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Stability in Mixed Linear Delay Levin-Nohel Integro-dynamic Equations on Time Scales

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ABSTRACT: In this paper we use the contraction mapping theorem to obtain asymptotic stability results about the zero solution for the following mixed linear delay Levin-Nohel integro-dynamic equation

$$x^{\triangle}(t) + \int_{t-r(t)}^{t} a(t,s)x(s)\Delta s + b(t)x(t-h(t)) = 0, \ t \in [t_0, \infty) \cap \mathbb{T},$$

where $f^{\triangle}$ is the $\triangle$-derivative on $\mathbb{T}$. An asymptotic stability theorem with a necessary and sufficient condition is proved. The results obtained here extend the work of Dung [13]. In addition, the case of the equation with several delays is studied.

Key Words: Fixed points, Delay integro-dynamic equations, Stability, Time scales.

Contents

1 Introduction **73**

2 Preliminaries **74**

3 Main results **76**

4 Levin-Nohel equations with several delays **83**

1. Introduction

The concept of time scales analysis is a fairly new idea. In 1988, it was introduced by the German mathematician Stefan Hilger in his Ph.D. thesis [14]. It combines the traditional areas of continuous and discrete analysis into one theory. After the publication of two textbooks in this area by Bohner and Peterson [7] and [8], more and more researchers were getting involved in this fast-growing field of mathematics.

The study of dynamic equations brings together the traditional research areas of differential and difference equations. It allows one to handle these two research areas at the same time, hence shedding light on the reasons for their seeming discrepancies. In fact, many new results for the continuous and discrete cases have been obtained by studying the more general time scales case (see [1,3,4,15] and the references therein).

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There is no doubt that the Lyapunov method have been used successfully to investigate stability properties of wide variety of ordinary, functional and partial equations. Nevertheless, the application of this method to problem of stability in differential equations with delay has encountered serious difficulties if the delay is unbounded or if the equation has unbounded term. It has been noticed that some of theses difficulties vanish by using the fixed point technic. Other advantages of fixed point theory over Lyapunov's method is that the conditions of the former are average while those of the latter are pointwise (see [2,5,6,9,10,11,12,13] and references therein).

In paper, we consider the following mixed linear Levin-Nohel integro-dynamic equation with variable delays

$$x^{\Delta}(t)+\int_{t-r(t)}^{t}a(t,s)x(s)\Delta s+b(t)x(t-h(t))=0,\ t\in[t_0,\infty)\cap\mathbb{T}, \tag{1.1}$$

with an assumed initial function

$$x\left(t\right)=\phi\left(t\right),\ t\in\left[m\left(t_0\right),t_0\right]\cap\mathbb{T},$$

where $\mathbb{T}$ is an unbounded above and below time scale and such that $0,t_0\in\mathbb{T}$, $\phi:\left[m\left(t_0\right),t_0\right]\cap\mathbb{T}\to\mathbb{R}$ is rd-continuous and

$$m\left(t_0\right)=\min\left(\inf\left\{t-r\left(t\right):t\in\left[t_0,\infty\right)\cap\mathbb{T}\right\},\inf\left\{t-h\left(t\right):t\in\left[t_0,\infty\right)\cap\mathbb{T}\right\}\right).$$

Throughout this paper, we assume that $b:[t_0,\infty)\cap\mathbb{T}\to\mathbb{R}$ is rd-continuous and $a:([t_0,\infty)\cap\mathbb{T})\times([m\left(t_0\right),\infty)\cap\mathbb{T})\to\mathbb{R}$ is rd-continuous. In order for the functions $x\left(t-r\left(t\right)\right)$ and $x\left(t-h\left(t\right)\right)$ to be well-defined over $[t_0,\infty)\cap\mathbb{T}$, we assume that $r,h:[t_0,\infty)\cap\mathbb{T}\to\mathbb{T}$ are positive rd-continuous, and that $id-r,id-h:[t_0,\infty)\cap\mathbb{T}\to\mathbb{T}$ are increasing mappings such that $(id-r)\left([t_0,\infty)\cap\mathbb{T}\right)$ and $(id-h)\left([t_0,\infty)\cap\mathbb{T}\right)$ are closed where id is the identity function.

Our purpose here is to use the contraction mapping theorem (see [16]) to show the asymptotic stability of the zero solution for Eq. (1.1). An asymptotic stability theorem with a necessary and sufficient condition is proved. In the special case $\mathbb{T}=\mathbb{R}$, Dung [13] shows the zero solution of (1.1) is asymptotically stable with a necessary and sufficient condition by using the contraction mapping theorem. Then, the results presented in this paper extend the main results in [13].

In Section 2, we present some preliminary material that we will need through the remainder of the paper. We will state some facts about the exponential function on a time scale. We present our main results on asymptotic stability in Section 3. A study of the general form of (1.1) (with several delays) is given in section 4.

2. Preliminaries

In this section, we consider some advanced topics in the theory of dynamic equations on a time scales. Again, we remind that for a review of this topic we direct the reader to the monographs of Bohner and Peterson [7] and [8].

