order to perform the monotonicity method.

Lemma 4.9. *Under assumptions (3.1)-(3.7), and let* (z_n) *be a sequence in* $W_0^{1,x}L_\varphi(Q_T)$ *such that:*

$$z_n \rightharpoonup z \quad for \quad \sigma(\Pi L_\varphi(Q_T), \Pi E_\psi(Q_T)), \tag{4.44}$$

$$(a(x,t,z_n,\nabla z_n)) \quad is\ bounded\ in\ (L_\psi(Q_T))^N, \tag{4.45}$$

$$\int_{Q_T} [a(x,t,z_n,\nabla z_n) - a(x,t,z_n,\nabla z\chi_s)][\nabla z_n - \nabla z\chi_s]dx\,dt \to 0, \tag{4.46}$$

as n *and* s *tend to* $+\infty$*, and where* χ_s *is the characteristic function of* $Q_s = \{(x,t) \in Q_T; |\nabla z| \le s\}$ *then,*

$$\nabla z_n \to \nabla z \quad a.e.\ in\ Q_T, \tag{4.47}$$

$$\lim_{n\to+\infty} \int_{Q_T} a(x,t,z_n,\nabla z_n)\nabla z_n dx\,dt = \int_{Q_T} a(x,t,z,\nabla z)\nabla z dx\,dt, \tag{4.48}$$

$$\varphi(x,|\nabla z_n|) \to \varphi(x,|\nabla z|) \quad in\ L^1(Q_T). \tag{4.49}$$

Proof: It is easily adapted from that given in [7]. □

Let $\upsilon_j \in \mathcal{D}(Q_T)$ be a sequence such that $\upsilon_j \to u$ in $W_0^{1,x}L_\varphi(Q_T)$ for the modular convergence.
This specific time regularization of $T_k(\upsilon_j)$ (for fixed $k \ge 0$) is defined as follows. Let $(\alpha_0^\mu)_\mu$ be a sequence of functions defined on Ω such that

$$\alpha_0^\mu \in L^\infty(\Omega) \cap W_0^1 L_\varphi(\Omega) \quad \text{for all} \quad \mu > 0, \tag{4.50}$$

$$\|\alpha_0^\mu\|_{L^\infty(\Omega)} \le k, \text{ for all } \quad \mu > 0,$$

and

$$\alpha_0^\mu \text{ converges to } \quad T_k(u_0) \text{ a.e. in } \quad \Omega \quad \text{ and } \quad \frac{1}{\mu}\|\alpha_0^\mu\|_{\varphi,\Omega} \text{ converges to } 0, \mu \to +\infty$$

For $k \ge 0$ and $\mu > 0$, let us consider the unique solution $(T_k(\upsilon_j))_\mu \in L^\infty(Q) \cap W_0^{1,x}L_\varphi(Q)$ of the monotone problem:

$$\frac{\partial (T_k(\upsilon_j))_\mu}{\partial t} + \mu((T_k(\upsilon_j))_\mu - T_k(\upsilon_j)) = 0 \text{ in } D'(\Omega),$$

$$(T_k(\upsilon_j))_\mu(t=0) = \alpha_0^\mu \text{ in } \Omega.$$

Remark that due to

$$\frac{\partial (T_k(\upsilon_j))_\mu}{\partial t} \in W_0^{1,x}L_\varphi(Q_T).$$

We just recall that,

$$(T_k(\upsilon_j))_\mu \to T_k(u) \quad \text{a.e. in} \quad Q_T, \quad \text{weakly} - * \quad \text{in} \quad L^\infty(Q_T)$$

$(T_k(\upsilon_j))_\mu \to (T_k(u))_\mu$ in $W_0^{1,x}L_\varphi(Q_T)$ for the modular convergence as $j \to +\infty$.
$(T_k(u))_\mu \to T_k(u)$ in $W_0^{1,x}L_\varphi(Q_T)$ for the modular convergence as $\mu \to +\infty$.

$$||(T_k(\upsilon_j))_\mu||_{L^\infty(Q_T)} \leq max(||(T_k(u))||_{L^\infty(Q_T)}, \quad ||\alpha_0^\mu||_{L^\infty(\Omega)}) \leq k,$$

for all $\mu > 0$, and for all $k > 0$. We introduce a sequence of increasing $\mathbf{C}^1(\mathbb{R})$-functions S_m such that

$$S_m(r) = 1 \text{ for } |r| \leq m, \; S_m(r) = m + 1 - |r|, \text{ for } m \leq |r| \leq m+1, S_m(r) = 0$$

for $|r| \geq m+1$ for any $m \geq 1$. And we denote by $\epsilon(n, \mu, \eta, j, m)$ the quantities such that

$$\lim_{m\to+\infty} \lim_{j\to+\infty} \lim_{\eta\to+\infty} \lim_{\mu\to+\infty} \lim_{n\to+\infty} \epsilon(n, \mu, \eta, j, m) = 0.$$

