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A New Generalization of Confluent Hypergeometric Function and Whittaker Function

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abstract: In this article, we introduce a further generalizations of the confluent hypergeometric function and Whittaker function by introducing an extra parameter in the extended confluent hypergeometric function defined by Parmar [15]. We also investigate some integral representations, some integral transforms, differential formulas and recurrence relations of these new generalizations.

Key Words: Whittaker function, Beta function, Extended beta function, Extended confluent hypergeometric function, Gauss hypergeometric function, Extended Gauss hypergeometric function.

Contents

1. Introduction

The Whittaker function $M_{k,\mu}(z)$ in terms of confluent hypergeometric function (or Kummer's function) of first kind (see [5], [6], also see [8]) is defined as

$$M_{k,\mu}(z) = z^{\mu+\frac{1}{2}}\ \exp\left(-\frac{z}{2}\right)\ \Phi(\mu-k+\frac{1}{2};\ 2\mu+1;z), \tag{1}$$

$$\left(\Re(\mu) > -\frac{1}{2} \text{ and } \Re(\mu\pm k) > -\frac{1}{2}\right).$$

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Very recently, Parmar [15] introduced and investigated some fundamental properties and characteristics of more generalized beta type function $B_\sigma^{(\alpha,\beta;m)}$ defined by

$$B_\sigma^{(\alpha,\beta;m)}(x,y) = \int_0^1 t^{x-1}\ (1-t)^{y-1}\ {}_1F_1\left(\alpha;\beta;\ -\frac{\sigma}{t^m(1-t)^m}\right) dt \tag{2}$$

$$(\Re(\sigma)>0,\ \Re(x)>0,\ \Re(y)>0,\ \Re(\alpha)>0,\ \Re(\beta)>0,\ \Re(m)>0).$$

When $m=1$, (2) reduces to the well-known generalized beta type function defined by Özergin et al. [7]:

$$B_\sigma^{(\alpha,\beta)}(x,y) = \int_0^1 t^{x-1}\ (1-t)^{y-1}\ {}_1F_1\left(\alpha;\beta;\ -\frac{\sigma}{t(1-t)}\right) dt \tag{3}$$

$$(\Re(\sigma)>0,\ \Re(x)>0,\ \Re(y)>0,\ \Re(\alpha)>0,\ \Re(\beta)>0).$$

For $\alpha=\beta$, (3) reduces to

$$B_\sigma(x,y) = B_\sigma^{(\alpha,\alpha)}(x,y) = \int_0^1 t^{x-1}\ (1-t)^{y-1}\ \exp\left(-\frac{\sigma}{t(1-t)}\right) dt\ (\Re(\sigma)>0), \tag{4}$$

which was introduced by Chaudhry et al. [12] in 1997. Clearly, classical beta function $B(x,y)$ is given by

$$B(x,y) = B_0(x,y) = B_0^{(\alpha,\beta)}(x,y).$$

Using (4), Chaudhry et al. [13] extended the Gauss hypergeometric function and confluent hypergeometric function as follows:

$$F_\sigma(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n\ B_\sigma(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!} \tag{5}$$

$$(\sigma \geq 0;\ |z|<1;\ \Re(c)>\Re(b)>0),$$

$$\Phi_\sigma(b;c;z) = \sum_{n=0}^{\infty} \frac{B_\sigma(b+n,c-b)}{B(b,c-b)}\ \frac{z^n}{n!} \tag{6}$$

$$(\sigma \geq 0;\ \Re(c)>\Re(b)>0)$$

and gave their Euler's type integral representation

$$F_\sigma(a,b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}\ (1-t)^{c-b-1}\ (1-zt)^{-a}\ \exp\left[-\frac{\sigma}{t(1-t)}\right] dt \tag{7}$$

$$(\sigma>0;\ \sigma=0 \text{ and } |\arg(1-z)|<\pi;\ \Re(c)>\Re(b)>0),$$

$$\Phi_\sigma(b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}\ (1-t)^{c-b-1}\ \exp\left[zt-\frac{\sigma}{t(1-t)}\right] dt \tag{8}$$

$$(\sigma > 0;\ \sigma = 0 \text{ and } \Re(c) > \Re(b) > 0).$$

By appealing $B_\sigma^{(\alpha,\beta)}(x,y)$, Özergin et al. [7] further extended Gauss hypergeometric function and confluent hypergeometric function by

$$F_\sigma^{(\alpha,\beta)}(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n\ B_\sigma^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!} \tag{9}$$

$$(\sigma \geq 0;\ |z| < 1;\ \Re(c) > \Re(b) > 0,\ \Re(\alpha) > 0,\ \Re(\beta) > 0),$$

$$\Phi_\sigma^{(\alpha,\beta)}(b;c;z) = \sum_{n=0}^{\infty} \frac{B_\sigma^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)}\ \frac{z^n}{n!} \tag{10}$$

$$(\sigma \geq 0;\ \Re(c) > \Re(b) > 0,\ \Re(\alpha) > 0,\ \Re(\beta) > 0)$$

and gave their Euler's type integral representation

$$F_\sigma^{(\alpha,\beta)}(a,b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}\ (1-t)^{c-b-1}\ (1-zt)^{-a} \times {}_1F_1\left(\alpha;\ \beta;\ -\frac{\sigma}{t(1-t)}\right) dt \tag{11}$$

$$(\sigma > 0;\ \sigma = 0 \text{ and } |\arg(1-z)| < \pi;\ \Re(c) > \Re(b) > 0,\ \Re(\alpha) > 0,\ \Re(\beta) > 0),$$

$$\Phi_\sigma^{(\alpha,\beta)}(b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}\ (1-t)^{c-b-1}\ e^{zt} \times {}_1F_1\left(\alpha;\ \beta;\ -\frac{\sigma}{t(1-t)}\right) dt \tag{12}$$

$$(\sigma > 0;\ \sigma = 0;\ \Re(c) > \Re(b) > 0,\ \Re(\alpha) > 0,\ \Re(\beta) > 0).$$

