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Multiple Solutions for a Critical $p(x)$-Kirchhoff Type Equations

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ABSTRACT: In this paper, by using the concentration–compactness principle of Lions for variable exponents and variational arguments, we obtain the existence and multiplicity solutions for a class of $p(x)$-Kirchhoff type equations with critical exponent.

Key Words: Critical exponent, p(x)-Kirchhoff problem, Variational methods.

Contents

1. Introduction and main results

In this work, we deal with the following nonlocal problem

$$\begin{cases} -M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\Delta_{p(x)}u = \lambda f(x,u) + |u|^{q(x)-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded subset of $\mathbb{R}^N (N \geq 1)$, with smooth boundary $\partial\Omega$, $\Delta_{p(x)}u = div(|\nabla u|^{p(x)-2}\nabla u)$ is the classical $p(x)$-Laplacian operator, λ is a positive parameter and $p(x)$, $q(x) \in C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) : h(x) > 1 \text{ for all } x \in \overline{\Omega}\}$ with

$$1 < p^- := \inf_{\overline{\Omega}} p(x) \leq p^+ := \sup_{\overline{\Omega}} p(x) < N$$
$$p^+ < q^- := \inf_{\overline{\Omega}} q(x) \leq q(x) \leq p^*(x),$$

where

$$p^*(x) = \frac{Np(x)}{N - p(x)}, \ \forall\, x \in \overline{\Omega},$$

and

$$\mathcal{A} := \left\{x \in \overline{\Omega};\ q(x) = p^*(x)\right\} \text{ is nonempty.}$$

$f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ and $M : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous functions that satisfy some conditions which will be stated later on.

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The $p(x)$-Laplacian operator possesses more complicated nonlinearities than the p-Laplacian operator, mainly due to the fact that it is not homogeneous. The study of various mathematical problems with variable exponent growth condition has been received considerable attention in recent years, we can for example refer to [1,10,14,20,26]. This great interest may be justified by their various physical applications. In fact, there are applications concerning elastic mechanics, electrorheological fluids [28], image restoration [11], dielectric breakdown, electrical resistivity and polycrystal plasticity [7] and continuum mechanics [4].

As it is well know, problem (1.1) is related to the stationary problem of a model introduced by Kirchhoff [24]. More precisely, Kirchhoff introduced a model given by the following equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$

which extends the classical D'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations. Latter, the study of Kirchhoff type equations has already been extended to the case involving the p-Laplacian

$$-M\left(\int_\Omega |\nabla u|^p dx\right)\Delta_p u = f(x,u) \text{ in } \Omega,$$

see [3,13,16]. However, There are many papers on the $p(x)$-Kirchhoff equation via the variational method, We refer the reader to [6,12,15,23,25] and the references therein for an overview on this subject.

In the present work, we will show the existence of infinitely many solutions for the nonlocal problem (1.1). The main theorems extend in several directions previous results recently appeared in the literature, see for example [2,19,21,22]. The difficulty in this case, is due to the lack of compactness of the embedding $W_0^{p(x)}(\Omega) \hookrightarrow L^{p^*(x)}(\Omega)$ and the Palais-Smale condition for the corresponding energy functional could not be checked directly. To deal with this difficulty, we use a version of the concentration–compactness lemma due to Lions for variable exponents [8].

Throughout the sequel, we make the following assumptions on $M(t)$ and $f(x,t)$:

(m_1) There exists $m_0 > 1$ such that $M(t) \geq m_0$ for all $t \geq 0$.

(m_2) There exists $\sigma > p^+/q^-$ such that

$$\widehat{M}(t) \geq \sigma M(t)t, \quad \forall\, t \geq 0,$$

where $\widehat{M}(t) = \int_0^t M(s)ds$.

(f_1) $f(x,t) = o(|t|^{p(x)-1})$ as $t \to 0$, uniformly for $x \in \Omega$.

(f_2) $f(x,t) = o(|t|^{q(x)-1})$ as $t \to +\infty$, uniformly for $x \in \Omega$.

(f_3) There exists $\theta \in (p^+/\sigma, q^-)$ such that

$$0 < \theta F(x,t) \leq t f(x,t), \quad \forall\, t \in \mathbb{R},\ a.e.\ x \in \Omega,$$

where $F(x,t) = \int_0^t f(x,s)ds$, and σ is given in assumption (m_2).

Our main results is as follows.

Theorem 1.1. *Suppose that $(m_1) - (m_2)$ and $(f_1) - (f_3)$ hold. Then, there exists $\lambda_* > 0$, such that problem (1.1) has at least one nontrivial solution for all $\lambda \geq \lambda_*$.*

To obtain the multiplicity of solutions of the problem (1.1), we give one more condition on $f(x,t)$:

(f_4) f is odd in t, i.e. $f(x,-t) = -f(x,t)$ for all $t \in \mathbb{R}$ and for all $x \in \overline{\Omega}$.