For fixed $k \geq 0$, let $W_{\mu,\eta}^{n,j} = T_\eta(T_k(u_n) - T_k(\upsilon_j)_\mu)^+$ and $W_{\mu,\eta}^{j} = T_\eta(T_k(u) - T_k(\upsilon_j)_\mu)^+$.
Multiplying the approximating equation by $\exp(G(u_n)))W_{\mu,\eta}^{n,j} S_m(u_n)$ and using the same technique in step 2 we obtain:

$$\begin{cases} \displaystyle\int_{Q_T} < \frac{\partial u_n}{\partial t} \exp(G(u_n)) W_{\mu,\eta}^{n,j} S_m(u_n)\, dx\, dt \\ \displaystyle + \int_{Q_T} a_n(x,t,u_n,\nabla u_n) \exp(G(u_n)) \nabla(W_{\mu,\eta}^{n,j}) S_m(u_n)\, dx\, dt \\ \displaystyle + \int_{Q_T} a_n(x,t,u_n,\nabla u_n) \nabla u_n \exp(G(u_n)) W_{\mu,\eta}^{n,j} S'_m(u_n)\, dx\, dt \\ \displaystyle - \int_{Q_T} \Phi_n(x,t,u_n) \exp(G(u_n)) \nabla(W_{\mu,\eta}^{n,j}) S_m(u_n)\, dx\, dt \\ \displaystyle - \int_{Q_T} \Phi_n(x,t,u_n) \nabla u_n \exp(G(u_n)) W_{\mu,\eta}^{n,j} S'_m(u_n)\, dx\, dt \\ \displaystyle \leq \int_{Q_T} f_n \exp(G(u_n)) W_{\mu,\eta}^{n,j} S_m(u_n)\, dx\, dt + \int_{Q_T} h(x,t) \exp(G(u_n)) W_{\mu,\eta}^{n,j} S_m(u_n)\, dx\, dt. \end{cases} \tag{4.51}$$

Now we pass to the limit in (4.51) for k real number fixed.
In order to perform this task we prove below the following results for any fixed $k \geq 0$:

$$\int_{Q_T} \frac{\partial u_n}{\partial t} \exp(G(u_n)) W_{\mu,\eta}^{n,j} S_m(u_n)\, dx\, dt \geq \epsilon(n, \mu, \eta, j) \qquad \text{for any } m \geq 1, \tag{4.52}$$

$$\int_{Q_T} \Phi_n(x,t,u_n) S_m(u_n) \exp(G(u_n)) \nabla(W_{\mu,\eta}^{n,j})\, dx\, dt = \epsilon(n, j, \mu) \quad \text{for any } m \geq 1, \tag{4.53}$$

$$\int_{Q_T} \Phi_n(x,t,u_n) \nabla u_n S'_m(u_n) \exp(G(u_n)) W_{\mu,\eta}^{n,j}\, dx\, dt = \epsilon(n, j, \mu) \qquad \text{for any } m \geq 1, \tag{4.54}$$

$$\int_{Q_T} a_n(x,t,u_n,\nabla u_n) \nabla u_n S'_m(u_n) \exp(G(u_n)) W_{\mu,\eta}^{n,j}\, dx\, dt \leq \epsilon(n, m), \tag{4.55}$$

$$\int_{Q_T} a_n(x,t,u_n,\nabla u_n) S_m(u_n)\exp(G(u_n))\nabla(W^{n,j}_{\mu,\eta})\,dx\,dt \leq C\eta + \epsilon(n,j,\mu,m), \tag{4.56}$$

$$\begin{aligned}&\int_{Q_T} f_n S_m(u_n)\exp(G(u_n))W^{n,j}_{\mu,\eta}\,dx\,dt\\ &+\int_{Q_T} h(x,t)\exp(G(u_n))W^{n,j}_{\mu,\eta}S_m(u_n)\,dx\,dt \leq C\eta + \epsilon(n,\eta),\end{aligned} \tag{4.57}$$

$$\int_{Q_T}\Big[a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u))\Big]\Big[\nabla T_k(u_n) - \nabla T_k(u)\Big]dx\,dt \to 0. \tag{4.58}$$

Proof of (4.52):

Lemma 4.10.

$$\int_{Q_T} \frac{\partial u_n}{\partial t}\exp(G(u_n))W^{n,j}_{\mu,\eta}S_m(u_n)dxdt \geq \epsilon(n,\mu,\eta,\eta,j) \quad m \geq 1. \tag{4.59}$$

Proof: Is a particular case of the proof in [4], with $b(x,u) = u$. □

Proof of (4.53): If we take $n > m+1$, we get

$$\begin{aligned}\Phi_n(x,t,u_n)\exp(G(u_n))S_m(u_n) &= \Phi(x,t,T_{m+1}(u_n))\exp(G(T_{m+1}(u_n)))\\ &\quad\times S_m(T_{m+1}(u_n)),\end{aligned}$$

then $\Phi_n(x,t,u_n)\exp(G(u_n))S_m(u_n)$ is bounded in $L_\psi(Q)$, thus, by using the pointwise convergence of u_n and Lebesgue's theorem we obtain

$$\Phi_n(x,t,u_n)\exp(G(u_n))S_m(u_n) \to \Phi(x,t,u)\exp(G(u))S_m(u),$$

with the modular convergence as $n \to +\infty$,
then

$$\Phi_n(x,t,u_n)\exp(G(u_n))S_m(u_n) \to \Phi(x,t,u)\exp(G(u))S_m(u),$$

for $\sigma(\prod L_\psi, \prod L_\varphi)$.
In the other hand $\nabla W^{n,j}_{\mu,\eta} = \nabla T_k(u_n) - \nabla(T_k(v_j))_\mu$ for $|T_k(u_n) - (T_k(v_j))_\mu| \leq \eta$ converge to $\nabla T_k(u) - \nabla(T_k(v_j))_\mu$ weakly in $(L_\varphi(Q_T))^N$, then

$$\begin{aligned}&\int_{Q_T} \Phi_n(x,t,u_n)\exp(G(u_n))S_m(u_n)\nabla W^{n,j}_{\mu,\eta}\,dx\,dt\\ &\to \int_{Q_T} \Phi(x,t,u)S_m(u)\exp(G(u))\nabla W^{j}_{\mu,\eta}\,dx\,dt, \text{ as } n \to +\infty\end{aligned}$$

By using the modular convergence of $W^j_{\mu,\eta}$ as $j \to +\infty$ and letting μ tends to infinity, we get (4.53).