By using $B_\sigma^{(\alpha,\beta;m)}(x,y)$, Parmar [15] defined a new generalization of extended Gauss hypergeometric function and confluent hypergeometric function as follows:

$$F_\sigma^{(\alpha,\beta;m)}(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n\ B_\sigma^{(\alpha,\beta;m)}(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!} \tag{13}$$

$$(\sigma \geq 0;\ |z| < 1;\ \Re(c) > \Re(b) > 0,\ \Re(\alpha) > 0,\ \Re(\beta) > 0,\ \Re(m) > 0),$$

$$\Phi_\sigma^{(\alpha,\beta;m)}(b;c;z) = \sum_{n=0}^{\infty} \frac{B_\sigma^{(\alpha,\beta;m)}(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!} \tag{14}$$

$$(\sigma \geq 0;\ \Re(c) > \Re(b) > 0,\ \Re(\alpha) > 0,\ \Re(\beta) > 0,\ \Re(m) > 0),$$

and gave their Euler's type integral representation

$$F_{\sigma}^{(\alpha,\beta;m)}(a,b;c;z)=\frac{1}{B(b,c-b)}\int_{0}^{1}t^{b-1}\,(1-t)^{c-b-1}\,(1-zt)^{-a}$$
$$\times {}_1F_1\left(\alpha;\ \beta;\ -\frac{\sigma}{t^{m}(1-t)^{m}}\right)dt \tag{15}$$
$$(\sigma>0;\ \sigma=0 \text{ and } |\arg(1-z)|<\pi;\ \Re(c)>\Re(b)>0,\ \Re(m)>0),$$

$$\Phi_{\sigma}^{(\alpha,\beta;m)}(b;c;z)=\frac{1}{B(b,c-b)}\int_{0}^{1}t^{b-1}\,(1-t)^{c-b-1}\,e^{zt}$$
$$\times {}_1F_1\left(\alpha;\ \beta;\ -\frac{\sigma}{t^{m}(1-t)^{m}}\right)dt \tag{16}$$
$$(\sigma>0;\ \sigma=0;\ \Re(c)>\Re(b)>0,\ \Re(m)>0).$$

On substituting $t=1-u$ in (16), Parmar [15] obtained the following new extension of Kummer's relation for the generalized extended confluent hypergeometric function of the first kind:

$$\Phi_{\sigma}^{(\alpha,\beta;m)}(b;c;z)=\exp(z)\ \Phi_{\sigma}^{(\alpha,\beta;m)}(c-b;c;-z). \tag{17}$$

For $\sigma=0$, (17) reduces to the Kummer's first formula for the classical confluent hypergeometric function [4].

Afterwards, Srivastava et al. [9] introduced a new generalized Gauss hypergeometric functions as follows:

$$F_{\sigma}^{(\alpha,\beta;m,n)}(a,b;c;z)=\sum_{k=0}^{\infty}(a)_k\frac{B_{\sigma}^{(\alpha,\beta;m,n)}(b+k,c-b)}{B(b,c-b)}\frac{z^k}{k!} \tag{18}$$

$$(|z|<1;\ \min\{\Re(\alpha),\Re(\beta),\Re(\mathrm{m}),\Re(\mathrm{n})\}>0;\ \Re(\mathrm{c})>\Re(\mathrm{b})>0;\ \Re(\sigma)\geq 0),$$

where the generalized beta function $B_{\sigma}^{(\alpha,\beta;m,n)}(x,y)$ is defined by

$$B_{\sigma}^{(\alpha,\beta;m,n)}(x,y)=\int_{0}^{1}t^{x-1}\,(1-t)^{y-1}\,{}_1F_1\left(\alpha;\ \beta;\ -\frac{\sigma}{t^{m}(1-t)^{n}}\right)dt \tag{19}$$

$$(\Re(\sigma)\geq 0;\ \min\{\Re(\alpha),\Re(\beta),\Re(\mathrm{x}),\Re(\mathrm{y})\}>0;\ \min\{\Re(\mathrm{m}),\Re(\mathrm{n})\}>0).$$

On substituting $m=n$ in (18) and (19), respectively, we get the extended Gauss hypergeometric function and extended beta function defined by Parmar [15].

2. Generalized extended confluent hypergeometric function

In this section, we give the definition of the generalized extended confluent hypergeometric function. An integral representation and a Kummer type relation of this function are also indicated.

Definition 2.1. The generalized extended confluent hypergeometric function (for $|z| < 1; \min\{\Re(\alpha), \Re(\beta), \Re(\mathrm{m}), \Re(\mathrm{n})\} > 0;\ \Re(\mathrm{c}) > \Re(\mathrm{b}) > 0;\ \Re(\sigma) \geq 0$) is denoted by $\Phi_\sigma^{(\alpha,\beta;m,n)}(b;c;z)$ and is defined as follows:

$$\Phi_\sigma^{(\alpha,\beta;m,n)}(b;c;z) = \sum_{k=0}^{\infty} \frac{B_\sigma^{(\alpha,\beta;m,n)}(b+k, c-b)}{B(b, c-b)} \frac{z^k}{k!} \tag{20}$$

Remark 2.2. On setting $m = n$, (20) reduces to the generalized extended confluent hypergeometric function defined by Parmar [15], which further for $n = 1$ gives the known extension of the confluent hypergeometric function given by Özergin et al. [7]. Further, if we set $\alpha = \beta$ and $m = n$ in (20) then we get the generalized confluent hypergeometric function defined by Lee et al. [3] and if we put $\alpha = \beta$ and $m = n = 1$ in (20) then we obtain the extended confluent hypergeometric function defined by Chaudhry et al. [13].

Integral representation: The integral representation of the generalized extended confluent hypergeometric function can be obtained by using the definition of generalized beta function defined by (19).

Theorem 2.3. For the generalized extended confluent hypergeometric function, we have the following integral representation:

$$\Phi_\sigma^{(\alpha,\beta;m,n)}(b;c;z) = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1}\ (1-t)^{c-b-1}\ e^{zt}$$

$$\times {}_1F_1\left(\alpha;\ \beta;\ -\frac{\sigma}{t^m(1-t)^n}\right) dt \tag{21}$$

$$(\sigma > 0;\ \sigma = 0;\ \Re(c) > \Re(b) > 0,\ \Re(m) > 0, \Re(n) > 0).$$

Proof. We have

$$\Phi_\sigma^{(\alpha,\beta;m,n)}(b;c;z) = \sum_{k=0}^{\infty} \frac{B_\sigma^{(\alpha,\beta;m,n)}(b+k, c-b)}{B(b, c-b)} \frac{z^k}{k!}$$

On using the integral representation of $B_\sigma^{(\alpha,\beta;m,n)}$, we arrive at

$$\Phi_\sigma^{(\alpha,\beta;m,n)}(b;c;z) = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1}\ (1-t)^{c-b-1}$$

$$\times {}_1F_1\left(\alpha;\ \beta;\ -\frac{\sigma}{t^m(1-t)^n}\right)\sum_{k=0}^{\infty}\frac{(zt)^k}{k!}\,dt$$

$$=\frac{1}{B(b,c-b)}\int_0^1 t^{b-1}\,(1-t)^{c-b-1}\,e^{zt}\,{}_1F_1\left(\alpha;\ \beta;\ -\frac{\sigma}{t^m(1-t)^n}\right)dt$$

This completes the proof.