Theorem 1.2. *Suppose that $(m_1) - (m_2)$ and $(f_1) - (f_4)$ hold. Then, there exists $\lambda_* > 0$, such that problem (1.1) has infinitely many weak solutions for all $\lambda \geq \lambda_*$.*

2. Abstract framework

Here, we state some interesting properties of the variable exponent Lebesgue and Sobolev spaces that will be useful to discuss problem (1.1). Every where below we consider $\Omega \subset \mathbb{R}^N$ to be a bounded domain with smooth boundary and $p(x) \in C_+(\overline{\Omega})$. Define the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R}\,\text{measurable} : \int_\Omega |u(x)|^{p(x)} dx < \infty \right\}.$$

This space endowed with the Luxemburg norm,

$$||u||_{L^{p(x)}(\Omega)} = \inf \left\{ \tau > 0 : \int_\Omega \left| \frac{u(x)}{\tau} \right|^{p(x)} dx \leq 1 \right\}$$

is a separable and reflexive Banach space. Denoting by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ we have the following Hölder type inequality

$$\int_\Omega |uv| dx \leq \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) ||u||_{L^{p(x)}(\Omega)} ||v||_{L^{p'(x)}(\Omega)}.$$

Now, we introduce the modular of the Lebesgue-Sobolev space $L^{p(x)}(\Omega)$ as the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)} dx, \quad \forall u \in L^{p(x)}(\Omega).$$

In the following proposition, we give some relations between the Luxemburg norm and the modular.

Proposition 2.1 ([17]). *If $u \in L^{p(x)}(\Omega)$ and $\{u_n\} \subset L^{p(x)}(\Omega)$, then following properties hold true:*
(1) $||u||_{L^{p(x)}(\Omega)} \leq 1 \Rightarrow ||u||^{p^+}_{L^{p(x)}(\Omega)} \leq \rho_{p(x)}(u) \leq ||u||^{p^-}_{L^{p(x)}(\Omega)}$;
(2) $||u||_{L^{p(x)}(\Omega)} \geq 1 \Rightarrow ||u||^{p^-}_{L^{p(x)}(\Omega)} \leq \rho_{p(x)}(u) \leq ||u||^{p^+}_{L^{p(x)}(\Omega)}$;
(3) $\lim_{n\to\infty} ||u_n||_{L^{p(x)}(\Omega)} = 0 \Leftrightarrow \lim_{n\to\infty} \rho_{p(x)}(u_n) = 0$;
(4) $\lim_{n\to\infty} ||u_n||_{L^{p(x)}(\Omega)} = \infty \Leftrightarrow \lim_{n\to\infty} \rho_{p(x)}(u_n) = \infty$.

Next, we define the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ by

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \right\},$$

endowed with the norm

$$||u||_{W^{1,p(x)}(\Omega)} = ||u||_{L^{p(x)}(\Omega)} + ||\nabla u||_{L^{p(x)}(\Omega)}.$$

We denote by $X := W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ with respect to the above norm. The space $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.

Proposition 2.2 ([17]).
(1) *If $r \in C_+(\overline{\Omega})$ and $r(x) < p^*(x)$ for all $x \in \overline{\Omega}$, then the embedding*

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$$

is compact and continuous.
(2) *There is a constant $C > 0$ such that*

$$||u||_{L^{p(x)}(\Omega)} \leq C||\nabla u||_{L^{p(x)}(\Omega)} \text{ for all } u \in X.$$

By (2) of Proposition 2.2, we see that $||u|| := ||\nabla u||_{L^{p(x)}(\Omega)}$ and $||.||_{W^{1,p(x)}(\Omega)}$ are equivalent norms in X. In the following, we will use $||.||$ instead of $||.||_{W^{1,p(x)}(\Omega)}$ on X. On this space, we consider the modular function $\rho_0 : X \to \mathbb{R}$ given by

$$\rho_0(u) = \int_\Omega |\nabla u|^{p(x)} dx.$$

Proposition 2.3. *Let $u \in X$ and $\{u_n\} \subset X$. Then, the same conclusion of Proposition 2.1 occurs considering $||.||$ and ρ_0.*

Proposition 2.4 ([18]). *The functional $\Lambda : X :\to \mathbb{R}$ given by $\Lambda(u) := \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx$ is well defined, even and C^1 in X, and The mapping Λ' is a strictly monotone, bounded, homeomorphism, and is of $(S+)$ type, namely*

$$u_n \rightharpoonup u \text{ and } \limsup_{n\to+\infty} \Lambda'(u_n)(u_n - u) \leq 0 \text{ implies } u_n \to u.$$

Definition 2.5. *We say that the functional $\varphi : X \to \mathbb{R}$ satisfies the Palais-Smale condition at level $c \in \mathbb{R}$ (briefly $(PS)_c$) on X, if any sequence $\{u_n\} \subset X$, such that $\varphi(u_n) \to c$ and $\varphi'(u_n) \to 0$ as $n \to \infty$, possesses a convergent subsequence.*

Our main tools are the classical Mountain Pass Theorem and its $\mathbb{Z}_2$–symmetric version recalled respectively in the next Theorems.

