On Representation Dimension of Tame Cluster Tilted Algebras *

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ABSTRACT: The aim of this work is to study the representation dimension of cluster tilted algebras. We prove that the weak representation dimension of tame cluster tilted algebras is equal to three. We construct a generator module that reaches the weak representation dimension, unfortunately this module is not always a cogenerator. We show for which algebras this module gives the representation dimension.

Key Words: Auslander generator, Weak representation dimension, Cluster-tilted algebra, Representation dimension.

Contents

1. Introduction

The representation dimension was introduced by Auslander [4] in the early seventies. Due to the connection of arbitrary artin algebras with representation finite artin algebras, it was expected that this notion would give a reasonable way of measuring how far an artin algebra is from being of representation finite type. In fact, the representation dimension characterizes artin algebras of finite representation type. Later, Iyama proved in [14] that the representation dimension of an artin algebra is always finite and Rouquier [17] constructed examples of algebras with

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arbitrarily large representation dimension. On the other hand, the representation dimension is a measure, in some way, of the complexity of the morphisms of the module category. The purpose of our work is to study the representation dimension of tame cluster tilted algebras.

Cluster algebras were introduced by Fomin and Zelevinsky [9]. Later, Buan, Marsh, Reineke, Reiten and Todorov [5] defined the cluster category and developed a tilting theory using a special class of objects, namely the cluster tilting objects. In [6], Buan, Marsh and Reiten introduced and studied the endomorphism algebras of such cluster-tilting objects in a cluster category. These algebras are known as cluster tilted algebras and are deeply connected to tilted and hereditary algebras. They also prove that the module category of a cluster tilted algebra is a proper quotient of the cluster category. This last fact motivates us to investigate the relationship between the module theory of cluster-tilted algebras and the module theory of hereditary algebras. In particular, one can obtain a lot of information about the module category of a cluster tilting algebra using the information provided by the torsion theory defined by the tilting module in the hereditary algebra.

In [11] we have studied the representation dimension of cluster concealed algebras and we proved that its representation dimension is equal to three. The aim of this paper is to extend the techniques developed there to construct a generator of the module category of tame cluster-tilted algebras.

Tame cluster tilted algebras are induced from tilting modules over tame hereditary algebras. Therefore, the torsion theory induced by a tilting module in module category of a tame hereditary algebra became a very useful tool. We study the representation dimension of tame cluster tilted algebras. More precisely, we prove that the weak representation dimension of tame cluster tilted algebras is equal to three. We construct a generator module which reaches the weak representation dimension, unfortunately the constructed module will not always be a cogenerator. Finally, we show when this generator module gives the representation dimension of a tame cluster tilted algebra.

Recently, García Elsener and Schiffler in [10] proved that the representation dimension of tame cluster tilted algebras is equal to three. The proof is based on a result of Bergh and Oppermann, that the category of Cohen Macaulay modules of this algebras is finite. Since the representation dimension is a measure of the complexity of the morphisms of the module category is still interesting to look for an Auslander generator of the module category. In this sense, the techniques introduced in this paper could be useful to know how the morphisms behave in the derived category.

In section 2, we introduce the notation we will use through the paper and recall some well known results about the Auslander-Reiten quiver of an hereditary algebra. We also recall some preliminary result about torsion theory for hereditary algebras. In section 3, we present our main theorem and prove various results required for the proof of our main theorems.

2. Preliminaries

Through this paper, all algebras will be finite dimensional algebras over an algebraically closed field and for an algebra A we will consider mod A the category of all finitely generated right A-modules and denote by ind$A$ the full subcategory of indecomposable modules in mod A.

Given an A-module M we denoted by add(M) the additive subcategory of mod A generated by M, whose objects consist in finite sums of direct summands of the module M.

We will denote by H a finite dimensional hereditary algebra and by $\mathcal{D}^b(H)$ the bounded derived category of H. Recall that the objects here are stalk complexes. We will identify the objects concentrated in degree zero with the corresponding H-modules. We will denote by $[\ ]$ the shift functor, thus any object in $\mathcal{D}^b(H)$ can be viewed as $X[i]$ with X in mod H and i an integer.

2.1. Representation dimension

We recall that an A-module M is a generator for mod A if for each $X \in$ mod A there exists an epimorphism $M' \to X$ with $M' \in$ add(M). Observe that A is a generator for mod A. Dually, we say that an A-module M is a cogenerator if for each $Y \in$ mod A there exists a monomorphism $Y \to M'$ with $M' \in$ add(M). Note that DA is a cogenerator for mod A. In particular, any module M containing every indecomposable projective and every indecomposable injective module as a summand is a generator-cogenerator module for mod A.

The original definition of representation dimension, which we will denote by rep.dim of an artin algebra A is due to Auslander. For more details on this topic, we refer the reader to [4]. The following is a useful characterization of representation dimension, in the case that A is a non semisimple algebra, also due to Auslander. The representation dimension of an artin algebra is given by

$$\text{rep.dim} A = \inf\{\text{gl.dim End}_A(M) \mid M \text{ is a generator-cogenerator for mod} A\}.$$

A module M that reaches the minimum in the above definition is called an **Auslander generator** and gl.dim End$_A(M) = $ rep.dim$A$ if M is an Auslander generator.

The representation dimension can also be defined in a functorial way, which will be more convenient for us. The next definition (see [3], [8], [7], [17]) will be very useful for the rest of this work.

Definition 2.1. *The representation dimension rep.dimA is the smallest integer $i \geq 2$ such that there is a module $M \in modA$ with the property that, given any A-module X,*

(a) there is an exact sequence

$$0 \to M^{i-1} \to M^{i-2} \to ... \to M^1 \xrightarrow{f} X \to 0$$

with $M^j \in add(M)$ such that the sequence

$$0 \to Hom_A(M, M^{i-1}) \to ... \to Hom_A(M, M^1) \to Hom_A(M, X) \to 0$$

is exact.

(b) there is a exact sequence

$$0 \to X \xrightarrow{g} M_1' \to M_2' \to ... \to M_{i-1}' \to 0$$

with $M_j' \in add(M)$ such that the sequence

$$0 \to Hom_A(M_{i-1}', M) \to ... \to Hom_A(M_1', M) \to Hom_A(X, M) \to 0$$

is exact.

Following [7], we say that the module M has the $i-1$-resolution property and that the sequence $0 \to M^{i-1} \to M^{i-2} \to ... \to M^1 \to X \to 0$ is an add$M$-approximation of X of length $i-1$. Observe, that $f : M^1 \to X$ is a right add(M)-approximation of X and $g : X \to M_1'$ is a left add(M)-approximation of X.

Remark 2.2. *Either condition (a) or (b) imply that gl.dim $End_A(M) \leq i$ ([8, Lemma 2.2]). Then, if $M \in modA$ and $i \geq 2$, the following statements are equivalent.*

- *M satisfies (a) and (b) of the definition.*

- *M satisfies (a) and M contains an injective cogenerator as a direct summand.*

- *M satisfies (b) and M contains a projective generator as a direct summand.*

Finally, we end this subsection with a weaker version of our last definition.

Definition 2.3. *The weak representation dimension w.rep.dimA is the smallest integer $i \geq 2$ such that there is a module $M \in modA$ with the property that, given any A-module X, there is an exact sequence*

$$0 \to M^{i-1} \to M^{i-2} \to ... \to M^1 \xrightarrow{f} X \to 0$$

with $M^j \in add(M)$ such that the sequence

$$0 \to Hom_A(M, M^{i-1}) \to ... \to Hom_A(M, M^1) \to Hom_A(M, X) \to 0$$

is exact.

As we observe before this condition implies that M contains a projective generator as a direct summand and that gl.dim $End_A(M) \leq i$. The weaker condition here is that it not requires M to be a cogenerator. In particular, w.repdim$A \leq$ repdim$A$.

2.2. Torsion theory for hereditary algebras

We start this section by giving the definition of tilting module over a hereditary algebra. For more details on tilting modules see [13]. Let H be an hereditary algebra with n non-isomorphic simple modules in mod$H$ and T an H-module. T is said to be a tilting module in mod H if it satisfies the following conditions:

(a) $\operatorname{Ext}^1_A(T,T) = 0$.

(b) T has exactly n non-isomorphic indecomposable direct summands.

A tilting module is said to be basic if all its direct summands are non-isomorphic. The endomorphism ring of a tilting module over a hereditary algebra is said to be a **tilted algebra**. In particular, hereditary algebras are tilted algebras.

We recall that a **path** from X to Y is a sequence $X = X_0 \to X_1 \to ... \to X_t = Y$ with $t > 0$ of non-zero non-isomorphisms between indecomposable modules. Given $X, Y \in \operatorname{ind}A$, we say that X is a **predecessor** of Y or that Y is a **successor** of X, provided that there exists a path from X to Y. A tilting module T is **convex** if, for a given pair of indecomposable summands of T, X,Y in add T, any path from X to Y contains only indecomposable modules in add T. Following [3], we say that a set Σ_T in mod A is a **complete slice** if $T = \bigoplus_{M \in \Sigma_T} M$ is a convex tilting module with $\operatorname{End}_A T$ hereditary. For the original definition of complete slice, we refer the reader to [15], [16].

For a given tilting module T in mod H there exists two full disjoint subcategories of mod H, namely

$$\mathcal{F}(T) = \{X \in \operatorname{mod}H \text{ such that } \operatorname{Hom}_H(T,X) = 0\}$$
$$\mathcal{T}(T) = \{X \in \operatorname{mod}H \text{ such that } \operatorname{Ext}^1_H(T,X) = 0\}$$

the free torsion class and the torsion class, respectively.