On substituting $t=1-u$ in (21), we obtain the following new extension of Kummer's relation for the generalized extended confluent hypergeometric function of the first kind:

$$\Phi_\sigma^{(\alpha,\beta;m,n)}(b;c;z)=\exp(z)\ \Phi_\sigma^{(\alpha,\beta;m,n)}(c-b;c;-z). \tag{22}$$

For $\sigma=0$, equation (22) reduces to the Kummer's first formula for the classical confluent hypergeometric function [4].

Remark 2.4. On taking $m=n$, (21) reduces to the integral representation of the generalized extended confluent hypergeometric function defined by Parmar [15], which further for $n=1$ gives the integral representation of the extended confluent hypergeometric function given by Özergin et al. [7]. Further, if we put $\alpha=\beta$ and $m=n$ in (21) then we get the integral representation of generalized confluent hypergeometric function defined by Lee et al. [3] and if we set $\alpha=\beta$ and $m=n=1$ in (21) then we obtain the integral representation of extended confluent hypergeometric function defined by Chaudhry et al. [13].

3. Derivative of $\Phi_\sigma^{(\alpha,\beta;m,n)}(b;c;z)$

The derivative of generalized extended confluent hypergeometric function $\Phi_\sigma^{(\alpha,\beta;m,n)}(b;c;z)$ with respect to the variable z in terms of a shift operator is obtained by using the following formulas:

$$B(b,c-b)=\frac{b}{c}B(b+1,c-b)\ \text{and}\ (\mathrm{a})_{\mathrm{n}+1}=\mathrm{a}(\mathrm{a}+1)_{\mathrm{n}}. \tag{23}$$

Theorem 3.1. For the generalized extended confluent hypergeometric function $\Phi_\sigma^{(\alpha,\beta;m,n)}(b;c;z)$, the following differentiation formula holds true:

$$\frac{d^r}{dz^r}\left[\Phi_\sigma^{(\alpha,\beta;m,n)}(b;c;z)\right]=\frac{(b)_r}{(c)_r}\ \Phi_\sigma^{(\alpha,\beta;m,n)}(b+r;c+r;z). \tag{24}$$

Proof. Taking the derivative of $\Phi_\sigma^{(\alpha,\beta;m,n)}(b;c;z)$ with respect to z, we obtain

$$\frac{d}{dz}\left[\Phi_\sigma^{(\alpha,\beta;m,n)}(b;c;z)\right]=\frac{d}{dz}\left[\sum_{r=0}^{\infty}\frac{B_\sigma^{(\alpha,\beta;m,n)}(b+r,c-b)}{B(b,c-b)}\frac{z^r}{r!}\right]$$

$$= \sum_{r=1}^{\infty} \frac{B_{\sigma}^{(\alpha,\beta;m,n)}(b+r, c-b)}{B(b, c-b)} \frac{z^{r-1}}{(r-1)!}.$$

Replacing r by $r+1$, we get

$$\frac{d}{dz}\left[\Phi_{\sigma}^{(\alpha,\beta;m,n)}(b;c;z)\right] = \frac{b}{c}\sum_{r=0}^{\infty} \frac{B_{\sigma}^{(\alpha,\beta;m,n)}(b+r+1, c-b)}{B(b+1, c-b)} \frac{z^{r}}{r!}$$
$$= \frac{b}{c}\left[\Phi_{\sigma}^{(\alpha,\beta;m,n)}(b+1;c+1;z)\right].$$

In a similar procedure, by induction, we can obtain the desired result.

4. Mellin transforms and transformation Formula

Certain interesting Mellin transforms and a transformation formula for the generalized extended confluent hypergeometric function are given in the following theorems:

Theorem 4.1. For the generalized extended confluent hypergeometric function, we have the following Mellin transform representation:

$$\int_0^{\infty} \sigma^{s-1}\ \Phi_{\sigma}^{(\alpha,\beta;m,n)}(b;c;z)d\sigma = \frac{\Gamma^{(\alpha,\beta)}(s)B(b+ms, c-b+ns)}{B(b,c-b)}$$
$$\times {}_1F_1(b+ms; c+(m+n)s; z). \tag{25}$$

Proof. To obtain Mellin transform, multiply both sides of (21) by σ^{s-1}, and integrating with respect to σ over the interval $[0,\infty)$, and changing the order of integral, we get

$$\int_0^{\infty} \sigma^{s-1}\ \Phi_{\sigma}^{(\alpha,\beta;m,n)}(b;c;z)d\sigma = \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}\ (1-t)^{c-b-1}\ e^{zt}$$
$$\times \left[\int_0^{\infty} \sigma^{s-1}{}_1F_1\left(\alpha;\ \beta;\ -\frac{\sigma}{t^m(1-t)^n}\right)d\sigma\right]dt.$$

Substitution of $u = \frac{\sigma}{t^m(1-t)^n}$ in the integral then leads to

$$\int_0^{\infty} \sigma^{s-1}{}_1F_1\left(\alpha;\ \beta;\ -\frac{\sigma}{t^m(1-t)^n}\right)d\sigma = \int_0^{\infty} u^{s-1}\ t^{ms}(1-t)^{ns}\ {}_1F_1(\alpha;\beta;-u)du$$
$$= t^{ms}(1-t)^{ns}\int_0^{\infty} u^{s-1}\ {}_1F_1(\alpha;\beta;-u)du$$
$$= t^{ms}(1-t)^{ns}\Gamma^{(\alpha,\beta)}(s),$$

where $\Gamma^{(\alpha,\beta)}(s)$ is the generalized Gamma function [7].