Theorem 2.6 (cf. [27]). *Let X be a real infinite dimensional Banach space and $\varphi \in C^1(X, \mathbb{R})$ such that $\varphi(0_X) = 0$ and satisfying the (PS) condition. Suppose that*

(I_1) *There are constants $\beta, \varrho > 0$ such that $\varphi(u) \geq \beta$ for all $u \in \partial B_\varrho \cap X$;*

(I_2) *There exists $e \in X$ with $\|e\| > \varrho$ such that $\varphi(e) < 0$.*

Then φ possesses a critical value $c \geq \beta$, which can be characterized as

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi(\gamma(t)) > 0,$$

where

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0,\ \varphi(\gamma(1)) < 0\}.$$

Theorem 2.7 (cf. [27]). *Let X be a real infinite dimensional Banach space and $\varphi \in C^1(X, \mathbb{R})$ be even, satisfying the Palais-Smale condition and $\varphi(0_X) = 0$. Suppose that condition (I_1) holds in addition to the following:*

(I_2') *For each finite dimensional subspace $X_1 \subset X$, the set $S_1 := \{u \in X_1 : \varphi(u) \geq 0\}$ is bounded in X.*

Then φ has an unbounded sequence of critical values.

Now, we recall an important version of concentration–compactness principal of Lions for variable exponents found in [8], which will be used in the proof of our main results.

Theorem 2.8 (cf. [8]). *Let $q(x)$ and $p(x)$ be two continuous functions such that*

$$1 < \inf_{x \in \Omega} p(x) \leq \sup_{x \in \Omega} p(x) < N \quad \text{and} \quad 1 \leq q(x) \leq p^*(x) \quad \text{in } \Omega.$$

Let $\{u_n\}_{n \in \mathbb{N}}$ be a weakly convergent sequence in $W_0^{1,p(x)}(\Omega)$ with weak limit u, and such that:

- *$|\nabla u_n|^{p(x)} \rightharpoonup \mu$ weakly-* in the sense of measures.*
- *$|u_n|^{q(x)} \longrightarrow \nu$ weakly-* in the sense of measures.*

Also assume that $\mathcal{A} = \{x \in \Omega \colon q(x) = p^*(x)\}$ is nonempty. Then, for some countable index set $\mathcal{J}$, we have:

$$\nu = |u|^{q(x)} + \sum_{j \in \mathcal{J}} \nu_j \delta_{x_j}, \quad \nu_j > 0$$

$$\mu \geq |\nabla u|^{p(x)} + \sum_{j \in \mathcal{J}} \mu_j \delta_{x_j}, \quad \mu_j > 0$$

$$S \nu_j^{1/p^*(x_j)} \leq \mu_j^{1/p(x_j)}, \quad \forall j \in \mathcal{J}.$$

where $\{x_j\}_{j\in\mathcal{J}} \subset \mathcal{A}$, δ_{x_j} *is the Dirac mass at* $x_j \in \overline{\Omega}$ *and* S *is the best constant in the Gagliardo-Nirenberg-Sobolev inequality for variable exponents, namely*

$$S = S_q(\Omega) := \inf_{\phi\in C_0^\infty(\Omega)} \frac{\|\phi\|}{\|\phi\|_{L^{q(x)}(\Omega)}}. \tag{2.1}$$

3. Proofs of main results

In the sequel, we use $c_i\,(i = 1, 2, ..)$, to denote the general nonnegative or positive constant. The energy functional $I_\lambda : X \mapsto \mathbb{R}$ corresponding to problem (1.1), defined as follows

$$I_\lambda(u) = \widehat{M}\Big(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx\Big) - \lambda\int_\Omega F(x,u)dx - \int_\Omega \frac{|u|^{q(x)}}{q(x)}dx.$$

Note that $I_\lambda \in C^1(X,\mathbb{R})$ with the derivatives given by

$$\begin{aligned}\langle I'_\lambda(u), v\rangle &= M\Big(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx\Big)\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v - \lambda\int_\Omega f(x,u)vdx\\ &\quad - \int_\Omega |u|^{q(x)-2}uvdx, \end{aligned}\tag{3.1}$$

for any u, $v \in X$, and the critical points of it are weak solutions of problem (1.1).

Now, we prove that the functional I_λ has the geometric features required by the Mountain Pass Theorem.

Lemma 3.1. *Under the conditions* (m_1), (f_1) *and* (f_2), *there exist* $\beta, \varrho > 0$ *such that* $I_\lambda(u) \geq \beta$ *for any* $u \in X$ *with* $||u|| = \varrho$.

Proof: By (f_1) and (f_2), it follows that for any $\varepsilon > 0$ there exists $C_\varepsilon = C(\varepsilon) > 0$ depending on ε such that

$$|F(x,t)| \leq \frac{\varepsilon}{p(x)}|t|^{p(x)} + \frac{C_\varepsilon}{q(x)}|t|^{q(x)} \text{ for all } (x,t) \in \overline{\Omega}\times\mathbb{R}. \tag{3.2}$$

Together with (m_1), we have

$$\begin{aligned} I_\lambda(u) &\geq m_0\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx - \frac{\lambda\varepsilon}{p^-}\int_\Omega |u|^{p(x)}dx - \frac{\lambda C_\varepsilon + 1}{q^-}\int_\Omega |u|^{q(x)}dx\\ &\geq \frac{m_0}{p^+}\int_\Omega |\nabla u|^{p(x)}dx - \frac{\lambda\varepsilon}{p^-}\int_\Omega |u|^{p(x)}dx - \frac{\lambda C_\varepsilon + 1}{q^-}\int_\Omega |u|^{q(x)}dx. \end{aligned}$$