Proof of (4.54):
For $n > m+1 > k$, we have $\nabla u_n S'_m(u_n) = \nabla T_{m+1}(u_n)$ a.e. in Q_T. By the almost every where convergence of u_n we have $\exp(G(u_n))W^{n,j}_{\mu,\eta} \to \exp(G(u))W^{j}_{\mu,\eta}$ in $L^\infty(Q_T)$ weak-* and since the sequence $(\Phi_n(x,t,T_{m+1}(u_n)))_n$ converge strongly in $E_\psi(Q_T)$ then

$$\Phi_n(x,t,T_{m+1}(u_n))\exp(G(u_n))\ W^{n,j}_{\mu,\eta} \to \Phi(x,t,T_{m+1}(u))\ \exp(G(u))W^{j}_{\mu,\eta},$$

converge strongly in $E_\psi(Q_T)$ as $n \to +\infty$. By virtue of $\nabla T_{m+1}(u_n) \to \nabla T_{m+1}(u)$ weakly in $(L_\varphi(Q_T))^N$ as $n \to +\infty$ we have

$$\int_{m\leq|u_n|\leq m+1} \Phi_n(x,t,T_{m+1}(u_n))\nabla u_n S'_m(u_n)\exp(G(u_n))W^{n,j}_{\mu,\eta}\,dx\,dt$$

$$\to \int_{m\leq|u|\leq m+1} \Phi(x,t,u))\nabla u \exp(G(u))W^{j}_{\mu,\eta}\,dx\,dt$$

as $n \to +\infty$ with the modular convergence of $W^{j}_{\mu,\eta}$ as $j \to +\infty$ and letting $\mu \to +\infty$ we get (4.54).
Proof of (4.55):
For (4.55), we have

$$\int_{Q_T} a_n(x,t,u_n,\nabla u_n)S'_m(u_n)\nabla u_n \exp(G(u_n))\exp(G(u_n))W^{n,j}_{\mu,\eta}\,dx\,dt$$

$$= \int_{m\leq|u_n|\leq m+1} a_n(x,t,u_n,\nabla u_n)S'_m(u_n)\nabla u_n \exp(G(u_n))W^{n,j}_{\mu,\eta}\,dx\,dt$$

$$\leq \eta C \int_{m\leq|u_n|\leq m+1} a_n(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt.$$

Using (4.39), we get

$$\int_{Q_T} a_n(x,t,u_n,\nabla u_n)S'_m(u_n)\nabla u_n \exp(G(u_n))W^{n,j}_{\mu,\eta}\,dx\,ds \leq \epsilon(n,\mu,m).$$

Proof of (4.57):
Since $S_m(r) \leq 1$ and $W^{n,j}_{\mu,\eta} \leq \eta$ we get

$$\int_{Q_T} f_n S_m(u_n)\exp(G(u_n))W^{n,j}_{\mu,\eta} \quad dx\,dt \leq \epsilon(n,\eta),$$

$$\int_{Q_T} h(x,t)\exp(G(u_n))W^{n,j}_{\mu,\eta} S_m(u_n)\,dx\,dt \leq C\eta.$$

Proof of (4.56):

$$\begin{aligned}
&\int_{Q_T} a_n(x,t,u_n,\nabla u_n) S_m(u_n)\exp(G(u_n))\nabla W^{n,j}_{\mu,\eta}\,dx\,dt\\
=\;& \int_{\{|u_n|\le k\}\cap\{0\le T_k(u_n)-T_k(\upsilon_j)_\mu)\le\eta\}} a_n(x,t,T_k(u_n),\nabla T_k(u_n))S_m(u_n)\exp(G(u_n))\\
&\times(\nabla T_k(u_n)-\nabla T_k(\upsilon_j)_\mu)\,dx\,dt\\
&-\int_{\{|u_n|>k\}\cap\{0\le T_k(u_n)-T_k(\upsilon_j)_\mu)\le\eta\}} a_n(x,t,u_n,\nabla u_n)S_m(u_n)\\
&\times\exp(G(u_n))\nabla T_k(\upsilon_j)_\mu\,dx\,dt. \qquad (4.60)
\end{aligned}$$