Thus we have

$$= \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}\ (1-t)^{c-b-1}\ e^{zt}\ \left[\int_0^\infty u^{s-1}{}_1F_1\,(\alpha;\ \beta;\ -u)\,du\right] dt$$

$$= \frac{\Gamma^{(\alpha,\beta)}(s)}{B(b,c-b)} \int_0^1 t^{b+ms-1}\ (1-t)^{c-b+ns-1}\ e^{zt} dt$$

$$= \frac{\Gamma^{(\alpha,\beta)}(s) B(b+ms, c-b+ns)}{B(b,c-b)}\ {}_1F_1[b+ms; c+(m+n)s; z].$$

This completes the proof.

Corollary 4.2. By the Mellin inversion formula, we have the following complex integral representation for $\Phi_\sigma^{(\alpha,\beta;m,n)}(b;c;z)$:

$$\Phi_\sigma^{(\alpha,\beta;m,n)}(b;c;z) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{\Gamma^{(\alpha,\beta)}(s) B(b+ms, c-b+ns)}{B(b,c-b)}$$

$$\times {}_1F_1(b+ms; c+(m+n)s; z)\sigma^{-s} ds. \tag{26}$$

Proof. Taking Mellin inversion of Theorem (4.1), we get the required result.

Theorem 4.3. For the generalized extended confluent hypergeometric function, we have the following transformation formula:

$$\Phi_\sigma^{(\alpha,\beta;m,n)}(b;c;z) = \exp(\mathrm{z})\Phi_\sigma^{(\alpha,\beta;\mathrm{m},\mathrm{n})}(\mathrm{c}-\mathrm{b};\mathrm{c};-\mathrm{z}). \tag{27}$$

Proof. Using the definition of the generalized extended confluent hypergeometric function, we have

$$\Phi_\sigma^{(\alpha,\beta;m,n)}(b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}\ (1-t)^{c-b-1}\ e^{zt}$$

$$\times {}_1F_1\left(\alpha;\ \beta;\ -\frac{\sigma}{t^m(1-t)^n}\right) dt$$

Replacing t by $t-1$, we get the result.

Remark 4.4. For $m = n$, (27) reduces to the extended Kummar's first formula defined by Parmar [15]. Clearly, for $\sigma = 0$, (27) reduces to the well known Kummer's first formula for the classical confluent hypergeometric function ${}_1F_1$.

5. Generalized Extended Whittaker Function

In this section, we give the definition of the generalized extended Whittaker function in terms of generalized extended confluent hypergeometric function. Some integral representations and a relation of this function are also derived.

Definition 5.1. The generalized extended Whittaker function for $\sigma \geq 0$ and $m \geq 1$, $n \geq 1$, $\Re(\alpha) > 0$ and $\Re(\beta) > 0$ is denoted by $M_{\sigma,k,\mu}^{(\alpha,\beta;m,n)}(z)$ and is defined as

$$M_{\sigma,k,\mu}^{(\alpha,\beta;m,n)}(z) = z^{\mu+\frac{1}{2}} \exp\left(-\frac{z}{2}\right) \Phi_{\sigma}^{(\alpha,\beta;m,n)}(\mu - k + \frac{1}{2};\ 2\mu + 1; z), \tag{28}$$

where $\Re(\mu) > -\frac{1}{2}$, $\Re(\mu \pm k) > -\frac{1}{2}$ and $\Phi_{\sigma}^{(\alpha,\beta;m,n)}$ is the generalized extended confluent hypergeometric function of the first kind defined by (21).

On setting $m = n = 1$ in (28), we obtain the following (presumably) a new representation of the extended Whittaker function:

$$M_{\sigma,k,\mu}^{(\alpha,\beta;1,1)}(z) = M_{\sigma,k,\mu}^{(\alpha,\beta)}(z) = z^{\mu+\frac{1}{2}} \exp\left(-\frac{z}{2}\right) \Phi_{\sigma}^{(\alpha,\beta)}(\mu - k + \frac{1}{2};\ 2\mu + 1; z), \tag{29}$$

where $\Re(\mu) > -\frac{1}{2}$, $\Re(\mu \pm k) > -\frac{1}{2}$.

Remark 5.2. On setting $m = n$ in (28), we obtain the generalized extended Whittaker function defined by Choi et al. [10]. Further, on setting $\alpha = \beta$, $m = n$ in (28), we get the extended Whittaker function given by Khan and Ghayasuddin [14], which further for $n = 1$ gives the extended Whittaker function due to Nagar et al. [2]. For $\sigma = 0$, (28) reduces to the classical Whittaker function defined by (1).

Integral representations: Certain interesting integral representations of the generalized extended Whittaker function $M_{\sigma,k,\mu}^{(\alpha,\beta;m,n)}(z)$ are given in the following theorem:

Theorem 5.3. Suppose that

$$\sigma > 0;\ \sigma = 0 \text{ and } \Re(\mu) > \Re(\mu \pm k) > -\frac{1}{2},\ \Re(m) > 0, \Re(n) > 0.$$

Each of the following integral formulas holds true:

$$M_{\sigma,k,\mu}^{(\alpha,\beta;m,n)}(z) = \frac{z^{\mu+\frac{1}{2}} \exp(-\frac{z}{2})}{B(\mu - k + \frac{1}{2}, \mu + k + \frac{1}{2})}$$

$$\times \int_0^1 t^{\mu-k-\frac{1}{2}} (1-t)^{\mu+k-\frac{1}{2}} e^{zt}\ {}_1F_1\left(\alpha;\ \beta;\ -\frac{\sigma}{t^m(1-t)^n}\right) dt; \tag{30}$$

$$M_{\sigma,k,\mu}^{(\alpha,\beta;m,n)}(z) = \frac{z^{\mu+\frac{1}{2}}\ \exp(\frac{z}{2})}{B(\mu-k+\frac{1}{2},\mu+k+\frac{1}{2})}$$

$$\times \int_0^1 u^{\mu+k-\frac{1}{2}}\ (1-u)^{\mu-k-\frac{1}{2}}\ e^{-zu}\ {}_1F_1\left(\alpha;\ \beta;\ -\frac{\sigma}{u^n(1-u)^m}\right) dt; \tag{31}$$

$$M_{\sigma,k,\mu}^{(\alpha,\beta;m,n)}(z) = \frac{(q-p)^{-2\mu}\ z^{\mu+\frac{1}{2}}\ \exp(-\frac{z}{2})}{B(\mu-k+\frac{1}{2},\mu+k+\frac{1}{2})} \times \int_p^q (u-p)^{\mu-k-\frac{1}{2}}\ (q-u)^{\mu+k-\frac{1}{2}}$$