Hence for ε sufficiently small, we get

$$I_\lambda(u) \geq \frac{m_0}{p^+}\int_\Omega |\nabla u|^{p(x)}dx - \frac{\lambda C_\varepsilon + 1}{q^-}\int_\Omega |u|^{q(x)}dx.$$

Due to the continuous embedding $X \hookrightarrow L^{q(x)}(\Omega)$, there exists $c_1 > 0$ such that

$$||u||_{L^{q(x)}(\Omega)} \leq c_1||u||.$$

We fix $\varrho \in (0,1)$ such that $\varrho < 1/c_1$. Then, by proposition 2.1 the following hold

$$I_\lambda(u) \geq \frac{m_0}{p^+}||u||^{p^+} - c_2||u||^{q^-} \quad \text{for all } u \in X \text{ with } \|u\| = \varrho.$$

Since $p^+ < q^-$, there exists $\beta > 0$ such that $I_\lambda(u) \geq \beta$ for $||u|| = \varrho$, where ϱ is chosen sufficiently small. □

Lemma 3.2. *Assume that conditions (f_2) and (m_2) hold. Then for all $\lambda > 0$, there exists a nonnegative function $e \in X$ such that $I_\lambda(e) < 0$ and $\|e\| > \varrho$, where ϱ is given in Lemma 3.1.*

Proof: Choose a nonnegative function $v_0 \in C_0^\infty(\Omega)$ with $\|v_0\| = 1$. By integrating (m_2), we obtain

$$\widehat{M}(t) \leq \frac{\widehat{M}(t_0)}{t_0^{1/\sigma}} t^{1/\sigma} = c_3 t^{1/\sigma} \quad \text{for all } t \geq t_0 > 0. \tag{3.3}$$

By using (f_2), for $\varepsilon > 0$ there is a constant $M_\varepsilon > 0$ such that

$$|F(x,t)| < \varepsilon |t|^{q(x)} + M_\varepsilon \quad \text{for all } \ (x,t) \in \overline{\Omega} \times \mathbb{R}. \tag{3.4}$$

Therefore, we find

$$I_\lambda(u) \leq \frac{c_3}{(p^-)^{1/\sigma}} \Big(\int_\Omega |\nabla u|^{p(x)} dx \Big)^{1/\sigma} + \left(\lambda\varepsilon - \frac{1}{q^+} \right) \int_\Omega |u|^{q(x)} dx + \lambda M_\varepsilon |\Omega|.$$

By choosing $\varepsilon = \frac{1}{2\lambda q^+}$, we obtain

$$I_\lambda(u) \leq \frac{c_3}{(p^-)^{1/\sigma}} \Big(\int_\Omega |\nabla u|^{p(x)} dx \Big)^{1/\sigma} - \frac{1}{2q^+} \int_\Omega |u|^{q(x)} dx + \lambda M_\varepsilon |\Omega|. \tag{3.5}$$

Then, for all $t \geq t_0$, according to Proposition 2.1 we get

$$I_\lambda(tv_0) \leq \frac{c_3}{(p^-)^{1/\sigma}} t^{p^+/\sigma} - c_4 t^{q^-} + \lambda M_\varepsilon |\Omega|.$$

Since $\sigma > p^+/q^-$, passing to the limit as $t \to \infty$ we obtain $I_\lambda(tv_0) < 0$. Hence, the assertion follows by taking $e = t_* v_0$, with $t_* > 0$ large enough. □

We discuss now the compactness property for the functional I_λ, given by the Palais–Smale condition at a suitable level. For this, from Lemmas 3.1 and 3.2, using a version of the Mountain Pass theorem due to Ambrosetti and Rabinowitz [5], without (PS) condition (see [29]), there exists a sequence $\{u_n\} \subset X$ such that

$$I_\lambda(u_n) \to c_\lambda \ \text{ and } \ I'_\lambda(u_n) \to 0,$$

where

$$c_\lambda = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\lambda(\gamma(t)) > 0, \tag{3.6}$$

with

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \, I_\lambda(\gamma(1)) < 0\}.$$

In the following Lemma, we shall prove an estimate for c_λ.

Lemma 3.3. *If the conditions* $(\mathrm{m}_1), (\mathrm{m}_2)$ *and* $(\mathrm{f}_1)-(\mathrm{f}_3)$ *hold, then*

$$\lim_{\lambda\to\infty} c_\lambda = 0.$$

Proof: Fix $\lambda > 0$ and let $e \in X$ be the function obtained by Lemma 3.2. Since I_λ satisfies the Mountain Pass geometry, there exists $t_\lambda > 0$ verifying $I_\lambda(t_\lambda e) = \max_{t\geq 0} I_\lambda(te)$. Hence, $\langle I'_\lambda(t_\lambda e), e\rangle = 0$ and by (3.1) we get

$$M\left(\int_\Omega \frac{t_\lambda^{p(x)}}{p(x)}|\nabla e|^{p(x)}dx\right)\int_\Omega t_\lambda^{p(x)}|\nabla e|^{p(x)}dx = \lambda\int_\Omega f(x,t_\lambda e)t_\lambda e\,dx + \int_\Omega t_\lambda^{q(x)}|e|^{q(x)}dx.$$