Furthermore, if T is a convex tilting module, then mod$H = \mathcal{F}(T) \bigcup \mathcal{T}(T)$. We have that, in this case, $\mathcal{F}(T)$ is closed under predecessors and $\mathcal{T}(T)$ is closed under successors.

We will say that the category $\mathcal{F}(T)(\mathcal{T}(T))$ is finite if and only if there exists a finite number of iso-classes of indecomposable modules in $\mathcal{F}(T)(\mathcal{T}(T))$, otherwise we say that the category $\mathcal{F}(T)(\mathcal{T}(T))$ is infinite.

Let us recall the following result from Happel and Ringel about tilting modules in tame hereditary algebras.

Lemma 2.4. *[13, Lema 3.1] Let H a tame hereditary algebra. If T is a tilting module over H then T has at least one non zero preprojective or preinjective direct summand.*

The previous lemma implies that a tilting module over an hereditary tame algebra is not regular. In fact, if we assume that all non regular summands of T are preprojectives we obtain the following description of the categories $\mathcal{T}(T)$ and $\mathcal{F}(T)$.

Proposition 2.5. *[13, Proposición (3.2)*]*
    Let H be a tame hereditary algebra, and T a tilting module over H. Then the following are equivalent:

(a) *$\mathcal{F}(T)$ is infinite.*

(b) *$\mathcal{F}(T)$ contains infinite preinjective indecomposable modules.*

(c) *There exists an indecomposable homogeneous simple module which belongs to $\mathcal{F}(T)$.*

(d) *All homogeneous modules belong to $\mathcal{F}(T)$.*

(e) *All preprojective modules belong to $\mathcal{F}(T)$.*

(f) *T has no non zero preprojective summand.*

Dualy, we have a similar description for a tilting module without preinjective direct summands.

Proposition 2.6. *[13, Proposición 3.2]*
    Let H be a tame hereditary algebra, and T a tilting module over H. Then the following are equivalent:

(a) *$\mathcal{T}(T)$ is infinite.*

(b) *$\mathcal{T}(T)$ contains infinite preprojective indecomposable modules.*

(c) *There exists an indecomposable homogeneous simple module which belongs to $\mathcal{T}(T)$.*

(d) *All homogeneous modules belong to $\mathcal{T}(T)$.*

(e) *All preprojective modules belong to $\mathcal{T}(T)$.*

(f) *T has no non zero preinjective summand.*

The following remark follows from (a) if and only if (f) and is very helpful.

Remark 2.7. *If T has preinjective direct summands then $\mathcal{T}(T)$ is finite. Moreover, if the only non regular direct summands of T are preinjective then $\mathcal{F}(T)$ is infinite.*

Analogously, if a tilting module T over a tame hereditary algebra has no non zero preinjective summand the roles of $\mathcal{F}(T)$ and $\mathcal{T}(T)$ are inverted, it is, $\mathcal{F}(T)$ is finite and $\mathcal{T}(T)$ is infinite.

The following proposition allow us to determine when to expect that the category $\mathcal{F}(T)$ be finite.

Proposition 2.8. *[2, Proposición 3.3]*

Let H be a representation infinite hereditary algebra and $(\mathcal{T}, \mathcal{F})$ a torsion pair over mod H. Then $\mathcal{T}$ contains infinite classes of isomorphisms of indecomposable preprojective modules if and only if $(\mathcal{T}, \mathcal{F})$ is induced by a tilting module without preinjective summands and $\mathcal{F}$ is finite.

Moreover, if H is wild, then $(\mathcal{T}, \mathcal{F})$ is induced by a preprojective tilting module.

Combining the last two results we obtain under what hypothesis a tilting module T induces a torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$ with one of the categories finite.

For every indecomposable regular module we have the cone determined by M, defined as follows.

Definition 2.9. *[19, Definición XVI. 1.4] Given an indecomposable H-module M in a regular component $\mathfrak{C}$ of the Auslander-Reiten quiver Γ_H of an hereditary algebra H, the cone determined by M is a full subquiver of $\mathfrak{C}$ with the following shape:*
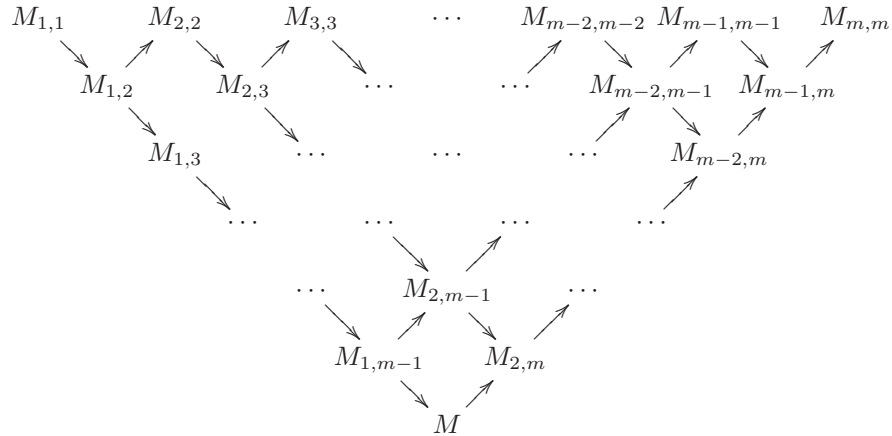
We denote by $\mathcal{C}(M)$ to the cone determined by M. We say that M is the vertex of $\mathcal{C}(M)$ and that the cone is of level m.

The next lemma is a bit technical but is really useful to deal with regular summands in a stable tube of a tilting T.

Lemma 2.10. *[19, XVII Lema 1.7]*

Let H be an hereditary algebra and $\mathcal{R}$ a stable tube in Γ_H. For any pair of indecomposable modules M, N in $\mathcal{R}$ such that:

$$Ext^1_H(M \oplus N, M \oplus N) = 0$$

one of the following conditions are satisfied:

a) $\mathcal{C}(M) \subset \mathcal{C}(N)$,

b) $\mathcal{C}(N) \subset \mathcal{C}(M)$ *or*

c) $\mathcal{C}(M) \bigcap \mathcal{C}(N) = \emptyset$, $\mathcal{C}(M) \bigcap \tau \mathcal{C}(N) = \emptyset$, *and* $\tau \mathcal{C}(M) \bigcap \mathcal{C}(N) = \emptyset$.

Let T be a tilting module with non zero regular summands over a tame hereditary algebra H. Let $T_1, ... T_n$ the regular summands of T which belongs to a stable tube $\mathcal{R}$ of Γ_H. Since $\text{Ext}^1_H(T,T) = 0$ by the previous lemma we can assume that $T_1, ..., T_n$ belongs to a disjoint union of cones, $\bigcup_{i=1}^{r} C(T'_i)$ where each one of the modules T'_i is a direct summand of T. Note that in general $r \leq n$. We will show later in section 4 that, under certain hypothesis, all the indecomposable regular modules in $\mathcal{T}(T) \bigcap \mathcal{R}$ belongs to the union of this cones.

Every one of the cones recently described have in addition the following property:

Proposition 2.11. *[18, XVII Proposición 2.1] Let H be a hereditary algebra and M an indecomposable module in a tube $\mathcal{R}$ such that $\mathcal{C}(M)$ is a cone of depth m. If T is a tilting module such that M is a direct summand of T then T have exactly m indecomposable direct summands in $\mathcal{C}(M)$.*

We will illustrate the results of this section with an example.

Example 2.12. *Let $H = KQ$ the hereditary algebra given by the quiver $Q = \tilde{\mathbb{D}}_{12}$.*
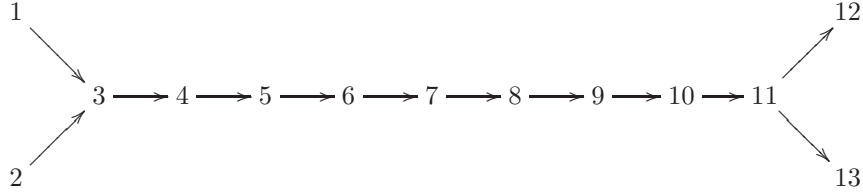
Since Q is an euclidean quiver , we know that the algebra H is a infinite representation algebra of tame type. Therefore we have that the Auslander-Reiten quiver of H, Γ_H has a preprojective component, a preinjective component and infinite regular component which consists in a family of orthogonal stable tubes. In fact, in this example we have a tube $\mathcal{R}$ of rank 10. The mouth of this tube consists in nine simple modules that we will denote by $E_i = S_{i+2}$ for $i = 1, ..., 9$ and the module E_{10} given by the representation:
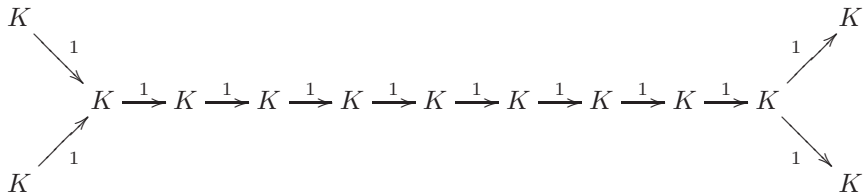
This modules also satisfy that $E_i = \tau E_{i+1}$ and $\tau E_1 = E_{10}$.