$$\times \exp\left[\frac{z(u-p)}{(q-p)}\right]\ {}_1F_1\left(\alpha;\ \beta;\ -\frac{\sigma\ (q-p)^{m+n}}{(u-p)^m(q-u)^n}\right) du; \tag{32}$$

$$M_{\sigma,k,\mu}^{(\alpha,\beta;m,n)}(z) = \frac{\exp\left(-\frac{z}{2}\right)\ z^{\mu+\frac{1}{2}}}{B(\mu-k+\frac{1}{2},\mu+k+\frac{1}{2})} \int_0^\infty u^{\mu-k-\frac{1}{2}}\ (1+u)^{-(2\mu+1)}$$

$$\times \exp\left(\frac{zu}{1+u}\right)\ {}_1F_1\left(\alpha;\ \beta;\ -\frac{\sigma\ (1+u)^{m+n}}{u^m}\right) du; \tag{33}$$

$$M_{\sigma,k,\mu}^{(\alpha,\beta;m,n)}(z) = \frac{2^{-2\mu}\ z^{\mu+\frac{1}{2}}}{B(\mu-k+\frac{1}{2},\mu+k+\frac{1}{2})} \int_{-1}^1 (1+u)^{\mu-k-\frac{1}{2}}\ (1-u)^{\mu+k-\frac{1}{2}}$$

$$\times \exp\left(\frac{zu}{2}\right)\ {}_1F_1\left(\alpha;\ \beta;\ -\frac{2^{m+n}\ \sigma}{(1+u)^m(1-u)^n}\right) du. \tag{34}$$

Proof. The use of (21) in (28) is seen to yield the integral representation (30). Setting $t = 1-u,\ t = \frac{u-p}{q-p}$ and $t = \frac{u}{1+u}$ in (30) yield (31),(32),(33) respectively. Setting $q = 1$ and $p = -1$ in (32) gives (34).

Remark 5.4. On setting $m = n$ in (30), (31), (32), (33) and (34), we obtain the integral representations of generalized extended Whittaker function defined by Choi et al. [10], which, on further setting $\alpha = \beta$ correspond with the integral representations for the extended Whittaker function given by Khan and Ghayasuddin [14]. The case $\alpha = \beta,\ m = n = 1$ of (30), (31), (32), (33) and (34) is seen to yield the integral representations of the generalized extended Whittaker function due to Nagar et al. [2].

Remark 5.5. On using (21) in the equation (31), we get

$$M_{\sigma,k,\mu}^{(\alpha,\beta;m,n)}(z) = z^{\mu+\frac{1}{2}}\ \exp\left(\frac{z}{2}\right)\ \Phi_\sigma^{(\alpha,\beta;m,n)}(\mu+k+\frac{1}{2};\ 2\mu+1;-z), \tag{35}$$

Thus it is seen that the generalized extended Whittaker function can also be expressed by (35).

Theorem 5.6. The following relation holds true:

$$M_{\sigma,k,\mu}^{(\alpha,\beta;m,n)}(-z) = (-1)^{\mu+\frac{1}{2}} \; M_{\sigma,-k,\mu}^{(\alpha,\beta;m,n)}(z). \tag{36}$$

$$(\sigma > 0;\ \sigma = 0 \text{ and } \Re(\mu) > \Re(\mu \pm k) > -\frac{1}{2},\ \Re(\alpha) > 0, \Re(\beta) > 0).$$

Proof. Replacing z by $-z$ in (28), we get

$$M_{\sigma,k,\mu}^{(\alpha,\beta;m,n)}(-z) = (-z)^{\mu+\frac{1}{2}} \; \exp\left(\frac{z}{2}\right) \; \Phi_{\sigma}^{(\alpha,\beta;m,n)}(\mu - k + \frac{1}{2};\ 2\mu + 1; -z). \tag{37}$$

Now using (22) in (37) and then writing the resulting expression by using (28), we get the desired result.

6. Integral transforms of $\mathbf{M_{\sigma,k,\mu}^{(\alpha,\beta;m,n)}(z)}$

Certain interesting integral transforms of the generalized extended Whittaker function $M_{\sigma,k,\mu}^{(\alpha,\beta;m,n)}(z)$ are given as follows:

Theorem 6.1. The following Mellin transformation holds true:

$$\int_0^\infty \sigma^{s-1} \; M_{\sigma,k,\mu}^{(\alpha,\beta;m,n)}(z) \; d\sigma$$

$$= \frac{\Gamma^{\,(\alpha,\beta)}(s) z^{\mu+\frac{1}{2}} \; \exp(-\frac{z}{2}) \; B(\mu - k + ms + \frac{1}{2}, \mu + k + ns + \frac{1}{2})}{B(\mu - k + \frac{1}{2}, \mu + k + \frac{1}{2})}$$

$$\times \Phi\left(\mu + ms - k + \frac{1}{2}; 2\mu + (m+n)s + 1; z\right) \tag{38}$$

$$\left(\Re(s) > 0,\ \Re(\mu \pm k) > -\frac{1}{2},\ \Re(\mu + ms - k) > -\frac{1}{2},\ \Re(\mu + ns + k) > -\frac{1}{2}\right).$$

Proof. Using the integral representation of $M_{\sigma,k,\mu}^{(\alpha,\beta;m,n)}(z)$ given by (30) and changing the order of integration, we get

$$\int_0^\infty \sigma^{s-1} \; M_{\sigma,k,\mu}^{(\alpha,\beta;m,n)}(z) \; d\sigma = \frac{z^{\mu+\frac{1}{2}} \; \exp(-\frac{z}{2})}{B(\mu - k + \frac{1}{2}, \mu + k + \frac{1}{2})} \int_0^1 t^{\mu-k-\frac{1}{2}} \; (1-t)^{\mu+k-\frac{1}{2}} \; e^{zt}$$

$$\times \left(\int_0^\infty \sigma^{s-1} {}_1F_1\left(\alpha;\ \beta;\ -\frac{\sigma}{t^m(1-t)^n}\right) d\sigma\right) dt.$$

Taking $u=\frac{\sigma}{t^m(1-t)^n}$, we get

$$\int_0^\infty \sigma^{s-1}{}_1F_1\left(\alpha;\ \beta; -\frac{\sigma}{t^m(1-t)^n}\right) = t^{ms}(1-t)^{ns}\int_0^\infty u^{s-1}{}_1F_1(\alpha;\ \beta;\ -u)du$$

$$= t^{ms}\ (1-t)^{ns}\ \Gamma^{\ (\alpha,\beta)}(s),$$

where $\Gamma_\sigma^{(\alpha,\beta)}(s)$ is the generalized gamma function defined by Özergin et al. [7].