Assume $t_\lambda \geq 1$ for convenience. By (m_2) and (f_3) it follows that

$$\begin{aligned}
t_\lambda^{q^-}\int_\Omega |e|^{q(x)}dx &\leq \lambda\int_\Omega f(x,t_\lambda e)t_\lambda e\,dx + \int_\Omega t_\lambda^{q(x)}|e|^{q(x)}dx\\
&= M\left(\int_\Omega \frac{t_\lambda^{p(x)}}{p(x)}|\nabla e|^{p(x)}dx\right)\int_\Omega t_\lambda^{p(x)}|\nabla e|^{p(x)}dx\\
&\leq \frac{p^+}{\sigma}\widehat{M}\Big(\int_\Omega \frac{t_\lambda^{p(x)}}{p(x)}|\nabla e|^{p(x)}dx\Big)\\
&\leq \frac{p^+c_3}{\sigma}\Big(\int_\Omega \frac{t_\lambda^{p(x)}}{p(x)}|\nabla e|^{p(x)}dx\Big)^{1/\sigma}\\
&\leq \frac{p^+c_3}{\sigma}t_\lambda^{p^+/\sigma}\Big(\int_\Omega \frac{1}{p(x)}|\nabla e|^{p(x)}dx\Big)^{1/\sigma}, \quad \text{with}\, t_\lambda > t_0. \qquad (3.7)
\end{aligned}$$

which implies that (t_λ) is bounded, since $q^- > p^+/\sigma$. So, there exists a sequence $\lambda_n \to +\infty$ and $t_* \geq 0$ such that $t_{\lambda_n} \to t_*$ as $n\to\infty$. By continuity of M, also $\left\{M\left(\int_\Omega \frac{t_{\lambda_n}^{p(x)}}{p(x)}|\nabla e|^{p(x)}dx\right)\right\}_n$ is bounded. Then , there exists $c_5 > 0$ such that

$$M\left(\int_\Omega \frac{t_{\lambda_n}^{p(x)}}{p(x)}|\nabla e|^{p(x)}dx\right)\int_\Omega t_{\lambda_n}^{p(x)}|\nabla e|^{p(x)} \leq c_5 \quad \text{for all } n\in\mathbb{N},$$

which yields,

$$\lambda_n\int_\Omega f(x,t_{\lambda_n}e)t_{\lambda_n}e\,dx + \int_\Omega t_{\lambda_n}^{q(x)}|e|^{q(x)}dx \leq c_5 \quad \text{for all } n\in\mathbb{N}. \qquad (3.8)$$

We assert that $t_* = 0$, otherwise, if $t_* > 0$ then by (f_3) and the Dominated Convergence Theorem, we obtain

$$\int_\Omega f(x,t_{\lambda_n}e)t_{\lambda_n}e\,dx \to \int_\Omega f(x,t_*e)t_*e\,dx > 0 \;\text{ as }\; n\to\infty.$$

Recalling that $\lambda_n \to \infty$, the last inequality (3.8) becomes

$$\lambda_n \int_\Omega f(x, t_{\lambda_n} e) t_{\lambda_n} e\, dx + \int_\Omega t_{\lambda_n}^{q(x)} |e|^{q(x)} dx \to +\infty, \quad \text{as } n \to \infty,$$

which is impossible, so $t_* = 0$ and $t_\lambda \to 0$ as $\lambda \to \infty$, since the sequence $\{\lambda_n\}_n$ is arbitrary. Now, let us consider the path $\gamma_*(t) = te$ for $t \in [0,1]$, which belongs to Γ. We get the following estimate

$$0 < c_\lambda \leq \max_{t\in[0,1]} I(\gamma_*(t)) = I(t_\lambda e) \leq \widehat{M}\left(\int_\Omega \frac{t_\lambda^{p(x)}}{p(x)} |\nabla e|^{p(x)} dx\right). \tag{3.9}$$

The fact that $\widehat{M}\left(\int_\Omega \frac{t_\lambda^{p(x)}}{p(x)} |\nabla e|^{p(x)} dx\right) \to 0$ as $\lambda \to \infty$ by continuity, and so by using also (3.9) we can conclude the proof. □

Now, we are ready to prove the Palais–Smale condition.

Lemma 3.4. *There exists $\lambda_* > 0$ such that for any $\lambda \geq \lambda_*$ the functional I_λ satisfies $(PS)_{c_\lambda}$ for $c_\lambda < \left(\frac{1}{\theta} - \frac{1}{q^-}\right) S^N m_0^{N/p^+}$.*

Proof: As a consequence of Lemma 3.3, there exists $\lambda_* > 0$ such that

$$c_\lambda < \left(\frac{1}{\theta} - \frac{1}{q^-}\right) S^N m_0^{N/p^+}, \quad \text{for every} \quad \lambda \geq \lambda_*, \tag{3.10}$$

where, S defined in (2.1). Now, fix $\lambda \geq \lambda^*$ and let us show that I_λ satisfies the $(PS)_{c_\lambda}$ condition. Indeed, assume that $\{u_n\} \subset X$ such that

$$I_\lambda(u_n) \to c_\lambda > 0 \quad \text{and} \quad I'_\lambda(u_n) \to 0.$$