Let $T = \oplus_{i=1}^{12} T_i$ be the tilting module given by the following direct summands:

$$T_1 = E_2^1,\ T_2 = E_2^3,\ T_3 = E_4^1,\ T_4 = E_2^4,\ T_5 = E_8^1,\ T_6 = E_7^2,\ T_7 = E_{10}^1$$

where E_j^i is the i-th module of the ray starting in E_j and the other summands are the injective modules:

$$T_8 = I_1,\ T_9 = I_2,\ T_{10} = I_7,\ T_{11} = I_{10},\ T_{12} = I_{12},\ T_{13} = I_{13}.$$

which are the indecomposable direct preinjective summands of T.

In this example, applying the proposition 2.6, we get that the torsion $\mathfrak{T}(T)$ is finite . We will only show how the torsion of T looks like in the tube $\mathcal{R}$.

The part of the Auslander-Reiten quiver corresponding to $\mathcal{R}$ is given by the following translation quiver where the upper vertices represents the modules E_i in the mouth of $\mathcal{R}$ ordered from lowest to highest respect the index i:
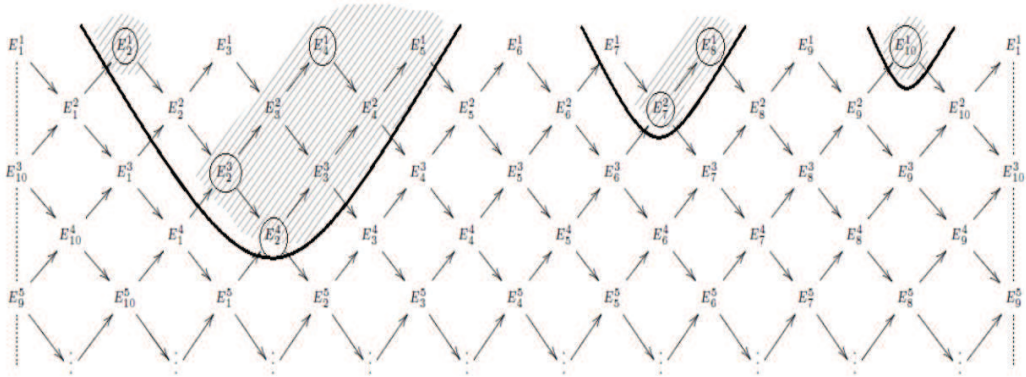
Here the vertices within the circles correspond to the summands of T. The shaded areas correspond to regular indecomposables modules in $\mathfrak{T}(T)$. We can see that the regular summands of T are distributed over 3 cones . The modules T_1, T_2 , T_3 and T_4 belong to the cone $\mathcal{C}(T_4)$ of level 4. The modules T_5 and T_6 are in the cone of level 2 determined by T_6 and finally T_7 belongs to the level 1 cone generated by itself . Furthermore, using the mesh relations of the Auslander - Reiten quiver on the component $\mathcal{R}$ and the fact that the rest of the summands of T are preinjective it can be shown that the rays generated by the regular simple modules $S_3 = E_1$, $S_4 = E_6$ and $S_7 = E_9$ are entirely contained in $\mathcal{F}(T)$. Somehow these rays separate the cones.

The rest of the indecomposable modules in $\mathfrak{T}(T)$ corresponds to preinjective modules. Moreover, due to the choice of T all the preprojective modules are in $\mathcal{F}(T)$, as well as all regular modules that do not belong to the tube $\mathcal{R}$.

2.3. Cluster categories and cluster tilted Algebras

For the convenience of the reader, we start this section recalling some definitions and results of cluster categories from [5]. Let $\mathcal{C}$ be the **cluster category** associated

to H given by $\mathcal{D}^b(H)/F$, where F is the composition functor $\tau_{\mathcal{D}}^{-1}[1]$. We represent by $\tilde{X}$ the class of an object X of $\mathcal{D}^b(H)$ in the cluster category. We recall that $\mathrm{Hom}_{\mathcal{C}}(\tilde{X}, \tilde{Y}) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{D}^b(H)}(X, F^i Y)$. We also recall that $S = \mathrm{ind} H \bigcup H[1]$ is a fundamental domain of $\mathcal{C}$. If X and Y are objects in the fundamental domain, then we have that $\mathrm{Hom}_{\mathcal{D}^b(H)}(X, F^i Y) = 0$ for all $i \neq 0, 1$. Therefore, by the definition of fundamental domain, any indecomposable object in $\mathcal{C}$ is of the form $\tilde{X}$ with $X \in S$.

We say that $\tilde{T}$ in $\mathcal{C}$ is a **tilting object** if $\mathrm{Ext}^1_{\mathcal{C}}(\tilde{T}, \tilde{T}) = 0$ and $\tilde{T}$ has a maximal number of non-isomorphic direct summands. A tilting object in $\mathcal{C}$ has a finite number of non isomorphic direct summands.

There exists the following nice correspondence between tilting modules and basic tilting objects. **Theorem** [5, Theorem 3.3.]

(a) *Let T be a basic tilting object in $\mathcal{C} = \mathcal{D}^b(H)/F$, where H is a hereditary algebra with n simple modules.*

    (i) *T is induced by a basic tilting module over a hereditary algebra H', derived equivalent to H.*

    (ii) *T has n indecomposable direct summands.*

(b) *Any basic tilting module over a hereditary algebra H induces a basic tilting object for $\mathcal{C} = \mathcal{D}^b(H)/F$.*

Let $\tilde{T}$ be a cluster tilting object in $\mathcal{C} = \mathcal{D}^b(H)/F$, if T is not a tilting module over H, then recall from the proof of [5, Theorem 3.3.] that it is possible to choose a different hereditary algebra H' derived equivalent to H such that T is a basic tilting module over H' and obviously $\mathcal{C} = \mathcal{D}^b(H')/F$.

In [6], Buan, Marsh and Reiten introduced the cluster-tilted algebras as follows. Let $\tilde{T}$ be a tilting object over the cluster category $\mathcal{C}$. We recall that B is a **cluster tilted algebra** if $B = \mathrm{End}_{\mathcal{C}}(\tilde{T})$. It is also known that, if $\tilde{T}$ is a tilting object in $\mathcal{C}$, the functor $\mathrm{Hom}_{\mathcal{C}}(\tilde{T}, \ )$ induces an equivalence of categories between $\mathcal{C}/\mathrm{add}(\tau \tilde{T})$ and $\mathrm{mod}\, B$. We will call this equivalence the [BMR]-equivalence.

Thus using the above equivalence, we can compute $\mathrm{Hom}_B(X', Y')$ in terms of the cluster category $\mathcal{C}$ as follows:

$$Hom_B(\mathrm{Hom}_{\mathcal{C}}(\tilde{T}, \tilde{X}), \mathrm{Hom}_{\mathcal{C}}(\tilde{T}, \tilde{Y})) \simeq \mathrm{Hom}_{\mathcal{C}}(\tilde{X}, \tilde{Y})/\mathrm{add}(\tau \tilde{T}),$$

where $X' = \mathrm{Hom}_{\mathcal{C}}(\tilde{T}, \tilde{X})$ and $Y' = \mathrm{Hom}_{\mathcal{C}}(\tilde{T}, \tilde{Y})$ are B-modules.

Remark 2.13. *Suppose $X' = Hom_{\mathcal{C}}(\tilde{T}, \tilde{X})$ and $Y' = Hom_{\mathcal{C}}(\tilde{T}, \tilde{Y})$. As direct consequence of the above stated we have that $Hom_B(X', Y') = 0$ if and only if for every $f : \tilde{X} \to \tilde{Y}$, f factors through $\mathrm{add}(\tau \tilde{T})$ in $\mathcal{C}$.*

3.   Resolutions for tame cluster tilted algebras

3.1. Note on finite torsion class hereditary algebras

In this section we will study the representation dimension of cluster tilted algebras given by a tilting module with finite torsion class. It follows from proposition

[2.8](#) that if H is an hereditary algebra of infinite representation type and T is a tilting module over H there is only two cases where the torsion class is finite. Either the algebra H is tame and T has no nonzero preprojectives summands or T is a preinjective tilting H-module. The cluster tilted algebras arising from the second case hypothesis are exactly the cluster concealed algebras. We known by [11] that such class of algebras has representation dimension three. Thus in this paper we will consider only the first case. We will show that the weak representation dimension of tame cluster tilted algebras is at most three.

3.2. Tame cluster tilted algebras

Let $B =\mathrm{End}_{\mathcal{C}}(\tilde{T})$ a tame cluster tilted algebra. By [5, Theorem3.3.], we know that $\tilde{T}$ is a cluster tilted object induced by a tilting module T over an hereditary algebra H where $\tilde{C}$ is the cluster category associated to H. We know that B is tame if and only if H is tame, therefore by [13] T has at least a non zero non regular summand. Moreover, due to the construction of cluster categories we can assume that all the non zero non regular indecomposables summands of T are either preprojective or preinjective. Thus by [13, Proposición (3.2),(3.2)*] we know that either $\mathcal{F}(T)$ is finite or $\mathcal{T}(T)$ is finite respectively.

In fact, if $\mathcal{T}$ is a cluster tilting object induced by an H-tilting module T with non zero preinjective and non zero regular summands such that $\mathcal{T}(T)$ is finite, then there exists a hereditary algebra H' derived equivalent to H and a H'-module T' with at least a non zero regular summand such that $\mathcal{F}(T')$ is finite in $\mathrm{mod}H'$. Even more, the cluster category induced by H and H' is the same and clearly T' induces the same cluster tilted object $\tilde{T}$ over $\mathcal{C}$. We will illustrate this situation in an example.