Since $a_n(x,t,T_{k+\eta}(u_n),\nabla T_{k+\eta}(u_n))$ is bounded in $(L_\psi(Q_T))^N$, there exist some $\varpi_{k+\eta}\in(L_\psi(Q_T))^N$ such that $a_n(x,t,T_{k+\eta}(u_n),\nabla T_{k+\eta}(u_n))\to\varpi_{k+\eta}$ weakly in $(L_\psi(Q_T))^N$. Consequently,

$$\begin{aligned}
&\int_{\{|u_n|>k\}\cap\{0\le T_k(u_n)-T_k(\upsilon_j)_\mu)\le\eta\}} a_n(x,t,u_n,\nabla u_n)S_m(u_n)\exp(G(u_n))\nabla T_k(\upsilon_j)_\mu\,dx\,dt\\
&=\int_{\{|u|>k\}\cap\{0\le T_k(u)-T_k(\upsilon_j)_\mu)\le\eta\}} S_m(u)\exp(G(u))\nabla T_k(\upsilon_j)_\mu\varpi_{k+\eta}\,dx\,dt+\epsilon(n), \qquad (4.61)
\end{aligned}$$

where we have used the fact that

$$\begin{aligned}
&S_m(u_n)\exp(G(u_n))\nabla T_k(\upsilon_j)_\mu)\chi_{\{|u_n|>k\}\cap\{0\le T_k(u_n)-T_k(\upsilon_j)_\mu)\le\eta\}}\\
&\to S_m(u)\exp(G(u))\nabla T_k(\upsilon_j)_\mu)\chi_{\{|u|>k\}\cap\{0\le T_k(u)-T_k(\upsilon_j)_\mu)\le\eta\}},
\end{aligned}$$

strongly in $(E_\varphi(Q_T))^N$.
Letting $j\to+\infty$, we obtain

$$\begin{aligned}
&\int_{\{|u|>k\}\cap\{0\le T_k(u)-T_k(\upsilon_j)_\mu)\le\eta\}} S_m(u)\exp(G(u))\nabla T_k(\upsilon_j)_\mu\varpi_{k+\eta}\,dx\,dt\\
&=\int_{\{|u|>k\}\cap\{0\le T_k(u)-T_k(u)_\mu)\le\eta\}} S_m(u)\exp(G(u))\nabla T_k(u)_\mu\varpi_{k+\eta}\,dx\,dt+\epsilon(n,j).
\end{aligned}$$

One easily has,

$$\int_{\{|u|>k\}\cap\{0\le T_k(u)-T_k(u)_\mu)\le\eta\}} S_m(u)\exp(G(u))\nabla T_k(u)_\mu\varpi_{k+\eta}\,dx\,dt=\epsilon(n,j,\mu).$$

By (4.51)-(4.57), (4.60) and (4.61) we obtain

$$\begin{aligned}
&\int_{\{|u_n|\le k\}\cap\{0\le T_k(u_n)-T_k(\upsilon_j)_\mu)|\le\eta\}} a_n(x,t,T_k(u_n),\nabla T_k(u_n))S_m(u_n)\exp(G(u_n))\\
&\times(\nabla T_k(u_n)-\nabla T_k(\upsilon_j)_\mu)\,dx\,dt\\
\le\;& C\eta+\epsilon(n,j,\mu,m),
\end{aligned}$$

we know that $\exp(G(u_n)) \geq 1$ and $S_m(u_n) = 1$ for $|u_n| \leq k$ then

$$\int_{\{|u_n|\leq k\}\cap\{0\leq T_k(u_n)-T_k(v_j)_\mu)|\leq\eta\}} a_n(x,t,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n) - \nabla T_k(v_j)_\mu)\,dx\,dt \leq C\eta + \epsilon(n,j,\mu,m). \tag{4.62}$$

Proof of (4.58):
Setting for $s > 0$, $Q^s = \{(x,t) \in Q : |\nabla T_k(u)| \leq s\}$ and $Q^s_j = \{(x,t) \in Q : |\nabla T_k(v_j)| \leq s\}$ and denoting by χ^s and χ^s_j the characteristic functions of Q^s and Q^s_j respectively, we deduce that letting $0 < \delta < 1$, define

$$\Theta_{n,k} = (a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)))(\nabla T_k(u_n) - \nabla T_k(u)).$$

For $s > 0$, we have

$$\begin{aligned} 0 &\leq \int_{Q^s} \Theta^\delta_{n,k}\,dx\,dt \\ &= \int_{Q^s} \Theta^\delta_{n,k}\chi_{|T_k(u_n)-T_k(v_j)_\mu|\leq\eta)}\,dx\,dt \\ &\quad + \int_{Q^s} \Theta^\delta_{n,k}\chi_{|T_k(u_n)-T_k(v_j)_\mu|>\eta)}\,dx\,dt. \end{aligned}$$

The first term of the right-side hand, with the Hölder inequality,

$$\begin{aligned} \int_{Q^s} \Theta^\delta_{n,k}\chi_{|T_k(u_n)-T_k(v_j)_\mu|\leq\eta)}\,dx\,dt &\leq (\int_{Q^s} \Theta_{n,k}\chi_{|T_k(u_n)-T_k(v_j)_\mu|\leq\eta)}\,dx\,dt)^\delta(\int_{Q^s} dx\,dt)^{1-\delta} \\ &\leq C_1(\int_{Q^s} \Theta_{n,k}\chi_{|T_k(u_n)-T_k(v_j)_\mu|\leq\eta)}\,dx\,dt)^\delta. \end{aligned}$$

Also using the Hölder inequality, the second term of the right-side hand is

$$\int_{Q^s} \Theta^\delta_{n,k}\chi_{|T_k(u_n)-T_k(v_j)_\mu|>\eta)}\,dx\,dt \leq (\int_{Q^s} \Theta_{n,k}\,dx\,dt)^\delta(\int_{|T_k(u_n)-T_k(v_j)_\mu|>\eta)} dx\,dt)^{1-\delta},$$

since $a(x,t,T_k(u_n),\nabla T_k(u_n))$ is bounded in $(L_\psi(Q_T))^N$, while $\nabla T_k(u_n)$ is bounded in $(L_\varphi(Q_T))^N$ then

$$\int_{Q^s} \Theta^\delta_{n,k}\chi_{|T_k(u_n)-T_k(v_j)_\mu|>\eta)}\,dx\,dt \leq C_2 meas\{(x,t) \in Q_T : |T_k(u_n) - T_k(v_j)_\mu| > \eta\}^{1-\delta}.$$