From (f_3), for n large enough, it follows from (m_1) and (m_2) that

$$\begin{aligned}
c_\lambda + 1 + ||u_n|| &\geq I_\lambda(u_n) - \frac{1}{\theta}\langle I'_\lambda(u_n), u_n\rangle \\
&= \widehat{M}\Big(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)} dx\Big) - \frac{1}{\theta} M\Big(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)} dx\Big) \int_\Omega |\nabla u_n|^{p(x)} dx \\
&+ \int_\Omega \left(\frac{1}{\theta} - \frac{1}{q(x)}\right) |u_n|^{q(x)} dx + \lambda \int_\Omega \left(\frac{1}{\theta} f(x, u_n) u_n - F(x, u_n)\right) dx \\
&\geq \Big(\frac{\sigma}{p^+} - \frac{1}{\theta}\Big) M\Big(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)} dx\Big) \int_\Omega |\nabla u_n|^{p(x)} dx \\
&+ \int_\Omega \left(\frac{1}{\theta} - \frac{1}{q(x)}\right) |u_n|^{q(x)} dx \\
&\geq m_0 \Big(\frac{\sigma}{p^+} - \frac{1}{\theta}\Big) \min\left(||u_n||^{p^-}, ||u_n||^{p^+}\right) + \int_\Omega \left(\frac{1}{\theta} - \frac{1}{q^-}\right) |u_n|^{q(x)} dx.
\end{aligned}$$

Once $p^- > 1$ and $\theta \in (p^+/\sigma, q^-)$, the above inequality gives that $\{u_n\}$ is bounded in X. Up to subsequence, we may assume that

$$\begin{aligned} &u_n \rightharpoonup u \text{ in } X,\\ &u_n \rightharpoonup u \text{ in } L^{q(x)}(\Omega),\\ &u_n \to \text{ in } L^{r(x)}(\Omega),\ r \in C_+(\overline{\Omega}),\ r(x) < p^*(x),\ \forall x \in \overline{\Omega}. \end{aligned}$$

Now, we claim that

$$u_n \to u \text{ in } L^{q(x)}(\Omega). \tag{3.11}$$

To prove this claim, we suppose that

$$|\nabla u_n|^{p(x)} \rightharpoonup \mu, \text{ and } |u_n|^{q(x)} \rightharpoonup \nu \text{ (weakly-* sense of measures).}$$

Using the the concentration compactness principale of Lions 2.8, there exists a countable index set $\mathcal{J}$, points $\{x_j\}_{j\in\mathcal{J}}$ in Ω and sequences $\{\mu_j\}_{j\in\mathcal{J}}, \{\nu_j\}_{j\in\mathcal{J}} \subset [0, +\infty)$, such that

$$\begin{aligned} &\mu \geq |\nabla u|^{p(x)} + \sum_{j\in\mathcal{J}} \mu_j \delta_{x_j} \text{ in } \Omega\\ &\nu = |u|^{q(x)} + \sum_{j\in\mathcal{J}} \nu_j \delta_{x_j} \text{ in } \Omega\\ &S\nu_j^{\frac{1}{p^*(x_j)}} \leq \mu_j^{\frac{1}{p(x_j)}} \text{ for all } j \in \mathcal{J}, \end{aligned} \tag{3.12}$$

where S defined as (2.1). Let $\phi \in C_0^\infty(\mathbb{R}^N)$ such that

$$0 \leq \phi \leq 1, \quad \phi \equiv 1 \text{ in } B_{1/2}(0), \quad \phi = 0 \text{ in } \mathbb{R}^N \backslash B_1(0).$$

For $\varepsilon > 0$ and $j \in \mathcal{J}$ denote

$$\phi_\varepsilon^j(x) = \phi\left(\frac{x - x_j}{\varepsilon}\right), \text{ for all } x \in \mathbb{R}^N.$$

Because $\{u_n\phi_\varepsilon^j\}$ is bounded for each $j \in \mathcal{J}$, $\langle I'_\lambda(u_n), u_n\phi_\varepsilon^j\rangle = o_n(1)$ that is

$$\begin{aligned} M\Big(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}dx\Big)\int_\Omega |\nabla u_n|^{p(x)}\phi_\varepsilon^j dx = &-M\Big(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}dx\Big)\int_\Omega |\nabla u_n|^{p(x)-2}u_n\nabla u_n\nabla\phi_\varepsilon^j dx\\ &+\int_\Omega |u_n|^{q(x)}\phi_\varepsilon^j dx + \lambda\int_\Omega f(x,u_n)u_n\phi_\varepsilon^j dx + o_n(1). \end{aligned} \tag{3.13}$$