So that, we have

$$\int_0^\infty \sigma^{s-1}\ M_{\sigma,k,\mu}^{(\alpha,\beta;m,n)}(z)\ d\sigma = \frac{\Gamma^{\ (\alpha,\beta)}(s)\ z^{\mu+\frac{1}{2}}\ \exp(-\frac{z}{2})}{B(\mu-k+\frac{1}{2},\mu+k+\frac{1}{2})}$$

$$\times\int_0^1 t^{\mu+ms-k-\frac{1}{2}}\ (1-t)^{\mu+ns+k-\frac{1}{2}}\ e^{zt}\ dt$$

$$= \frac{\Gamma^{\ (\alpha,\beta)}(s)z^{\mu+\frac{1}{2}}\ \exp(-\frac{z}{2})\ B(\mu-k+ms+\frac{1}{2},\mu+k+ns+\frac{1}{2})}{B(\mu-k+\frac{1}{2},\mu+k+\frac{1}{2})B(\mu-k+ms+\frac{1}{2},\mu+k+ns+\frac{1}{2})}$$

$$\times\int_0^1 t^{\mu+ms-k+\frac{1}{2}-1}\ (1-t)^{\{2\mu+(m+n)s+1\}-\left(\mu+ms-k+\frac{1}{2}\right)-1}\ e^{zt}\ dt,$$

On using the integral representation of confluent hypergeometric function ${}_1F_1$ or Φ in the above equation, we get the required result.

Remark 6.2. The case $n=m$ of (38) on arranging the resulting expression in terms of classical Whittaker function, is seen to yield the Mellin transform of the extended Whittaker function given by Choi et al. [[10], p.6535, eq.(27)].

Theorem 6.3. The following formula holds true:

$$\int_0^\infty z^{a-1}\ e^{-pz}M_{\sigma,k,\mu}^{(\alpha,\beta;m,n)}(bz)\ dz = \frac{b^{\mu+\frac{1}{2}}\ \Gamma(a+\mu+\frac{1}{2})}{(p+\frac{b}{2})^{a+\mu+\frac{1}{2}}}$$

$$\times F_\sigma^{(\alpha,\beta;m,n)}\left(a+\mu+\frac{1}{2},\ \mu-k+\frac{1}{2};\ 2\mu+1;\ \frac{2b}{2p+b}\right), \tag{39}$$

$$(\sigma\geq 0, 2p>b>0, \Re(a+\mu)>-\frac{1}{2})$$

where $F_\sigma^{(\alpha,\beta;m,n)}(a,b;c;z)$ is the generalized extended Gauss hypergeometric function defined by (18).

Proof. On using the integral representation of $M_{\sigma,k,\mu}^{(\alpha,\beta;m,n)}(z)$ on the left hand side of (39) given by (30), and changing the order of integration, and integrating with respect to z by using the definition of gamma function, we arrive at

$$\int_0^\infty z^{a-1}\, e^{-pz}\, M_{\sigma,k,\mu}^{(\alpha,\beta;m,n)}(bz)\, dz = \frac{b^{\mu+\frac{1}{2}}\, \Gamma(a+\mu+\frac{1}{2})}{(p+\frac{b}{2})^{a+\mu+\frac{1}{2}}\, B(\mu-k+\frac{1}{2},\mu+k+\frac{1}{2})}$$

$$\times \int_0^1 t^{\mu-k-\frac{1}{2}}(1-t)^{\mu+k-\frac{1}{2}} \left(1-\frac{2bt}{2p+b}\right)^{-(a+\mu+\frac{1}{2})} {}_1F_1\left(\alpha;\ \beta;\ -\frac{\sigma}{t^m(1-t)^n}\right) dt. \tag{40}$$

On using the integral representation of $F_\sigma^{(\alpha,\beta;m,n)}(a,b;c;z)$ (which can be easily obtained by using the integral representation of extended beta function given by (19) in (18)) in (40), yield the desired result.

Corollary 6.4. If we put $b=a=1$ in (39), we obtain following special case.

$$\int_0^\infty e^{-pz}\, M_{\sigma,k,\mu}^{(\alpha,\beta;m,n)}(z)\, dz = \frac{2^{\mu+\frac{3}{2}}\, \Gamma(\mu+\frac{3}{2})}{(2p+1)^{\mu+\frac{3}{2}}}$$

$$\times F_\sigma^{(\alpha,\beta;m,n)}\left(\mu+\frac{3}{2},\ \mu-k+\frac{1}{2};\ 2\mu+1;\ \frac{2}{2p+1}\right). \tag{41}$$

Theorem 6.5. The following Hankel transformation holds true:

$$\int_0^\infty z\, M_{\sigma,k,\mu}^{(\alpha,\beta;m,n)}(z)\, J_\nu(az)\, dz = \frac{\Gamma(\mu+\nu+\frac{5}{2})}{(a^2+\frac{1}{4})^{\frac{\mu}{2}+\frac{5}{4}}}$$

$$\times \sum_{s=0}^\infty \frac{B_\sigma^{(\alpha,\beta;m,n)}(\mu-k+\frac{1}{2}+s,\mu+k+\frac{1}{2})\ (\mu+\nu+\frac{5}{2})_s}{B(\mu-k+\frac{1}{2},\mu+k+\frac{1}{2})\ (a^2+\frac{1}{4})^{\frac{s}{2}}\ s!}\ P_{\mu+n+\frac{3}{2}}^{-\nu}\left(\frac{1}{\sqrt{4a^2+1}}\right) \tag{42}$$

$$(\Re(\mu\pm k) > -\frac{1}{2}, \Re(\mu+\nu) > -\frac{5}{2}),$$

where $P_\mu^\nu(z)$ is the Legendre function [8].