We obtain,

$$\begin{aligned} \int_{Q^s} \Theta^\delta_{n,k}\,dx\,dt &\leq C_1(\int_{Q^s} \Theta_{n,k}\chi_{|T_k(u_n)-T_k(v_j)_\mu|\leq\eta)}\,dx\,dt)^\delta \\ &\quad + C_2 meas\{(x,t) \in Q_T : |T_k(u_n) - T_k(v_j)_\mu| > \eta\}^{1-\delta}. \end{aligned}$$

On the other hand,

$$\int_{Q^s} \Theta_{n,k}\chi_{|T_k(u_n)-T_k(v_j)_\mu|\leq\eta)}\,dx\,dt$$

$$\leq \int_{|T_k(u_n)-T_k(\upsilon_j)_\mu|\leq\eta)} (a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)) \times(\nabla T_k(u_n) - \nabla T_k(u)\chi_s)\,dx\,dt.$$

For each $s > r, r > 0,$ one has

$$0 \leq \int_{Q^r\cap\{|T_k(u_n)-T_k(\upsilon_j)_\mu|\leq\eta)\}} (a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u))) \times(\nabla T_k(u_n) - \nabla T_k(u))\,dx\,dt$$

$$\leq \int_{Q^s\cap\{|T_k(u_n)-T_k(\upsilon_j)_\mu|\leq\eta)\}} (a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u))) \times(\nabla T_k(u_n) - \nabla T_k(u))\,dx\,dt$$

$$= \int_{Q^s\cap\{|T_k(u_n)-T_k(\upsilon_j)_\mu|\leq\eta)\}} (a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)) \times(\nabla T_k(u_n) - \nabla T_k(u)\chi_s)\,dx\,dt$$

$$\leq \int_{Q\cap\{|T_k(u_n)-T_k(\upsilon_j)_\mu|\leq\eta)\}} (a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi^s)) \times(\nabla T_k(u_n) - \nabla T_k(u)\chi^s)\,dx\,dt$$

$$= \int_{|T_k(u_n)-T_k(\upsilon_j)_\mu|\leq\eta} (a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(\upsilon_j)\chi_j^s)) \times(\nabla T_k(u_n) - \nabla T_k(\upsilon_j)\chi_j^s)\,dx\,dt$$

$$+ \int_{|T_k(u_n)-T_k(\upsilon_j)_\mu|\leq\eta} a(x,t,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(\upsilon_j)\chi_j^s - \nabla T_k(u)\chi^s)\,dx\,dt$$

$$+ \int_{|T_k(u_n)-T_k(\upsilon_j)_\mu|\leq\eta} (a(x,t,T_k(u_n),\nabla T_k(\upsilon_j)\chi_j^s) - a(x,t,T_k(u_n),\nabla T_k(u)\chi^s)) \nabla T_k(u_n)\,dx\,dt$$

$$- \int_{|T_k(u_n)-T_k(\upsilon_j)_\mu|\leq\eta} a(x,t,T_k(u_n),\nabla T_k(\upsilon_j)\chi_j^s)\nabla T_k(\upsilon_j)\chi_j^s\,dx\,dt$$

$$+\int_{|T_k(u_n)-T_k(\upsilon_j)_\mu|\leq\eta} a(x,t,T_k(u_n),\nabla T_k(u)\chi^s)\nabla T_k(u)\chi^s)\,dx\,dt$$
$$= I_1(n,j,s)+I_2(n,j)+I_3(n,j)+I_4(n,j,\mu)+I_5(n,\mu).$$

We go to the limit as n, j, μ, and $s\to+\infty$

$$I_1=\int_{|T_k(u_n)-T_k(\upsilon_j)_\mu|\leq\eta} a(x,t,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla T_k(\upsilon_j)_\mu)\,dx\,dt$$

$$-\int_{|T_k(u_n)-T_k(\upsilon_j)_\mu|\leq\eta} a(x,t,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(\upsilon_j)\chi_j^s-\nabla T_k(\upsilon_j)_\mu)\,dx\,dt$$

$$-\int_{|T_k(u_n)-T_k(\upsilon_j)_\mu|\leq\eta} a(x,t,T_k(u_n),\nabla T_k(\upsilon_j)\chi_j^s))(\nabla T_k(u_n)-\nabla T_k(\upsilon_j)\chi_j^s))\,dx\,dt.$$

Using (4.62), the first term of the right-hand side, we get

$$\int_{|T_k(u_n)-T_k(\upsilon_j)_\mu|\leq\eta} a(x,t,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla T_k(\upsilon_j)_\mu)\,dx\,dt$$

$$\leq C\eta+\epsilon(n,m,j,s)-\int_{|u|>k\cap|T_k(u)-T_k(\upsilon_j)_\mu|\leq\eta} a(x,t,T_k(u),0)\nabla T_k(\upsilon_j)_\mu\,dx\,dt$$
$$\leq C\eta+\epsilon(n,m,j,\mu).$$