Example 3.1. *Let H the algebra given by the dynkin diagram $\tilde{\mathbb{D}}_5$. We consider the tilting module $T = \tau^{-2}P_5 \oplus \tau^{-2}P_6 \oplus \tau^{-2}P_4 \oplus \tau^{-2}P_1 \oplus \tau^{-2}P_2 \oplus S_3$. This module has 5 indecomposable preprojective summands and the indecomposable regular summand S_3.*

Let $B =End_{\mathcal{C}}(\tilde{T})$. *B is given by the following quiver:*
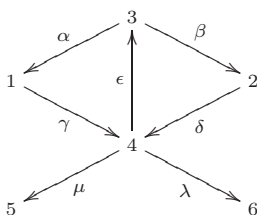
with the relations $\epsilon\alpha = \epsilon\beta = \gamma\epsilon = \delta\epsilon = 0$ and $\alpha\gamma = \beta\delta$.

In this example we have that the category $\mathcal{F}(T)$ is finite since T has no non zero preinjective summands.

Now consider the following tilting H-module $T' = \tau^2 I_5 \oplus \tau^2 I_6 \oplus \tau^2 I_4 \oplus \tau^2 I_1 \oplus \tau^2 I_2 \oplus S_3$. Unlike the previous module, here all the indecomposable summands

are preinjective with the exception of the regular summand S_3 and the torsion associated with this module is finite.

Finally, we can see that this module induces the same cluster tilted algebra $B = \text{End}_{\mathcal{C}}(\tilde{T}) \simeq \text{End}_{\mathcal{C}}(\tilde{T}')$ and therefore the algebra B can be studied beginning with any of the H-modules T or T' . As shown in this example here the algebras H and H' coincide. In general this is not necessarily true. One must consider all possible changes in orientation on the ordinary quiver of H in order to obtain an adequate algebra H'. Also note that since we only consider changes of orientation the algebra H' is always derived equivalent to H.

From now on, we will consider only tame cluster tilted algebra $\text{End}_{\mathcal{C}}(\tilde{T})$ is induced by a tilting module T such that every indecomposable summand of T is either preinjective or regular. Thus by lemma 2.6 we know that the category $\mathcal{T}(T)$ is finite. We write $T = T_I \oplus T_R$ where T_I is a preinjective module and T_R is a regular module. We recall that under these hypotheses T_I must be a nonzero summand of T. We will only consider the case when $T_R \neq 0$ as we discussed before.

Since we work under the assumption that $\mathcal{T}(T)$ is finite we can define the following H-module. Let $\mathcal{T}$ the direct sum of all indecomposable non isomorphic modules in $\mathcal{T}(T)$. Thus applying the $\hom_{\mathcal{C}}(\tilde{T}, \ )$ to $\mathcal{T}$ we obtain the B-module $\mathcal{T}' = \text{Hom}_{\mathcal{C}}(\tilde{T}, \tilde{\mathcal{T}})$. We will show that we only need the modules $\mathcal{T}'$ and H' to get an approximation of all, except for finitely many, indecomposable modules in mod B.

Since our objective is to compute the weak representation dimension of B, we need to construct a generator module M' for mod B.

We start constructing a transjective object $\tilde{M}_1$ in $\mathcal{C}$. First, let $\mathcal{Y}$ be the the direct sum of all indecomposable preinjective modules, up to isomorphisms, in $\mathcal{T}(T)$. Recall that if S is a tilting module given by a complete slice $\mathcal{S}$ then an indecomposable module belongs to $\mathcal{T}(S)$ if and only if is a successor of $\mathcal{S}$. Since $\mathcal{T}(T)$ is finite, it is possible to choose a preinjective slice $\mathcal{S}$ such that $\mathcal{Y}$ belongs to $\mathcal{T}(S)$. Since $\mathcal{S}$ is in the preinjective component then $\mathcal{T}(S)$ is also finite. We fix a preinjective slice $\mathcal{S}$ with such property. Hence we can define the following module.

Let $N_1 = \oplus_{X \in \mathcal{T}(S)} X$ with X indecomposable. Let $\tilde{M}_1 = \tilde{N}_1 \oplus \tilde{H} \oplus \tilde{H}[1]$. Finally we set $M'_1 = \hom_{\mathcal{C}}(\tilde{T}, \tilde{M}_1)$.

We have the following alternative description for M'_1. Let H_1 be the hereditary algebra $\tau_{\mathcal{D}}^{-2} H$. Then, the B-module M'_1 is obtained by applying the functor $\hom_{\mathcal{C}}(\tilde{T}, \ )$ to the object $\tilde{M}_1$ in $\mathcal{C}_{H_1}$. Here M_1 is the H_1-module given by $\oplus_{X \in \mathcal{T}(\Sigma)} X$ where Σ is the same complete slice choosen for mod H but instead considered as H_1-module and $\mathcal{T}_{H_1}(\Sigma) \subset \text{mod } H_1$. Note that the object M_1 in $D^b(H)$ is obtained as the direct sum of all indecomposable objects which are both successors of Σ and predecessors of the slice $\tau_{\mathcal{D}}^{-1} H[1]$. This mean that the B-module M'_1 is between two local slices (see [1]) in mod B. This way to visualize the module M'_1 is not only more simple but will also help to simplify some technical details in some proofs.

Before we complete the definition of the module M', we need to establish the following definition:

Definition 3.2. *Let E be a regular module in a stable tube $\mathcal{T}_\lambda$. We will call **edge of***

the cone $\mathcal{C}(E)$ *and denoted by* $\mathcal{C}(E \to)$ *to the intersection of the coray containing* E *in* $\mathcal{T}_\lambda$ *and the cone* $\mathcal{C}(E)$.

Note that the edge of the cone cogenerates all modules in $\mathcal{C}(E)$ and that E generates all modules in the edge of the cone.

Let $T_1,...,T_n$ be all the indecomposables regular summands of T_R. For each T_i we set $\mathcal{C}(T_i)$, $i = 1, ..., n$ the cone of T_i. We have the following observations:

i) Every summand $T_1,...,T_n$ belong to at least one of the cones $\mathcal{C}(T_i)$ but possible more than one.

ii) Every **cone** is obviously determined by an indecomposable summand of T_R.

iii) The edge of the cone $\mathcal{C}(T_i \to)$ is entirely contained in $\mathcal{T}(T)$ for every $i = 1, ..., n$.

We define W_i as the regular H-module given by $\bigoplus_{X \in \text{ind}(\mathcal{C}(T_i))} X$.

Since we are assuming that T_R has n regular indecomposable summands, then we have r modules $W_1,...,W_r$ with the above properties. Moreover , we can choose by lemma 2.10 regular summands of T, (rearranging the index of the summands) $T_1, ..., T_m$ such that we can replace i) by:

i') Every one of the summands $T_1,...,T_n$ belongs to one and only one of the cones $\mathcal{C}(T_j)$ for $j = 1, ..., m$.

We call this cones **maximal** respect to property i). More over, it is possible to determine the exact amount of indecomposable summands of T in each of this cones by only knowing the level of the vertex T_j in the stable tube.

Let M_2' be $(W_1)' \oplus ... \oplus (W_m)'$ where each $(W_j)'$ denotes the image of the object $\tilde{W}_j$ by the equivalence $\text{hom}_{\mathcal{C}}(\tilde{T}, \ )$. Note that this module is not transjective neither regular in mod B since it have at least an indecomposable projective direct summand which does not lies in the transjective component of B neither in an stable tube of mod$B$ (therefore is not a regular module).

Finally, the B-module M' is given by $M' = M_1' \oplus M_2'$. We will show that M' satisfies (a) of definition 2.1 for $i = 3$.

3.3. The approximation resolution

The aim of this subsection is to study when is possible to obtain an exact sequence:

$$0 \to M_1 \to M_0 \xrightarrow{f'} X' \to 0$$

for each X' in ind B where $M_i \in \text{add}(M')$ for $i = 0, 1$.

If $X' \in \text{add}(M')$, the existence of such sequence is trivial. Recall that any indecomposable module X' in mod B can be written as $X' = \text{Hom}_{\mathcal{C}}(\tilde{T}, \tilde{X})$ with X in ind $H \bigcup H[1]$ the fundamental domain of $\mathcal{C}$. Since $H[1]$ is a direct summand of

M', we will only study the case where $X' = \operatorname{Hom}_{\mathcal{C}}(\tilde{T}, \tilde{X})$ with $X \in \operatorname{ind} H$, it is, X' will always be obtained from a H-module X.

Let X be in $\operatorname{ind} H$. Associated to X we have the modules tX, X/tX and the exact sequence given by

$$0 \to tX \xrightarrow{i} X \xrightarrow{p} X/tX \to 0.$$

We will construct a new short exact sequence from a minimal projective resolution of X/tX.

Let

$$0 \to K \to P_{X/tX} \xrightarrow{\pi} X/tX \to 0$$

be a minimal projective resolution of X/tX then $\pi : P_{X/tX} \to X/tX$ is a projective cover of X/tX.

Since $p : X \to X/tX$ is an epimorphism, there exists a morphism $g : P_{X/tX} \to X$ such that $pg = \pi$. We set $f := \left(\begin{array}{cc} i, & g \end{array} \right)$ and we have the following commutative diagram:

$$
\begin{array}{ccccccccc}
& & 0 & & & & 0 & & \\
& & \uparrow & & & & \uparrow & & \\
0 & \longrightarrow & tX & \xrightarrow{\ i\ } & X & \xrightarrow{\ p\ } & X/tX & \longrightarrow & 0 \\
& & \uparrow{\scriptstyle Id} & & \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle \pi} & & \\
0 & \longrightarrow & tX & \longrightarrow & tX \oplus P_{X/tX} & \longrightarrow & P_{X/tX} & \longrightarrow & 0 \\
& & \uparrow & & & & \uparrow & & \\
& & 0 & & & & K & & \\
& & & & & & \uparrow & & \\
& & & & & & 0 & &
\end{array}
$$

Moreover, is easy to check that the columns and rows are exact sequences.