Proof. By using (20) and (28), expanding $M_{\sigma,k,\mu}^{(\alpha,\beta;m,n)}(z)$ in terms of generalized extended beta function and changing the order of integration and summation, we get

$$\int_0^\infty z\, M_{\sigma,k,\mu}^{(\alpha,\beta;m,n)}(z)\, J_\nu(az)\, dz = \sum_{s=0}^\infty \frac{B_\sigma^{(\alpha,\beta;m,n)}(\mu-k+\frac{1}{2}+s,\mu+k+\frac{1}{2})}{B(\mu-k+\frac{1}{2},\mu+k+\frac{1}{2})\ s!}$$

$$\times \int_0^\infty z^{\mu+s+\frac{3}{2}}\, e^{-\frac{z}{2}}\, J_\nu(az)\, dz.$$

Using the known formula (see [[1]; p.182(9)]):

$$\int_0^\infty e^{-pt}\, t^\mu\, J_\nu(at)\, dt = \Gamma(\mu+\nu+1)\, r^{-\mu-1}\, P_\mu^{-\nu}\left(\frac{p}{r}\right),$$

where $\Re(\mu+\nu) > -1$, $r = (p^2+a^2)^{\frac{1}{2}}$ and $P_\mu^\nu(z)$ is the Legendre function [8],

in the above expression and after some simplification, we get the desired result.

7. Derivative of $\mathbf{M}_{\boldsymbol{\sigma},\mathbf{k},\boldsymbol{\mu}}^{(\boldsymbol{\alpha},\boldsymbol{\beta};\mathbf{m},\mathbf{n})}(\mathbf{z})$

Theorem 7.1. For the generalized extended Whittaker function $M_{\sigma,k,\mu}^{(\alpha,\beta;m,n)}(z)$, the following differential formula holds true:

$$\frac{d^r}{dz^r}\left[e^{\frac{z}{2}}\, z^{-\mu-\frac{1}{2}}\, M_{\sigma,k,\mu}^{(\alpha,\beta;m,n)}(z)\right] = \frac{(\mu-k+\frac{1}{2})_r}{(2\mu+1)_r}\, e^{\frac{z}{2}}\, z^{-\mu-\frac{r}{2}-\frac{1}{2}}\, M_{\sigma,k-\frac{r}{2},\mu+\frac{r}{2}}^{(\alpha,\beta;m,n)}(z). \tag{43}$$

Proof. By using (24), we have

$$\frac{d^r}{dz^r}[\Phi_\sigma^{(\alpha,\beta;m,n)}(b;\; c;\; z)] = \frac{(b)_r}{(c)_r}\, \Phi_\sigma^{(\alpha,\beta;m,n)}(b+r;\; c+r;\; z). \tag{44}$$

Now, using the definition of generalized extended Whittaker function on the left hand side of (43), we get

$$\frac{d^r}{dz^r}\left[e^{\frac{z}{2}}\, z^{-\mu-\frac{1}{2}}\, M_{\sigma,k,\mu}^{(\alpha,\beta;m,n)}(z)\right] = \frac{d^r}{dz^r}[\Phi_\sigma^{(\alpha,\beta;m,n)}(\mu-k+\frac{1}{2};\; 2\mu+1;\; z)].$$

By applying (44) in the above expression and then using the definition of generalized extended Whittaker function given by (28), yield the required result.

8. Recurrence relations of $M_{\sigma,k,\mu}^{(\alpha,\beta;m,n)}(z)$ and concluding remarks

Theorem 8.1. The following relations for the generalized extended Whittaker function $M_{\sigma,k,\mu}^{(\alpha,\beta;m,n)}(z)$ holds true:

$$\begin{aligned}
&\text{(i) } (\beta-\alpha)\, M_{\sigma,k,\mu}^{(\alpha-1,\beta;m,n)}(z) + (2\alpha-\beta) M_{\sigma,k,\mu}^{(\alpha,\beta;m,n)}(z) \\
&-\frac{\sigma z^{\mu+\frac{1}{2}}\exp(-\frac{z}{2}) B(\mu-m-k+\frac{1}{2}, \mu-n+k+\frac{1}{2})}{B(\mu-k+\frac{1}{2}, \mu+k+\frac{1}{2})} \\
&\times\Phi\left(\mu-m-k+\frac{1}{2}; 2\mu-(m+n)+1; z\right) - \alpha\, M_{\sigma,k,\mu}^{(\alpha+1,\beta;m)}(z) = 0;
\end{aligned} \tag{45}$$

$$\text{(ii) } \beta(\beta-1) M_{\sigma,k,\mu}^{(\alpha,\beta-1;m,n)}(z) - \beta(\beta-1) M_{\sigma,k,\mu}^{(\alpha,\beta;m,n)}(z)$$

$$+\frac{\sigma\beta z^{\mu+\frac{1}{2}}\exp(-\frac{z}{2})B(\mu-m-k+\frac{1}{2},\mu-n+k+\frac{1}{2})}{B(\mu-k+\frac{1}{2},\ \mu+k+\frac{1}{2})}$$

$$\times\Phi\left(\mu-m-k+\frac{1}{2};2\mu-(m+n)+1;z\right)$$

$$-\frac{\sigma(\beta-\alpha)z^{\mu+\frac{1}{2}}\exp(-\frac{z}{2})B(\mu-m-k+\frac{1}{2},\mu-n+k+\frac{1}{2})}{B(\mu-k+\frac{1}{2},\mu+k+\frac{1}{2})}$$

$$\times\Phi\left(\mu-m-k+\frac{1}{2};2\mu-(m+n)+1;z\right)=0; \tag{46}$$

$$\text{(iii)}\ (1+\alpha-\beta)M^{(\alpha+1,\beta;m,n)}_{\sigma,k,\mu}(z)+(\beta-1)\ M^{(\alpha,\beta-1;m,n)}_{\sigma,k,\mu}(z)=0; \tag{47}$$

$$\text{(iv)}\ \beta\ M^{(\alpha,\beta;m,n)}_{\sigma,k,\mu}(z)-\beta\ M^{(\alpha-1,\beta;m,n)}_{\sigma,k,\mu}(z)$$

$$+\frac{\sigma\ z^{\mu+\frac{1}{2}}\ \exp(-\frac{z}{2})B(\mu-m-k+\frac{1}{2},\ \mu-n+k+\frac{1}{2})}{B(\mu-k+\frac{1}{2},\ \mu+k+\frac{1}{2})}$$

$$\times\Phi\left(\mu-m-k+\frac{1}{2};2\mu-(m+n)+1;z\right)=0; \tag{48}$$

$$\text{(v)}\ \alpha\beta\ M^{(\alpha,\beta;m,n)}_{\sigma,k,\mu}(z)-\frac{\sigma\beta z^{\mu+\frac{1}{2}}\exp(-\frac{z}{2})B(\mu-m-k+\frac{1}{2},\ \mu-n+k+\frac{1}{2})}{B(\mu-k+\frac{1}{2},\ \mu+k+\frac{1}{2})}$$