The second term of the right-hand side tends to

$$\int_{|T_k(u)-T_k(\upsilon_j)_\mu|\leq\eta} \varpi_k(\nabla T_k(\upsilon_j)\chi_j^s-\nabla T_k(\upsilon_j)_\mu)\,dx\,dt,$$

since $a(x,t,T_k(u_n),\nabla T_k(u_n))$ is bounded in $(L_\psi(Q_T))^N$, there exist some $\varpi_k\in(L_\psi(Q_T))^N$ such that (for a subsequence still denoted by u_n

$$a(x,t,T_k(u_n),\nabla T_k(u_n))\to\varpi_k \quad \text{in} \quad (L_\varphi(Q_T))^N \quad \text{for} \quad \sigma(\Pi L_\psi,\Pi E_\varphi).$$

In view of the fact that

$$(\nabla T_k(\upsilon_j)\chi_j^s-\nabla T_k(\upsilon_j)_\mu)\chi_{|T_k(u_n)-T_k(\upsilon_j)_\mu|\leq\eta}$$
$$\to(\nabla T_k(\upsilon_j)\chi_j^s-\nabla T_k(\upsilon_j)_\mu)\chi_{|T_k(u)-T_k(\upsilon_j)_\mu|\leq\eta},$$

strongly in $(E_\varphi(Q_T))^N$ as $n\to+\infty$.
The third term of the right-hand side tends to

$$\int_{|T_k(u)-T_k(\upsilon_j)_\mu|\leq\eta} a(x,t,T_k(u),\nabla T_k(\upsilon_j)\chi_j^s))(\nabla T_k(u)-\nabla T_k(\upsilon_j)\chi_j^s))\,dx\,dt.$$

Since

$$a(x,t,T_k(u_n),\nabla T_k(\upsilon_j)\chi_j^s))\chi_{|T_k(u_n)-T_k(\upsilon_j)_\mu|\le\eta}$$
$$\to a(x,t,T_k(u),\nabla T_k(\upsilon_j)\chi_j^s))\chi_{|T_k(u)-T_k(\upsilon_j)_\mu|\le\eta},$$

in $(E_\psi(Q_T))^N$ while

$$(\nabla T_k(u_n)-\nabla T_k(\upsilon_j)\chi_j^s))\to(\nabla T_k(u)-\nabla T_k(\upsilon_j)\chi_j^s)),$$

in $(L_\varphi(Q_T))^N$ for $\sigma(\Pi L_\psi,\Pi E_\varphi)$.
Passing to limit as $j\to+\infty$ and $\mu\to+\infty$ and using Lebesgue's theorem, we have

$$I_1\le C\eta+\epsilon(n,j,s,\mu).$$

For what concerns I_2, by letting $n\to+\infty$, we have

$$I_2\to\int_{|T_k(u)-T_k(\upsilon_j)_\mu|\le\eta)}\varpi_k(\nabla T_k(\upsilon_j)\chi_j^s-\nabla T_k(u)\chi^s)\,dx\,dt.$$

Since $a(x,t,T_k(u_n),\nabla T_k(u_n))\rightharpoonup\varpi_k$ in $(L_\psi(Q_T))^N$, for $\sigma(\Pi L_\psi,\Pi E_\varphi)$, while

$$(\nabla T_k(\upsilon_j)\chi_j^s-\nabla T_k(u)\chi^s)\chi_{|T_k(u_n)-T_k(\upsilon_j)_\mu|\le\eta}$$
$$\to(\nabla T_k(\upsilon_j)\chi_j^s-\nabla T_k(u)\chi^s)\chi_{|T_k(u)-T_k(\upsilon_j)_\mu|\le\eta},$$

strongly in $(E_\varphi(Q_T))^N$.
Passing to limit $j\to+\infty$, and using Lebesgue's theorem, we have

$$I_2=\epsilon(n,j).$$

Similar ways as above give

$$I_3=\epsilon(n,j).$$

$$I_4=\int_{|T_k(u)-T_k(u)_\mu|\le\eta)}a(x,t,T_k(u),\nabla T_k(u))\nabla T_k(u)\,dx\,dt+\epsilon(n,j,\mu,s,m).$$

$$I_5=\int_{|T_k(u)-T_k(u)_\mu|\le\eta)}a(x,t,T_k(u),\nabla T_k(u))\nabla T_k(u)\,dx\,dt+\epsilon(n,j,\mu,s,m).$$

Finally, we obtain,

$$\int_{Q^s}\Theta_{n,k}\,dx\,dt\le C_1(C\eta+\epsilon(n,\mu,\eta,m))^\delta+C_2(\epsilon(n,\mu,))^{1-\delta}.$$

Which yields, by passing to the limit sup over n, j, μ , s and η

$$\int_{Q^r}\Big[(a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u)))(\nabla T_k(u_n)-\nabla T_k(u))\Big]^\delta\,dx\,dt=\epsilon(n).$$

Thus, passing to a subsequence if necessary, $\nabla u_n \to \nabla u$ a.e. in Q^r, and since r is arbitrary,

$$\nabla u_n \to \nabla u \ , \quad \text{a.e. in} \quad Q_T.$$

Step 5: Equi-integrability of the nonlinearity sequence

We shall prove that $H_n(x,t,u_n,\nabla u_n) \to H(x,t,u,\nabla u)$ strongly in $L^1(\Omega)$.