Applying the snake lemma, we obtain a short exact sequence:

$$0 \to K \to tX \oplus P_{X/tX} \xrightarrow{f} X \to 0. \tag{3.1}$$

Our objective is to study when this short exact sequences remains exact in mod B. That is, we will show under which hypothesis this short exact sequence induces a short exact sequence

$$0 \to K' \to (tX)' \oplus (P_{X/tX})' \xrightarrow{f'} X' \to 0$$

in mod B.

Proposition 3.3. *Let $X' = Hom_{\mathbb{C}}(\tilde{T}, \tilde{X})$ with $X \in mod\ H$. Then the morphism $f : tX \oplus P_{X/tX} \xrightarrow{f} X$ induces an epimorphism $(tX)' \oplus P'_{X/tX} \to X'$. More over, if $X \notin \mathcal{T}(T)$ then the sequence*

$$0 \to K' \to (tX)' \oplus P'_{X/tX} \to X' \to 0$$

is exact in modB.

Before we proceed with the proof of proposition 3.3, we will discuss some details that show up in the proof.

First, we need to assure when a morphism in $\mathcal{C}$ induces a non zero morphism in mod B. Recall that

Remark 3.4. *If $X' = Hom_{\mathbb{C}}(\tilde{T}, \tilde{X})$ and $Y' = Hom_{\mathbb{C}}(\tilde{T}, \tilde{Y})$ are indecomposable in mod B, then a morphism f' is non zero if and only if the morphism $\tilde{f} : \tilde{X} \to \tilde{Y}$ does not factorizes through $add(\tau\tilde{T})$ in $\mathcal{C}$.*

The next theorem gives us necessary and sufficient condition to determine when the morphism $h : \tilde{X} \to \tilde{E}[1]$ factors through $add(\tau\tilde{T})$ in $\mathcal{C}$ where E is an arbitrary H-module.

Theorem 3.5. *Let T be a tilting module over modH without no non zero preprojective direct summand and let Σ be a preprojective complete slice in mod H. Let X be an H-module such that $\tau X \in \mathcal{T}(\Sigma)$ and there exists a triangle*

$$E \to M \to X \xrightarrow{h} E[1]$$

in $D^b(H)$ with E in add (Σ).

Therefore the induced morphism $h : \tilde{X} \to \tilde{E}[1]$ factors through $add(\tau\tilde{T})$ in $\mathcal{C}$ if and only if there exists a morphism $k : X \to T'$ with $T' \in add(\tau\tilde{T})$ such that $hi = 0$ where i is the canonic inclusion of Ker k in X.

Proof:

Let X be an H-module that does not belong to $add(\Sigma)$ and assume that we have a triangle

$$E \to M \to X \xrightarrow{h} E[1].$$

Proceeding as in [11, proposition 3.2], let $H_1 = \tau_{\mathcal{D}}^{-1}\Sigma$, it is easy to see that $X, \tau T, T$ can be identified with H_1-modules and that the slice $\Sigma[1]$ can be identified with the complete slice formed by the indecomposable injective H_1-modules.

Suppose that we have a non zero morphism $k : X \to T'$ with $T' \in add(\tau T)$. Let K be the kernel of k, Ker k and let i be the inclusion of kernel of k in X, then we have that

$$0 \to K \xrightarrow{i} X \xrightarrow{k} T'$$

is exact in mod H_1. We have that $E[1]$ is an injective module considered as H_1-module, then the sequence:

$$\operatorname{Hom}_{H_1}(T', E[1]) \xrightarrow{k^*} \operatorname{Hom}_{H_1}(X, E[1]) \xrightarrow{i^*} \operatorname{Hom}_{H_1}(Kerk, E[1]) \to 0$$

is exact in mod H_1, where $k^* = \operatorname{Hom}_{H_1}(k, E[1])$ e $i^* = \operatorname{Hom}_{H_1}(i, E[1])$. Since $Imk^* = \operatorname{Ker} i^*$, $hi = 0$ if and only if exists a non zero morphism, $l : T' \to E[1]$ such that $h = kl$. Therefore, we have the following diagram:

Hence, h factors through $\operatorname{add}(\tau T)$ if and only if $hi = 0$. □

This gives us a necessary and sufficient condition under which the exact sequence 3.1 induces a short exact sequence such as the one described in proposition 3.5.

Remark 3.6. *Recall from [11], that an epimorphism $f : M \to N$ in mod H induces an epimorphism in mod B if and only if all morphism $\tilde{h} : \tilde{N} \to \tau ker\tilde{f}$ factors through $\operatorname{add}(\tau \tilde{T})$ in the cluster category $\mathcal{C}$.*

Therefore, as an immediate consequence of the previous theorem we can conclude that any triangle of the form $K \to E \to X \to K[1]$ satisfying the condition described in 3.6, induces an epimorphism $E' \to X'$ in mod B. We may establish a dual version of this theorem which gives conditions necessary and sufficient for a triangle induces a monomorphism in mod B.

Note that the condition required in the Theorem 3.5, $hi = 0$, it is satisfied trivially if X' is of the form $\operatorname{Hom}_{\mathcal{C}}(\tilde{T}, X)$, with $X \in \mathcal{F}(T)$. In fact, since $\mathcal{F}(T) = \operatorname{Cogen}(\tau T)$, exists a monomorphism $k : X \to T'$ with $T' \in \operatorname{add}(\tau T)$. Hence, $\operatorname{Ker} k = 0$ and therefore $i = 0$. Which implies that $hi = 0$ for every h. This shows that in this particular case, the sequence described in the Proposition 3.3 is similar to the described in [11, proposition 3.2].

We proceed with proof of proposition 3.3.

Proof of proposition 3.3:

Let X' be such that X is a non projective indecomposable H-module which does not belong to $\mathcal{T}(T)$. Observe that if X is projective we have that $X' \in \operatorname{add}(M')$ and there is nothing to prove.

Recall that, in the beginning of this section, we had constructed an exact sequence in mod H of the form:

$$0 \to K \to tX \oplus P_{X/tX} \xrightarrow{f} X \to 0$$

.

This exact sequences induces a triangle in $\mathcal{D}^b(H)$:

$$K \to tX \oplus P_{X/tX} \to X \xrightarrow{h} K[1].$$

We want to show that this triangle induces an exact sequence in mod B. We will prove that $f' : (tX)' \oplus P'_{X/tX} \to X'$ is an epimorphism. As we state before it suffices to show that $h : \tilde{X} \to \tau \tilde{K}$ factors through $\mathrm{add}(\tau \tilde{T})$ in $\mathcal{C}$.

Since $\mathrm{Cogen}(\tau T) = \mathcal{F}(T)$, and $X/tX \in \mathcal{F}(T)$ we know that there exists a monomorphism $j : X/tX \to T'$ with $T' \in \mathrm{add}(\tau T)$. Since we assume that $X \notin \mathcal{T}(T)$, $X/tX \neq 0$ and therefore $p : X \to X/tX$ is a non zero morphism. Hence, the composition jp gives rise to a non zero morphism $k : X \to T'$. More over, note that $\mathrm{Ker}\, k = tX$. let $i : tX \to X$ be the canonical inclusion. Recall that K is the kernel of the projective cover of X/tX and hence is a projective module. Let Σ be the projective complete slice, by Theorem 3.5, it suffices to prove that the composition $hi : tX \to K[1]$ is zero.

From the exact sequence (3.1) , we have that $h \left(\begin{array}{cc} i, & g \end{array} \right) = \left(\begin{array}{cc} hi, & hg \end{array} \right) = 0$. Hence $hi = hg = 0$.

Therefore $f' : (tX)' \oplus P'_{X/tX} \to X'$ is an epimorphism in mod B.

We will show that this epimorphism can be completed to an exact sequence:

$$0 \to (K)' \to (tX)' \oplus P'_{X/tX} \to X' \to 0$$

with $K' \in \mathrm{add}(M')$.

First, we start with the minimal projective resolution of X in mod H:

$$0 \to K \xrightarrow{j} tX \oplus P_{X/tX} \to X \to 0$$

This short exact sequence induces the following triangle:

$$\tilde{K} \xrightarrow{\tilde{j}} t\tilde{X} \oplus \tilde{P}_{X/tX} \to \tilde{X}$$

in the cluster category $\mathcal{C}$.

Applying the functor $\mathrm{Hom}_C(\tilde{T}, \ )$ to this triangle we obtain the following sequence in $\mathcal{C}/\mathrm{add}(\tau \tilde{T})$:

$$\mathrm{Hom}_C(\tilde{T}, \tilde{K}) \xrightarrow{\tilde{j}^*} \mathrm{Hom}_C(\tilde{T}, t\tilde{X} \oplus \tilde{P}_{X/tX}) \to \dots$$

We will show that the morphism $\tilde{j}^*$ given by composing with $\tilde{j}$, it is, $\tilde{j}^* = \mathrm{Hom}_C(\tilde{T}, \tilde{j})$ is a monomorphism.