$$\times\Phi\left(\mu-m-k+\frac{1}{2};2\mu-(m+n)+1;z\right)$$

$$+\frac{\sigma(\beta-\alpha)z^{\mu+\frac{1}{2}}\exp(-\frac{z}{2})B(\mu-m-k+\frac{1}{2},\ \mu-n+k+\frac{1}{2})}{B(\mu-k+\frac{1}{2},\ \mu+k+\frac{1}{2})}$$

$$\times\Phi\left(\mu-m-k+\frac{1}{2};2\mu-(m+n)+1;z\right)-\alpha\beta\ M^{(\alpha+1,\beta;m,n)}_{\sigma,k,\mu}(z)=0; \tag{49}$$

$$\text{(vi)}\ (\alpha-1)\ M^{(\alpha,\beta+1;m,n)}_{\sigma,k,\mu}(z)-\frac{\sigma z^{\mu+\frac{1}{2}}\exp(-\frac{z}{2})B(\mu-m-k+\frac{1}{2},\ \mu-n+k+\frac{1}{2})}{B(\mu-k+\frac{1}{2},\ \mu+k+\frac{1}{2})}$$

$$\times\Phi\left(\mu-m-k+\frac{1}{2};2\mu-(m+n)+1;z\right)+(\beta-\alpha)\ M^{(\alpha-1,\beta;m,n)}_{\sigma,k,\mu}(z)$$

$$-(\beta-1)\ M^{(\alpha,\beta-1;m,n)}_{\sigma,k,\mu}(z)=0. \tag{50}$$

Proof (i). We have the following recurrence relation of the confluent hypergeometric function ${}_1F_1$ (see [11, p.19])

$$(b-a)\,{}_1F_1(a-1;\ b;\ z)+(2a-b)\,{}_1F_1(a;\ b;\ z)+z\,{}_1F_1(a;\ b;\ z)-a\,{}_1F_1(a+1;\ b;\ z)=0. \tag{51}$$

With the help of above relation (51), we derive

$$\begin{aligned}
&\frac{(\beta-\alpha)\ z^{\mu+\frac{1}{2}}\exp(-\frac{z}{2})}{B(\mu-k+\frac{1}{2},\ \mu+k+\frac{1}{2})}\int_0^1 t^{\mu-k-\frac{1}{2}}(1-t)^{\mu+k-\frac{1}{2}}\ e^{zt}\\
&\qquad\times{}_1F_1\left(\alpha-1;\ \beta;\ -\frac{\sigma}{t^m(1-t)^n}\right)dt\\
&+\frac{(2\alpha-\beta)\ z^{\mu+\frac{1}{2}}\exp(-\frac{z}{2})}{B(\mu-k+\frac{1}{2},\ \mu+k+\frac{1}{2})}\int_0^1 t^{\mu-k-\frac{1}{2}}(1-t)^{\mu+k-\frac{1}{2}}\ e^{zt}\\
&\qquad\times{}_1F_1\left(\alpha;\ \beta;\ -\frac{\sigma}{t^m(1-t)^n}\right)dt\\
&-\frac{\sigma\ z^{\mu+\frac{1}{2}}\exp(-\frac{z}{2})}{B(\mu-k+\frac{1}{2},\ \mu+k+\frac{1}{2})}\int_0^1 t^{\mu-m-k-\frac{1}{2}}(1-t)^{\mu-n+k-\frac{1}{2}}\ e^{zt}\\
&\qquad\times{}_1F_1\left(\alpha;\ \beta;\ -\frac{\sigma}{t^m(1-t)^n}\right)dt\\
&-\frac{\alpha\ z^{\mu+\frac{1}{2}}\exp(-\frac{z}{2})}{B(\mu-k+\frac{1}{2},\ \mu+k+\frac{1}{2})}\int_0^1 t^{\mu-k-\frac{1}{2}}(1-t)^{\mu+k-\frac{1}{2}}\ e^{zt}\\
&\qquad\times{}_1F_1\left(\alpha+1;\ \beta;\ -\frac{\sigma}{t^m(1-t)^n}\right)dt=0.
\end{aligned}$$

By using the integral representations of $M_{\sigma,k,\mu}^{(\alpha,\beta;m)}(z)$ and Φ in the above expression, we get our first relation (45). Similarly, we can derive (46), (47), (48), (49) and (50) by using the following recurrence relations of ${}_1F_1$ (see [11, p.19]):

$$\begin{aligned}
&b(b-1)\ {}_1F_1(a;\ b-1;\ z)-b(b-1)\ {}_1F_1(a;\ b;\ z)-bz\ {}_1F_1(a;\ b;\ z)\\
&\qquad+(b-a)z\ {}_1F_1(a;\ b+1;\ z)=0; &(52)\\
&(1+a-b)\ {}_1F_1(a;\ b;\ z)-a\ {}_1F_1(a+1;\ b;\ z)+(b-1)\ {}_1F_1(a;\ b-1;\ z)=0; &(53)\\
&b\ {}_1F_1(a;\ b;\ z)-b\ {}_1F_1(a-1;\ b;\ z)-z\ {}_1F_1(a;\ b+1;\ z)=0; &(54)\\
&ab\ {}_1F_1(a;\ b;\ z)+bz\ {}_1F_1(a;\ b;\ z)-(b-a)z\ {}_1F_1(a;\ b+1;\ z)\\
&\qquad-ab\ {}_1F_1(a+1;\ b;\ z)=0; &(55)\\
&(a-1)\ {}_1F_1(a;\ b;\ z)+z\ {}_1F_1(a;\ b;\ z)+(b-a)\ {}_1F_1(a-1;\ b;\ z)
\end{aligned}$$

$$-(b-1)\ {}_1F_1(a;\ b-1;\ z) = 0. \tag{56}$$

In the present investigation, we have made an attempt here to extend the generalized extended confluent hypergeometric function and the generalized extended Whittaker function of first kind. Further, we have provided an elegant extensions of the usual properties of classical confluent hypergeometric function and the classical Whittaker function of first kind. Most of the special functions of mathematical physics, such as modified Bessel function, Hermite and Laguerre polynomials can be expressed in terms of Whittaker function. Therefore numerous generating functions involving the extensions and generalizations of the Whittaker function are capable of playing important roles in the theory of special functions of applied mathematics and mathematical physics.

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