By Hölder inequality and the boundedness of $\{u_n\}$, we have that

$$\begin{aligned}
0 &\leq \lim_{\varepsilon\to 0}\lim_{n\to\infty}\Big|\int_\Omega |\nabla u_n|^{p(x)-2}u_n\nabla u_n\nabla\phi_\varepsilon^j dx\Big| \\
&\leq \lim_{\varepsilon\to 0}\lim_{n\to\infty}\Big(\int_\Omega |u_n|^{p(x)}|\nabla\phi_\varepsilon^j|^{p(x)}dx\Big)^{1/p(x)}\Big(\int_\Omega |\nabla u_n|^{p(x)}dx\Big)^{\frac{p(x)-1}{p(x)}} \\
&\leq c_6\lim_{\varepsilon\to 0}\Big(\int_{B_\varepsilon(x_j)}|u|^{p(x)}|\nabla\phi_\varepsilon^j|^{p(x)}dx\Big)^{1/p(x)} \\
&\leq c_6\lim_{\varepsilon\to 0}\Big(\int_{B_\varepsilon(x_j)}|u|^{p^*(x)}dx\Big)^{1/p^*(x)}\Big(\int_{B_\varepsilon(x_j)}|\nabla\phi_\varepsilon^j|^N dx\Big)^{1/N} = 0. \quad (3.14)
\end{aligned}$$

By the compactness lemma of Strauss [9], we obtain

$$\lim_{n\to+\infty}\int_\Omega f(x,u_n)u_n\phi_\varepsilon^j dx = \int_\Omega f(x,u)u\phi_\varepsilon^j dx. \qquad (3.15)$$

Since ϕ_ε^j has compact support, going to the limit $n\to\infty$ in (3.13), from (3.14) and (3.15), we obtain

$$\begin{aligned}
m_0\int_\Omega \phi_\varepsilon^j d\mu \leq{}& c_6\Big(\int_{B_\varepsilon(x_j)}|u|^{p^*(x)}dx\Big)^{1/p^*(x)}\Big(\int_{B_\varepsilon(x_j)}|\nabla\phi_\varepsilon^j|^N dx\Big)^{1/N} \\
&+\int_\Omega \phi_\varepsilon^j d\nu + \lambda\int_{B_\varepsilon(x_j)} f(x,u)u\phi_\varepsilon^j dx
\end{aligned}$$

Letting $\varepsilon\to 0$, we derive $m_0\mu_j\leq\nu_j$. Therefore

$$S\nu_j^{\frac{1}{p^*(x_j)}} \leq \mu_j^{\frac{1}{p(x_j)}} \leq \Big(\frac{\nu_j}{m_0}\Big)^{\frac{1}{p(x_j)}}.$$

Then either $\nu_j = 0$ or $\nu_j\geq S^N m_0^{N/p(x_j)}$. We assert that $\nu_j = 0$ for each j. If not, assume that $\nu_j\geq S^N m_0^{N/p(x_j)}$ for some j, then because $\{u_n\}$ is a Palais-Smale sequence, we have

$$\begin{aligned}
c_\lambda &= I_\lambda(u_n) - \frac{1}{\theta}\langle I_\lambda'(u_n),u_n\rangle + o_n(1) \\
&= \widehat{M}\Big(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}dx\Big) - \frac{1}{\theta}M\Big(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}dx\Big)\int_\Omega |\nabla u_n|^{p(x)}dx + o_n(1) \\
&+ \int_\Omega \left(\frac{1}{\theta}-\frac{1}{q(x)}\right)|u_n|^{q(x)}dx + \lambda\int_\Omega\left(\frac{1}{\theta}f(x,u_n)u_n - F(x,u_n)\right)dx \\
&\geq m_0\Big(\frac{\sigma}{p^+}-\frac{1}{\theta}\Big)\int_\Omega |\nabla u_n|^{p(x)}dx + \int_\Omega\left(\frac{1}{\theta}-\frac{1}{q(x)}\right)|u_n|^{q(x)}dx \\
&+ \lambda\int_\Omega\left(\frac{1}{\theta}f(x,u_n)u_n - F(x,u_n)\right)dx + o_n(1)
\end{aligned}$$

From (f_3) and the fact that $\theta \in (p^+/\sigma, q^-)$ we obtain

$$c_\lambda \geq \int_\Omega \left(\frac{1}{\theta} - \frac{1}{q^-}\right) |u_n|^{q(x)} dx + o_n(1).$$

Using (3.12), it follows that

$$\begin{aligned} c_\lambda &\geq \lim_{n\to+\infty} \int_\Omega \left(\frac{1}{\theta} - \frac{1}{q^-}\right) |u_n|^{q(x)} dx \\ &\geq \left(\frac{1}{\theta} - \frac{1}{q^-}\right) \left(\int_\Omega |u|^{q(x)} dx + \sum_{j\in\mathcal{J}} \nu_j\right) \\ &\geq \left(\frac{1}{\theta} - \frac{1}{q^-}\right) \nu_j \\ &\geq \left(\frac{1}{\theta} - \frac{1}{q^-}\right) S^N m_0^{N/p(x_j)}. \end{aligned}$$

Hence for $m_0 > 1$

$$c_\lambda \geq \left(\frac{1}{\theta} - \frac{1}{q^-}\right) S^N m_0^{N/p^+},$$

which contradicts (3.10), and so $\nu_j = 0$ for all $j \in \mathcal{J}$. Hence

$$\lim_{n\to+\infty} \int_\Omega |u_n|^{q(x)} dx = \int_\Omega |u|^{q(x)} dx.$$

This implies $\lim\limits_{n\to+\infty} \int_\Omega |u_n - u|^{q(x)} dx = 0$, thanks to Proposition 2.1, we deduce

$$u_n \to u \text{ in } L^{q(x)}(\Omega),$$

which proves claim (3.11).