Let $\tilde{h} \in \mathrm{Hom}_C(\tilde{T}, \tilde{K})$ such that $\tilde{h} \in \mathrm{Ker}\, \tilde{j}^*$, it is, $\tilde{j}\tilde{h} = 0$. Since $\tilde{h} \in \mathrm{Hom}_C(\tilde{T}, \tilde{K})$ we know that $\tilde{h} = (h_1, h_2)$ with $h_1 : T \to K$ y $h_2 : T \to \tau^{-1}K[1]$. Since we choose T without preprojective summands, then $\mathrm{Hom}_H(T, K) = 0$ and $h_1 = 0$. Hence $\tilde{h} = h_2 : T \to \tau^{-1}K[1]$.

Since $Imj \subset P_{X/tX}$, the composition $\tilde{j}\tilde{h} \in \mathrm{Hom}(\tilde{T}, \tilde{P}_{X/tX})$. More over, the morphism $\tilde{j}\tilde{h}$ is induced by the morphism $Fjh_2 : T \to \tau^{-1}[1] \to \tau^{-1}P_{X/tX}[1]$ in $\mathcal{D}^b(H)$, where $F = \mathrm{Hom}_C(\tilde{T}, \ )$

By hypothesis $\tilde{j}\tilde{h} = 0$ in $\mathcal{C}/\mathrm{add}(\tau T)$, therefore $Fjh_2 : T \to \tau^{-1}P_{X/tX}[1]$ must factor through τT en $\mathcal{D}^b(H)$. But, since T is a tilting module, $\mathrm{Hom}_H(T, \tau T) = 0$ and thus this factorization implies that $Fjh_2 = 0$.

We will prove that Fj is a monomorphism in mod $\tau_D^{-2}H$.

First, we will show that $\tau^{-1}j$ is a monomorphism. Let $E_X = tX \oplus P_{X/tX}$
Applying D the dual functor to the short exact sequence (3.1) we obtain:

$$0 \to DX \xrightarrow{j} DE_X \to DK \to 0$$

Applying $\mathrm{Hom}(\quad, H)$ to the last sequence we have:

$$\mathrm{Hom}(DK, H) \to \mathrm{Hom}(DE_X, H) \to \mathrm{Hom}(DX, H) \to$$

$$\to \mathrm{Ext}^1(DK, H) \to \mathrm{Ext}^1(DE_X, H)$$

Since X has no injective summands (Recall that $X \notin \mathfrak{T}(T)$) therefore DX
has no projective summands and hence $\mathrm{Hom}(DX, H) = 0$. This implies that
$\mathrm{Ext}^1(DK, H) \to \mathrm{Ext}^1(DE_X, H)$ is a monomorphism and since DK and DE_X
has no projective summands we conclude that $\mathrm{Ext}^1(DK, H) = trDK = \tau^{-1}K$
and $\mathrm{Ext}^1(DE_X, H) = trDE_X = \tau^{-1}E_X$. Hence $\tau^{-1}j : \tau^{-1}K \to \tau^{-1}E_X$ is a
monomorphism and $Fj : \tau^{-1}K[1] \to \tau^{-1}P_{X/tX}[1]$ is a monomorphism in mod
$\tau_{\mathcal{D}}^{-2}H$.

Finally, we get that $h_2 = 0$ since Fj is a monomorphism and $Fjh_2 = 0$.
Therefore $\tilde{h} = 0$ and $\tilde{j}^*$ is a monomorphism.

Thus we have proved that the sequence:

$$0 \to (K)' \to (tX)' \oplus P'_{X/tX} \xrightarrow{f} X' \to 0$$

is exact in mod B □.

Last proposition shows that the chosen module M' satisfies the first part of
property (a) on the definition 2.1 for every $X' \in \mathrm{mod}\ B$ of the form $X' =$
$\mathrm{Hom}_{\mathcal{C}}(\tilde{T}, \tilde{X})$ such that $X \notin \mathfrak{T}(T)$. But, since we are only considering the case
in that $\mathfrak{T}(T)$ is finite, then we only have a finite number of indecomposable mod-
ules (up to isomorphism) which are not of this form. Let $\mathcal{G}' = \oplus_{\mathrm{ind}\ B} X'$ such that
$X' = \mathrm{Hom}_{\mathcal{C}}(\tilde{T}, \tilde{X})$ with $X \in \mathfrak{T}(T)$.

We will show that $\mathcal{G}'$ is in fact a direct summand of M' and thus we have a
short exact sequence:

$$0 \to M_1 \to M_0 \xrightarrow{f'} X' \to 0$$

with $M_i \in \mathrm{add}(M')$, $i = 0, 1$ for every $X' \notin \mathrm{add}(M')$ in mod B.

Theorem 3.7. *Let H be a tame hereditary algebra and T a tilting H-module
without preprojective direct summands. Let X be a non zero indecomposable regular
module in $\mathfrak{T}(T)$. Then there exists an indecomposable regular summand of T, E,
such that $X \in \mathcal{C}(E)$.*

Proof:

Suppose X is a non zero indecomposable regular module in $\mathfrak{T}(T)$. Then since H
is tame there exists a stable tube $\mathfrak{T}_\lambda$ such that $X \in \mathfrak{T}_\lambda$. More over, since T has no
preprojective summands and any two regular components of the Auslander-Reiten

quiver of H are orthogonal, the fact that $X \in \mathcal{T}(T)$ implies that there is at least an indecomposable regular summand of T in $\mathcal{T}_\lambda$. Otherwise, $\mathrm{Hom}_H(T, \mathcal{T}_\lambda) = 0$, which is a contradiction to the fact that $X \in \mathcal{T}(T) \bigcap \mathcal{T}_\lambda$.

Let $T_1, ..., T_r$ be the indecomposable regular summands of T in $\mathcal{T}_\lambda$. By Lemma 2.10, we know that there are summands $E_1, ..., E_m$ of T with every other summand $T_i \in \mathcal{C}(E_j)$ for some j and the cones $\mathcal{C}(E_j), j = 1, ..., m$ are maximal with respect to this property. It is, $\mathcal{C}(E_j) \bigcap \mathcal{C}(E_i) = \emptyset$ if and only if $i \neq j$.

If $X \in \mathcal{C}(E_j)$ for some j, we set $E = E_j$. Otherwise, suppose X does not belong to any of this cones. We will show that there is only one of the maximal cones $\mathcal{C}(E_j)$ such that $\mathrm{Hom}_H(\mathcal{C}(E_j), X)$ is not zero.

Assume that $\mathrm{Hom}_H(\mathcal{C}(E_{j_1}), X)$ and $\mathrm{Hom}_H(\mathcal{C}(E_{j_2}), X)$ are both non zero. Since $\mathrm{Hom}_H(\mathcal{C}(E_{j_1}), X) \neq 0$ there exists an indecomposable module $N_1 \in \mathcal{C}(E_{j_1})$ in the mouth of the stable tube $\mathcal{T}_\lambda$ such that X belongs to the ray starting in N_1. Analogously, since $\mathrm{Hom}_H(\mathcal{C}(E_{j_2}), X) \neq 0$ there exists an indecomposable module $N_2 \in \mathcal{C}(E_{j_2})$ in the mouth of the stable tube $\mathcal{T}_\lambda$ such that X belongs to the ray starting in N_2. Then $N_1 = N_2$ because there is a unique ray in $\mathcal{T}_\lambda$ containing X. Therefore $X \in \mathcal{C}(E_{j_1}) \bigcap \mathcal{C}(E_{j_2}) \neq \emptyset$ and by Lemma 2.10 $\mathcal{C}(E_{j_1}) = \mathcal{C}(E_{j_2})$.

Since $X \in \mathcal{T}(T)$ we have that $\mathrm{Hom}_H(\mathcal{C}(E_j), X) \neq 0$ for some j. Let E be the direct summand of T such that $\mathrm{Hom}_H(\mathcal{C}(E), X) \neq 0$. Let $N \in \mathcal{C}(E)$ be the indecomposable module in the mouth of the stable tube $\mathcal{T}$ such that X is contained in the ray starting in N. We will call this ray $\mathcal{R}$. Since $X \in \mathcal{T}(T)$ there exists an epimorphism $h : T^n \to X$. We are assuming that X does not belong to the cone $\mathcal{C}(E)$. Using the lifting property of the Auslander-Reiten sequences the morphism h must factorize by a module C in the mouth of the stable tube $\mathcal{T}_\lambda$, by a module M_1 in the edge of the cone $\mathcal{C}(E)$, such that there exists an epimorphism $E \to M_1$, or a combination of both of this cases. More over, using the mesh relations of the Auslanter-Reiten quiver, specifically the mesh of $\mathcal{T}_\lambda$ we have:

(a) If h factors through the mouth of $\mathcal{T}_\lambda$, we may assume that $C = N$ since X belongs to the ray starting in N. By [12, Lema 3.7] h must factor through $\mathcal{R}$, but since $X \notin \mathcal{C}(E)$, there exists a module M in $\mathcal{C}(E \to)$ such that $M \in \mathrm{Gen}$ $(E) \bigcap \mathcal{R}$.

(b) If h factors through a module in the edge of $\mathcal{C}(E)$, it is, $\mathrm{Gen}(E) \bigcap \mathcal{C}(E)$, then h should factor through a module in the same ray of X.

We conclude that in any case the morphism h must factor through M and therefore h applies in X through a composition of morphisms in the ray $\mathcal{R}$. Which is a contradiction, since h is an epimorphism and all morphisms morphisms in the ray are monomorphism. Hence, $X \in \mathcal{C}(E)$. $\square$

Finally, we will apply the last theorem to prove that $\mathcal{G}'$ is a summand of M'. In fact, suppose that G' is an indecomposable direct summand of $\mathcal{G}'$. Then $G' = \mathrm{Hom}_\mathcal{C}(\tilde{T}, \tilde{G})$ with G an indecomposable H-module such that $G \in \mathcal{T}(T)$. Therefore we have only two possibilities for G, it is a regular or it is a preinjective module.