Consider $g_0(u_n) = \displaystyle\int_0^{u_n} \rho(s)\chi_{\{s>h\}}ds$ and multiply (4.6) by $\exp(G(u_n))g_0(u_n)$, we get

$$\begin{aligned}\int_\Omega \widetilde{T}_h(u_n)(T)dx &+ \int_{Q_T} a(x,u_n,\nabla u_n)\nabla(\exp(G(u_n))g_0(u_n))dxdt \\ &+ \int_{Q_T} \Phi_n(x,u_n,\nabla u_n)\nabla(\exp(G(u_n))g_0(u_n))dxdt \\ &+ \int_{Q_T} H_n(x,t,u_n,\nabla u_n)\exp(G(u_n))g_0(u_n))dxdt \\ \le (\int_h^{+\infty} \rho(s)dx)\exp\Big(\frac{\|\rho\|_{L^1(\mathbb{R})}}{\alpha'}\Big)&\Big[\|f\|_{L^1(Q_T)} + \|u_0\|_{L^1(\Omega)} + \|h(.,.)\|_{L^1(Q_T)}\Big].\end{aligned}$$

where $\widetilde{T}_h(r) = \displaystyle\int_0^r g_0(s)\exp(G(s))ds \ge 0$,

then using same technique in step 2 we can have

$$\int_{\{u_n>h\}} \rho(u_n)\varphi(x,\nabla u_n)dxdt \le C(\int_h^{+\infty} \rho(s)dx).$$

Since $\rho \in L^1(\mathbb{R})$, we get

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n>h\}} \rho(u_n)\varphi(x,\nabla u_n)dxdt = 0.$$

Similarly, let $g_0(u_n) = \displaystyle\int_{u_n}^0 \rho(s)\chi_{\{s<-h\}}dx$ in (4.6), we have also

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n<-h\}} \rho(u_n)\varphi(x,\nabla u_n)dxdt = 0.$$

We conclude that

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{|u_n|>h\}} \rho(u_n)\varphi(x,\nabla u_n)dxdt = 0. \tag{4.63}$$

Let $D \subset \Omega$ then

$$\begin{aligned}\int_D \rho(u_n)\varphi(x,\nabla u_n)dxdt &\le \max_{\{|u_n|\le h\}}(\rho(x))\int_{D\cap\{|u_n|\le h\}} \varphi(x,\nabla u_n)dxdt \\ &\quad + \int_{D\cap\{|u_n|>h\}} \rho(u_n)\varphi(x,\nabla u_n)dxdt.\end{aligned}$$

Consequently $\rho(u_n)\varphi(x,\nabla u_n)$ is equi-integrable. Then $\rho(u_n)\varphi(x,\nabla u_n)$ converge to $\rho(u)\varphi(x,\nabla u)$ strongly in $L^1(\mathbb{R})$. By (3.5), we get our result.

Step 6: We show that u satisfies (3.8)

Firstly show that $u \geq \zeta$ a.e. in Q_T.

(4.20) and (4.31) we get

In fact, from (4.20) and (4.31) we get

$$0 \leq \int_{Q_T} T_n(u_n-\zeta)^- dxdt \leq \frac{c_1}{n}.$$

Let n tends to $+\infty$ we obtain

$$\int_{Q_T} (u-\zeta)^- dxdt = 0,$$

then $(u-\zeta)^- = 0$ a.e. in Q_T; thus $u \geq \zeta$ a.e. in Q_T.

Secondly passing Now to the limit in (4.65) to show that u satisfies the equation (3.8).

Let $v \in W_0^1 L_\varphi(Q_T) \cap L^\infty(Q_T)$ such that $\frac{\partial v}{\partial t} \in W^{-1,x}L_\psi(Q_T) + L^1(Q_T)$, then by theorem (2.14) we can take

$$\begin{gathered} \overline{v} = v \quad \text{on} \quad Q_T, \\ \overline{v} \in W^{1,x}L_\varphi(\Omega\times\mathbb{R}) \cap L^1(\Omega\times\mathbb{R}) \cap L^\infty(\Omega\times\mathbb{R}), \\ \frac{\partial \overline{v}}{\partial t} \in W^{-1,x}L_\psi(Q_T) + L^1(Q_T), \end{gathered}$$

and there exists $v_j \in \mathcal{D}(\Omega\times\mathbb{R})$ such that

$$v_j \to \overline{v} \quad \text{in} \quad W_0^{1,x}L_\varphi(\Omega\times\mathbb{R}) \quad \text{and} \quad \frac{\partial v_j}{\partial t} \to \frac{\partial \overline{v}}{\partial t} \in W^{-1,x}L_\psi(Q_T) + L^1(Q_T), \tag{4.64}$$

for the modular convergence in $W_0^1 L_\varphi(Q_T)$, with

$$\|v_j\|_{L^\infty(Q_T)} \leq (N+2)\|v\|_{L^\infty(Q_T)}.$$

Pointwise multiplication of the approximate equation ((4.6)) by $T_k(u_n - v_j)$, we get

$$\left\{ \begin{aligned} &\int_0^T < \frac{\partial u_n}{\partial s}, T_k(u_n-v_j) > ds + \int_{Q_T} a_n(x,s,u_n,\nabla u_n))\nabla T_k(u_n-v_j)dxds \\ &+ \int_{Q_T} \Phi_n(x,s,u_n)\nabla T_k(u_n-v_j)dxds + \int_Q T_n(u_n-\zeta)^- sg_{\frac{1}{n}}(u_n)T_k(u_n-v_j)dxds \\ &+ \int_{Q_T} H_n(x,s,u_n,\nabla u_n)\nabla T_k(u_n-v_j)dxds = \int_{Q_T} f_n T_k(u_n-v_j)dxds. \end{aligned} \right. \tag{4.65}$$

We pass to the limit as in (4.65), n tend to $+\infty$ and j tend to $+\infty$.