Now, using again the Hölder type inequality, it follows that

$$\lim_{n\to+\infty} \int_\Omega |u_n|^{q(x)-2} u_n (u_n - u) dx = 0, \text{ and } \lim_{n\to+\infty} \int_\Omega f(x, u_n)(u_n - u) dx = 0. \tag{3.16}$$

On the other hand, we have

$$\begin{aligned} \langle I'(u_n), u_n - u\rangle &= M\Big(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)} dx\Big) \int_\Omega |\nabla u_n|^{p(x)-2} \nabla u_n \nabla (u_n - u) dx \\ &\quad - \int_\Omega |u_n|^{q(x)-2} u_n (u_n - u) dx - \lambda \int_\Omega f(x, u_n)(u_n - u) dx \\ &= o_n(1). \end{aligned} \tag{3.17}$$

Moreover, from the continuity of M and the boundedness of $\{u_n\}$ in X, we may find $c_8 \geq 0$ such that

$$M\Big(\int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)} dx\Big) \to M(c_8) > 0.$$

Combining the last limit with (3.16) and (3.17)

$$\lim_{n\to+\infty}\Lambda'(u_n)(u_n-u)dx=\lim_{n\to+\infty}\int_\Omega|\nabla u_n|^{p(x)-2}\nabla u_n\nabla(u_n-u)dx=0,$$

from where it follows by Proposition 2.4

$$u_n\to u \ \text{ in } \ X. \tag{3.18}$$

This completes the proof of the Lemma. □

Proof: [**Proof of Theorem** 1.1] Lemmas 3.1, 3.2 and 3.4 guarantee that for any $\lambda\geq\lambda_*$ the functional I_λ satisfies all the assumptions of the Mountain Pass Theorem 2.6. Hence, from (3.18) for any $\lambda\geq\lambda_*$ one has

$$I'_\lambda(u)=0 \ \text{ and } \ I_\lambda(u)=c_\lambda>0.$$

i.e., u is a nontrivial weak solution of (1.1). □

Proof: [**Proof of Theorem** 1.2] We will use a $\mathbb{Z}_2$-symmetric version of the Mountain Pass Theorem 2.7, to accomplish the proof of Theorem 1.2. By assumption (f_4), the functional I_λ is even. Considering the proof of Theorem 1.1, we need only check the condition (I'_2). In fact, by (3.5) one has

$$I_\lambda(u)\leq\frac{c_3}{(p^-)^{1/\sigma}}\Big(\int_\Omega|\nabla u|^{p(x)}dx\Big)^{1/\sigma}-\frac{1}{2q^+}\int_\Omega|u|^{q(x)}dx+\lambda M_\varepsilon|\Omega|.$$

Let $u\in X$ be arbitrary but fixed. We define

$$\Omega_<=\{x\in\Omega;\,|u(x)|<1\},\ \text{ and }\ \Omega_\geq=\Omega\backslash\Omega_<.$$

Then we obtain

$$\begin{aligned}I_\lambda(u)&\leq\frac{c_3}{(p^-)^{1/\sigma}}\Big(\int_\Omega|\nabla u|^{p(x)}dx\Big)^{1/\sigma}-\frac{1}{2q^+}\int_\Omega|u|^{q(x)}dx+\lambda M_\varepsilon|\Omega|\\&\leq\frac{c_3}{(p^-)^{1/\sigma}}\Big(\int_\Omega|\nabla u|^{p(x)}dx\Big)^{1/\sigma}-\frac{1}{2q^+}\int_{\Omega_\geq}|u|^{q^-}dx+\lambda M_\varepsilon|\Omega|\\&\leq\frac{c_3}{(p^-)^{1/\sigma}}\Big(\int_\Omega|\nabla u|^{p(x)}dx\Big)^{1/\sigma}-\frac{1}{2q^+}\int_\Omega|u|^{q^-}dx+\frac{1}{2q^+}\int_{\Omega_<}|u|^{q^-}dx\\&+\lambda M_\varepsilon|\Omega|\\&\leq\frac{c_3}{(p^-)^{1/\sigma}}\Big(\int_\Omega|\nabla u|^{p(x)}dx\Big)^{1/\sigma}-\frac{1}{2q^+}\int_\Omega|u|^{q^-}dx+C.\end{aligned}$$

The functional $|.|_{q^-}:X\to\mathbb{R}$ defined by

$$|u|_{q^-}=\Big(\int_\Omega|u|^{q^-}\Big)^{1/q^-}$$

is a norm in X. Let X_1 be a fixed finite dimensional subspace of X. Then $|.|_{q^-}$ and $\|.\|$ are equivalent norms on X_1, so there exists a positive constant $c_9 = c_9(X_1)$ such that

$$\|u\|^{q^-} \leq c_9 |u|_{q^-}^{q^-}, \quad \text{for all } u \in X_1.$$

Assume $\|u\| > 1$ for convenience. According to Proposition 2.1, for any $u \in S_1$ we obtain

$$0 \leq I_\lambda(u) \leq \frac{c_3}{(p^-)^{1/\sigma}} \|u\|^{p^+/\sigma} - \frac{1}{2c_9 q^+} \|u\|^{q^-} + C.$$

since $q^- > p^+/\sigma$ we conclude that S_1 is bounded in X. □

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