If G is preinjective, is easy to see from the choose of the complete slice Σ that $G \in \mathcal{T}(\Sigma)$ and therefore G' is a direct summand of M_1'.

If G is regular, by Theorem 3.7, there exists a direct summand of T, E such that $G \in \mathcal{C}(E)$. Therefore $G \in \mathcal{C}(E)$. Then it follows from the definition of M_2' that G' results a direct summand of M'.

Note that in the proof of theorem 3.7 we have proved the following lemma:

Lemma 3.8. *Let $M \in \mathcal{C}(E)$ and $N \notin \mathcal{C}(E)$. Every non zero morphism from $M \to N$ factors trough $\mathcal{C}(E \to)$.*

4. Weak representation dimension of tame cluster tilted algebras

For each B-module $X' \notin \mathrm{add}(M')$, we will call M_X' the B-module $P_{X/tX}' \oplus (tX)'$. In the previous section, we have constructed an epimorphism $f' : M_X' \to X'$. We will begin this section studying the properties of this morphism. In particular, we want to show that the following result holds:

Theorem 4.1. *The epimorphism $f' : M_X' \to X'$ constructed in Proposition 3.3 is an $\mathrm{add}(M')$-approximation of X'.*

We will concentrate for the rest of this section in proving this result. In order to do this, we will study the behavior of the morphism f' in relation of the functor $\mathrm{Hom}_B(M'', \ )$, where M'' is in $\mathrm{add}(M')$.

We begin by showing the following lemma.

Lemma 4.2. *The morphism $f' : M_X' \to X'$ is an $\mathrm{add}(H' \oplus \mathcal{T}')$-approximation for every $X' \notin \mathrm{add}(M')$.*

Proof:

We are going to split this proof in the following two cases:

a) any morphism of B-modules from $\mathrm{add}(H')$ to X' factors through f'.

b) any morphism of B-modules from $\mathrm{add}(\mathcal{T}')$ to X' factors through f'.

We start with the first case. Let P' be in $\mathrm{add}(H')$ and let $0 \neq h' : P' \to X'$. Without lose of generality, we may assume that $P' = \mathrm{Hom}_{\mathcal{C}}(\tilde{T}, \tilde{P})$ with P an H-module projective indecomposable. Then, there exists $\tilde{h} \in \mathrm{Hom}_{\mathcal{C}}(\tilde{P}, \tilde{X})$ such that $h' = \mathrm{Hom}_{\mathcal{C}}(\tilde{T}, \tilde{h})$.

In fact, we can identify $\tilde{h}$ with a morphism t of H-modules. Since P is projective and X in mod H we have that $\mathrm{Hom}_{\mathcal{D}}(P, \tau^{-1}X[1]) = 0$, therefore $\mathrm{Hom}_{\mathcal{C}}(\tilde{P}, \tilde{X}) = \mathrm{Hom}_{\mathcal{D}}(P, X) = \mathrm{Hom}_H(P, X)$. Let $t : P \to X$ such that $\tilde{t} = \tilde{h}$.

Consider the canonical exact sequence in mod H and the composition of the morphisms p and t:

$$
\begin{array}{ccccccccc}
& & & & P & & & & \\
& & & & \downarrow{\scriptstyle t} & & & & \\
0 & \longrightarrow & tX & \overset{i}{\longrightarrow} & X & \overset{p}{\longrightarrow} & (X/tX) & \longrightarrow & 0
\end{array}
$$

Since P is projective, the morphism $pt : P \to X/tX$ must factor through the minimal projective cover of X/tX, it is, factors through $\pi : P_{X/tX} \to X/tX$, the projective cover of X/tX, which implies that there exists $j : P \to P_{X/tX}$ such that $pt = \pi j$.

Recall that by the construction of g, $\pi = pg$. Hence, we have that $pt = pgj$. Therefore $p(t - gj) = 0$ and by the kernel universal property the morphism $t - gj$ must factors through $\operatorname{Ker} p = tX$. Hence, there exists $k : P \to tX$ such that $t - gj = ik$. Thus, we have proof that $t = gj + ik = f \begin{pmatrix} k \\ j \end{pmatrix}$, it is, t factors through M_X in mod H.

We define $l := \begin{pmatrix} k \\ j \end{pmatrix}$ and let $l' = \operatorname{Hom}_C(\tilde{T}, \tilde{l})$. Then is easy to see that $h' = f'l'$. Note that l' must be a non zero morphism in mod B, since otherwise h' it would be null. This completes the proof of lemma corresponding to the case (a).

Lets focus on case (b). Let G be in $\operatorname{add}(\mathcal{T}')$, we want to show that $h' : G' \to X'$ it factors trough $f' : M'_X \to X'$.

Suppose that we have $h' : G' \to X'$. Then exists a morphisms $\tilde{h} \in \operatorname{Hom}_{\mathcal{C}}(\tilde{G}, \tilde{X})$ such that $h' = \operatorname{Hom}_{\mathcal{C}}(\tilde{T}, \tilde{h})$. We know that

$$\operatorname{Hom}_{\mathcal{C}}(\tilde{G}, \tilde{X}) = \operatorname{Hom}_{\mathcal{D}}(H)(G, X) \oplus \operatorname{Hom}_{\mathcal{D}}(H)(G, \tau^{-1}X[1]).$$

Let $\tilde{h} = (h_1, h_2) : \tilde{G} \to \tilde{X}$ with $h_1 : G \to X$ y $h_2 : G \to \tau^{-1}X[1]$. It suffices to show that this morphism factors through $\tilde{f} = (f, 0)$ where $f : M_X \to X$ in $\operatorname{mod} H$. It is, there exists $\tilde{g} = (g_1, g_2)$ such that $\tilde{g}\tilde{f} = \tilde{h}$.

We will show that there exists a morphism $\tilde{h} = (h_1, h_2)$ such that the following diagram results commutative:

$$
\begin{array}{c}
M_X \oplus FM_X \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & Ff \end{pmatrix}} X \oplus FX \\
{\scriptstyle (g_1, g_2)} \Big\uparrow \quad {\scriptstyle (h_1, h_2)} \nearrow \\
G
\end{array}
$$

Then we will have that $\tilde{g}\tilde{f} = (g_1, g_2) \begin{pmatrix} f & 0 \\ 0 & Ff \end{pmatrix} = (g_1 f, Ff g_2)$, so it suffices to prove that any morphism h en $\mathcal{D}^b(H)$, such that $h : G \to X$ it factors trough f in $\mathcal{D}^b(H)$ and any morphism $h : G \to \tau^{-1}X[1]$ it factors through Ff in $\mathcal{D}^b(H)$.

Suppose $h : G \to X$. Since $G \in \mathcal{T}(T)$, $\operatorname{Hom}_H(G, X) = \operatorname{Hom}_H(G, tX)$ then we have that h factors through tX trivially. Therefore, h' factors trivially over $(tX)'$ which is a summand of M'_X. This concludes the proof for this case.

Now, assume that we have a morphism in $\mathcal{D}^b(H)$, $h : G \to FX$. We will show that this case can be reduced to case (a).

We start with a projective resolution of X in mod H

$$0 \to K_X \to P_X \xrightarrow{\pi_X} X \to 0,$$

This exact sequence induces a triangle in $\mathcal{D}^b(H)$:

$$K_X \to P_X \overset{\pi_X}{\to} X \to K_X[1]$$

Now, applying the functor F to this triangle, we obtain another triangle:

$$FK_X \to FP_X \overset{F\pi_X}{\to} FX \to FK_X[1]$$

Finally, we apply the functor $\mathrm{Hom}_{\mathcal{D}}(G,\ )$ to the last triangle to obtain the long exact sequence:

$$\cdots \to \mathrm{Hom}_{\mathcal{D}}(G, FP_X) \overset{(F\pi_X)^*}{\to} \mathrm{Hom}_{\mathcal{D}}(G, FX) \to \mathrm{Hom}_{\mathcal{D}}(G, FK_X[1]) \to \cdots$$

where the morphism $(F\pi_X)^*$ denotes the composition by $(F\pi_X)$.

Note that $\mathrm{Hom}_{\mathcal{D}}(G, FK_X[1]) = \mathrm{Ext}_H^2(G, \tau^{-1}K_X) = 0$ since H is an hereditary algebra. Then, the morphism $(F\pi_X)^*$ is an epimorphism. Therefore, since $h : G \to FX$, there exists $k : G \to FP_X$ such that $h = (F\pi_X)k$.

We have prove that h factors through FH and we have a morphism $Fq : FP_X \to FX$ where $P_X \in \mathrm{add}(H')$. Then proceeding similarly to the case (a) over $q : P_X \to X$ we have that q' and therefore h' factors through M'_X.

$\square$

This lemma will be helpful to show that the morphism f' is an $\mathrm{add}(M')$-approximation for every B-module X'.