Limit of the first term of (4.65):

The first term can be written

$$\begin{aligned}\int_0^T < \frac{\partial u_n}{\partial s}, T_k(u_n - v_j) > ds &= \int_0^T < \frac{\partial (u_n - v_j)}{\partial s}, T_k(u_n - v_j) > ds \\ &\quad + \int_0^T < \frac{\partial v_j}{\partial s}, T_k(u_n - v_j) > ds \\ &= S_k(u_n(T) - v_j(T)) - S_k(u_n(0) - v_j(0)) \\ &\quad + \int_0^T < \frac{\partial v_j}{\partial s}, T_k(u_n - v_j) > ds.\end{aligned}$$

We pass to the limit as $n \to +\infty$ and $j \to +\infty$ we can easily deduce

$$\begin{aligned}\int_0^T < \frac{\partial u_n}{\partial s}, T_k(u_n - v_j) > ds &\to \int_\Omega S_k(u_n(T) - v(T))dx - \int_\Omega S_k(u_n(0) - v(0))dx \\ &\quad + \int_0^T < \frac{\partial v}{\partial s}, T_k(u_n - v) > ds.\end{aligned}$$

- We can follow same way in [8] to prove that

$$\begin{aligned}&\liminf_{j\to\infty} \liminf_{n\to\infty} \int_{Q_T} a(x, s, u_n, \nabla u_n)\nabla T_k(u_n - v_j)dxds \\ \geq &\int_{Q_T} a(x, s, u, \nabla u)\nabla T_k(u - v)dxds.\end{aligned}$$

- For $n \geq k + (N+2)\|v\|_{L^\infty(Q_T)}$

$$\Phi_n(x, s, u_n)\nabla T_k(u_n - v_j) = \Phi(x, s, T_{k+(N+2)\|v\|_{L^\infty(Q_T)}}(u_n))\nabla T_k(u_n - v_j).$$

The pointwise convergence of u_n to u as n tends to $+\infty$ and (3.4), then

$$\begin{aligned}&\Phi(x, s, T_{k+(N+2)\|v\|_{L^\infty(Q_T)}}(u_n))\nabla T_k(u_n - v_j \rightharpoonup \\ &\Phi(x, s, T_{k+(N+2)\|v\|_{L^\infty(Q_T)}}(u))\nabla T_k(u - v_j),\end{aligned}$$

weakly for $\sigma(\Pi L_v, \Pi L_\psi)$.
In a similar way, we obtain

$$\begin{aligned}&\lim_{j\to\infty} \int_{Q_T} \Phi(x, s, T_{k+(N+2)\|v\|_{L^\infty(Q_T)}}(u))\nabla T_k(u - v_j)dxds \\ &= \int_{Q_T} \Phi(x, s, T_{k+(N+2)\|v\|_{L^\infty(Q_T)}}(u))\nabla T_k(u - v)dxds \\ &= \int_{Q_T} \Phi(x, s, u)\nabla T_k(u - v)dxds.\end{aligned}$$

- Limit of $H_n(x,s,u_n,\nabla u_n)T_k(u_n-v_j)$:
Since $H_n(x,s,u_n,\nabla u_n)$ converge strongly to $H(x,t,u,\nabla u)$ in $L^1(Q_T)$ and and the pointwise convergence of u_n to u as $n\to+\infty$, it is possible to prove that $H_n(x,s,u_n,\nabla u_n)T_k(u_n-v_j)$ converge to $H(x,s,u,\nabla u)T_k(u-v_j)$ in $L^1(Q_T)$ and

$$\lim_{j\to\infty}\int_{Q_T} H(x,s,u,\nabla u)T_k(u-v_j)dxds = \int_{Q_T} H(x,s,u,\nabla u)T_k(u-v)dxds.$$

- Since f_n converge strongly to f in $L^1(Q_T)$, and $T_k(u_n-v_j)\to T_k(u-v_j)$ weakly* in $L^\infty(Q_T)$, we have $\int_{Q_T} f_nT_k(u_n-v_j)dxds \to \int_{Q_T} fT_k(u-v_j)dxds$ as $n\to\infty$ and also we have $\int_{Q_T} fT_k(u-v_j)dxds \to \int_{Q_T} fT_k(u-v)dxds$ as $j\to\infty$.
Finally we know that $\int_Q T_n(u_n-\zeta)^- sg_{\frac{1}{n}}(u_n)T_k(u_n-v_j)dxds\geq 0$, thus

$$\begin{cases} \int_\Omega S_k(u(T)-v(T))dx + \int_0^T < \frac{\partial v}{\partial s}, T_k(u-v) > ds \\ + \int_{Q_T} a(x,s,u,\nabla u)\nabla T_k(u-v)dxds + \int_{Q_T} \Phi(x,s,u)\nabla T_k(u-v)dxds \\ + \int_{Q_T} H(x,s,u,\nabla u)T_k(u-v)dxds \\ \leq \int_{Q_T} fT_k(u-v)dxds - \int_\Omega S_k(u_0-v(x,0))dx. \end{cases}$$

As a conclusion, the proof of Theorem (4.1) is complete.

□

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