Proposition 4.3. *Let $X' \notin \mathrm{add}(M')$ and let $f' : M'_X \to X'$. Then any morphism of $\mathrm{add}(M')$ to X' it factors through f'.*

Proof:

We will split the proof in two cases:

a) Any morphism from $\mathrm{add}(M'_1)$ to X' factors through f'.

b) Any morphism from $\mathrm{add}(M'_2)$ to X' factors through f'.

a) Let Y' be a module indecomposable in $\mathrm{add}(M'_1)$. Consider H_1 the algebra hereditary derived equivalent to H, given by identifying the complete slice Σ in mod H with the complete projective slice modH_1. We can assume that $Y' = \mathrm{Hom}_{\mathcal{C}}(\tilde{T}, \tilde{Y})$ with Y an H_1-module indecomposable in $\mathcal{F}(FH)$ identifying FH with the corresponding tilting module in mod H_1.

Let $0 \neq h' : Y' \to X'$. Then, there exists $\tilde{h} \in \mathrm{Hom}_{\mathcal{C}}(\tilde{Y}, \tilde{X})$ such that $h' = \mathrm{Hom}_{\mathcal{C}}(\tilde{T}, \tilde{h})$. Since $X' \notin \mathrm{add}(M')$, we may assume that X is a module in $\mathcal{F}(\Sigma) \subset$ mod$H$. Therefore $\mathrm{Hom}_H(Y, X) = 0$ and $\mathrm{Hom}_{\mathcal{C}}(\tilde{Y}, \tilde{X}) = \mathrm{Hom}_{\mathcal{D}}(Y, \tau^{-1}X[1])$.

Let $h : Y \to \tau^{-1}X[1]$. As we saw before, we know that it is possible to identify Y with a preprojective H_1-module with $Y \in \mathcal{F}(FH)$.

Since we suppose that $X' \notin \operatorname{add}(M')$ then we can assume that $X \in \mathcal{F}(\Sigma)$. We have two cases to analyze: $\tau^{-1}X \notin \operatorname{add}(\Sigma)$ and then $\tau^{-1}X \in \mathcal{F}(\Sigma)$ or $\tau^{-1}X \in \operatorname{add}(\Sigma)$.

If $\tau^{-1}X \notin \operatorname{add}(\Sigma)$ it is also possible to identify $\tau^{-1}X[1]$ with an H_1-module. Moreover, we have that $\tau^{-1}X[1] \in \mathcal{T}(FH)$ as H_1-module and that $Y \in \mathcal{F}(FH)$ therefore, since FH is a complete slice in $\operatorname{mod} H_1$ any morphism from Y to $\tau^{-1}X[1]$, in particular h must factor through FH. Thus, the morphism h' in $\operatorname{mod} B$ factors through a B-module in $\operatorname{add}(H')$. Then the proof of this cases can be deduced from case (a) of Lemma 4.2.

It only left to see what happens with the case $\tau^{-1}X \in \operatorname{add}(\Sigma)$. Suppose that $Y \in \operatorname{add}(\Sigma)$, then $\operatorname{Hom}_{\mathcal{D}}(Y, \tau^{-1}X[1]) = \operatorname{Ext}^1_H(Y, \tau^{-1}X) = 0$ since Σ is a tilting module and therefore $\operatorname{Ext}^1_H(\Sigma, \Sigma) = 0$. Then we can assume that $Y \notin \operatorname{add}(\Sigma)$. Finally, defining H_2 as $\tau^{-1}_{\mathcal{D}}H_1$ we proceed analogously to the previous case. This finishes the proof of case a).

b) Let X' be an indecomposable B-module such that $X' \notin \operatorname{add}(M')$. Let Y' be an indecomposable module in $\operatorname{add}(W'_i)$ for some i. We write $X' = \operatorname{Hom}_{\mathcal{C}}(\tilde{T}, \tilde{X})$ and $Y' = \operatorname{Hom}_{\mathcal{C}}(\tilde{T}, \tilde{Y})$. We may assume that $Y \notin \mathcal{T}(T)$. Recall that, by the definition of W_i, There exists a regular summand of the tilting module T, which we denote by T_i such that Y must be a regular module in the cone $\mathcal{C}(T_i)$.

As we did before, we have two cases to analyse: $h: Y \to X$ or $h: Y \to \tau^{-1}X[1]$.

We will show that any morphism $h: Y \to X$ factors necessarily by $\mathcal{T}(T)$ and therefore, as we prove in the proof of case (b) of Lemma 4.2, by M_X.

Since $X' \notin \operatorname{add}(M')$ then X does not belong to the cone $\mathcal{C}(T_i)$. Therefore, by corollary 3.8 we have that the morphism h must factor through a module in $\mathcal{C}(T_i \to) \subset \mathcal{T}(T)$. Then, the proof of this case follows from case (b) of the previous Lemma.

All we need to analyze the case in which $h: Y \to \tau^{-1}X[1]$. Proceeding analogously to the case (b) of the previous lemma we can see that in this case h must factors through FH and therefore the proof of this case follows from case a) of Lemma 4.2. This completes the proof of the proposition. $\square$

Summarizing, we have proved the following result:

Theorem 4.4. *Let H be a tame hereditary algebra of infinite representation type. Let $T = T_I \oplus T_R$ a tilting module with T_I a non zero preinjective module and T_R a regular module. Let B the cluster tilted algebra given by $B = End_{\mathcal{C}}(\tilde{T})$. Then $w.rep.dim\ B = 3$.*

Proof:

As a direct consequence of proposition 3.3 and proposition 4.3 we have that $\operatorname{gl.dim} \operatorname{End}_B(M') \leq 3$. Since the module M' is generator and B is of infinite representation type, we conclude that $w.rep.dim\ B = 3$. $\square$

Corollary 4.5. *Let B a tame cluster tilted algebra. Then $w.rep.dim B = 3$.*

Proof:

Suppose that B is a tame cluster tilted algebra. Then there exists an hereditary algebra H of tame representation type and a tilting module T such that $B = \text{End}_{\tilde{C}}(\tilde{T})$. Since H is tame, we know that T has at least a non zero non regular summand, otherwise the algebra B would be wild. Suppose that T has no non zero regular summands then by theorem [11, theorem 3.6] repdim $B =$ w.rep.dim$B = 3$.

Suppose T has at least one regular summand, as we discuss previously there is no lose of generality in assuming that the non regular summands of T are all preinjective. By Lemma 2.6 this implies that $\mathcal{T}(T)$ results finite. Thus, by theorem 4.4 w.rep.dim$B = 3$.

$\square$

4.1. About representation dimension of tame cluster tilted algebras

The module M' utilized in the previous section to compute the weak representation dimension is not necessarily a cogenerator for mod B. We will illustrate this fact in the following example.

Example 4.6. *Let T' be the tilting module given in example 3.1. Consider the cluster tilted algebra given by $B = End_{\mathcal{C}}(\tilde{T}')$. We will show that there exists an injective indecomposable B-module which is not a summand of M'.*

First, we observe that S_3 is the only indecomposable regular summand of T', and it lies in the mouth of the stable tube of rank 3 in Γ_H:
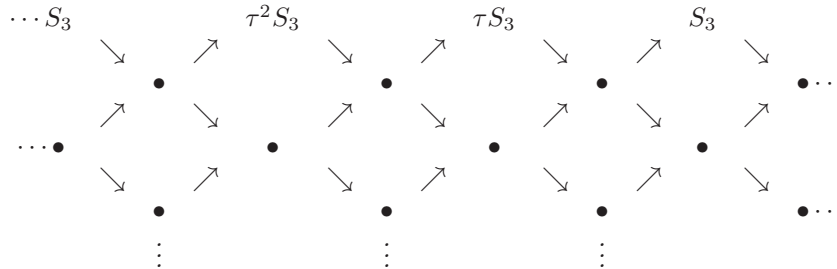
Applying the $Hom_{\mathcal{C}}(T, \quad)$ we obtain a description of the corresponding component in Γ_B by suppressing the vertex associated to τS_3 in Γ_H. Note that the B-module $(\tau^2 S_3)'$ is an indecomposable injective module, and it is not transjective in mod B. Moreover, the only indecomposable summand of M'_2, is the projective B-module corresponding to vertex 3, $(S_3)'$. therefore, in this case, M' is not a cogenerator for mod B.

Corollary 4.7. *Let B a tame cluster tilted algebra. If every non transjective injective module in B lies in $add(M'_2)$ then rep.dim $B = 3$.*

Proof:

Assume that every indecomposable injective module which are not transjectives lies in $\text{add}(M_2')$. Recall that every injective module in B is of the form $\text{Hom}_{\mathcal{C}}(\tilde{T}, \tau^2\tilde{T})$. It is possible to choose Σ in mod H such that every preinjective module of the form $\tau^2 T$ are in $\mathcal{T}(\Sigma)$. This means that it is possible to construct M_1' with this Σ. This implies that every preinjective injective B-module is in $\text{add}(M_1')$. Hence M' is a cogenerator for mod B. Thus, in this case, follows as an immediately consequence of our last theorem that rep.dim $B = 3$. $\square$

Finally, we illustrate the situation of last corollary in the following example:

Example 4.8. *Let H be the hereditary algebra given by the quiver $\tilde{\mathbb{D}}_4$. Consider the tilting H-module $T = I_5 \oplus I_4 \oplus I_1 \oplus I_2 \oplus S_3$. Here every indecomposable summand of T are preinjectives with the exception of S_3 and is immediate to see that the torsion associated to this module is finite. Let B given by $End_{\mathcal{C}}(\tilde{T})$. We know that S_3 is a regular module in an homogeneous stable tube in Γ_H. Then $\tau^2 S_3 = S_3$. Therefore the module $M_2' = (S_3)'$ is a projective-injective module in mod B. More over, choose $\Sigma = \tau^2 DH$. Hence, by the last corollary we conclude that rep.dim $B = 3$.*

Acknowledgments

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