A time scale $\mathbb{T}$ is a closed nonempty subset of $\mathbb{R}$. For $t \in \mathbb{T}$ the forward jump operator σ, and the backward jump operator ρ, respectively, are defined as $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) = \sup\{t \in \mathbb{T} : s < t\}$. These operators allow elements in the time scale to be classified as follows. We say t is right scattered if $\sigma(t) > t$ and right dense if $\sigma(t) = t$. We say t is left scattered if $\rho(t) < t$ and left dense if $\rho(t) = t$. The graininess function $\mu : \mathbb{T} \to [0, \infty)$, is defined by $\mu(t) = \sigma(t) - t$ and gives the distance between an element and its successor. We set $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$. If $\mathbb{T}$ has a left scattered maximum M, we define $\mathbb{T}^k = \mathbb{T} \diagdown \{M\}$. Otherwise, we define $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right scattered minimum m, we define $\mathbb{T}_k = \mathbb{T} \diagdown \{m\}$. Otherwise, we define $\mathbb{T}_k = \mathbb{T}$.

Let $t \in \mathbb{T}^k$ and let $f : \mathbb{T} \to \mathbb{R}$. The delta derivative of $f(t)$, denoted $f^{\triangle}(t)$, is defined to be the number (when it exists), with the property that, for each $\epsilon > 0$, there is a neighborhood U of t such that

$$\left| f(\sigma(t)) - f(s) - f^{\triangle}(t)[\sigma(t) - s] \right| \le \epsilon |\sigma(t) - s|,$$

for all $s \in U$. If $\mathbb{T} = \mathbb{R}$ then $f^{\triangle}(t) = f'(t)$ is the usual derivative. If $\mathbb{T} = \mathbb{Z}$ then $f^{\triangle}(t) = \triangle f(t) = f(t+1) - f(t)$ is the forward difference of f at t.

A function f is right dense continuous (rd-continuous), $f \in C_{rd} = C_{rd}(\mathbb{T}, \mathbb{R})$, if it is continuous at every right dense point $t \in \mathbb{T}$ and its left-hand limits exist at each left dense point $t \in \mathbb{T}$. The function $f : \mathbb{T} \to \mathbb{R}$ is differentiable on $\mathbb{T}^k$ provided $f^{\triangle}(t)$ exists for all $t \in \mathbb{T}^k$.

We are now ready to state some properties of the delta-derivative of f. Note $f^{\sigma}(t) = f(\sigma(t))$.

Theorem 2.1 ([7, Theorem 1.20])**.** *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$ and let α be a scalar.*

(i) $(f + g)^{\triangle}(t) = g^{\triangle}(t) + f^{\triangle}(t)$.

(ii) $(\alpha f)^{\triangle}(t) = \alpha f^{\triangle}(t)$.

(iii) The product rules

$$\begin{aligned} (fg)^{\triangle}(t) &= f^{\triangle}(t)\, g(t) + f^{\sigma}(t)\, g^{\triangle}(t), \\ (fg)^{\triangle}(t) &= f(t)\, g^{\triangle}(t) + f^{\triangle}(t)\, g^{\sigma}(t). \end{aligned}$$

(iv) If $g(t)\, g^{\sigma}(t) \neq 0$ then

$$\left(\frac{f}{g}\right)^{\triangle}(t) = \frac{f^{\triangle}(t)\, g(t) - f(t)\, g^{\triangle}(t)}{g(t)\, g^{\sigma}(t)}.$$

The next theorem is the chain rule on time scales ([7, Theorem 1.93], Theorem 1.93).

Theorem 2.2 (Chain Rule)**.** *Assume $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. Let $\omega : \widetilde{\mathbb{T}} \to \mathbb{R}$. If $\nu^{\triangle}(t)$ and $\omega^{\widetilde{\triangle}}(\nu(t))$ exist for $t \in \mathbb{T}^k$, then $(\omega \circ \nu)^{\triangle} = \left(\omega^{\widetilde{\triangle}} \circ \nu\right) \nu^{\triangle}$.*

In the sequel we will need to differentiate and integrate functions of the form $f(t - r(t)) = f(\nu(t))$ where, $\nu(t) := t - r(t)$. Our next theorem is the substitution rule ([7, Theorem 1.98], Theorem 1.98).

Theorem 2.3 (Substitution)**.** *Assume $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := \nu(T)$ is a time scale. If $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous function and ν is differentiable with rd-continuous derivative, then for $a, b \in T$,*

$$\int_a^b f(t)\, \nu^{\triangle}(t) \triangle t = \int_{\nu(a)}^{\nu(b)} \left(f \circ \nu^{-1}\right)(s)\, \widetilde{\triangle} s.$$

A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive provided $1 + \mu(t) p(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive rd-continuous function $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$. The set of all positively regressive functions $\mathcal{R}^+$, is given by $\mathcal{R}^+ = \{f \in \mathcal{R} : 1 + \mu(t) f(t) > 0 \text{ for all } t \in \mathbb{T}\}$.

Let $p \in \mathcal{R}$ and $\mu(t) \neq 0$ for all $t \in \mathbb{T}$. The exponential function on $\mathbb{T}$ is defined by

$$e_p(t, s) = \exp\left(\int_s^t \frac{1}{\mu(\tau)} \log(1 + \mu(\tau) p(\tau)) \Delta\tau\right).$$

It is well known that if $p \in \mathcal{R}^+$, then $e_p(t, s) > 0$ for all $t \in \mathbb{T}$. Also, the exponential function $y(t) = e_p(t, s)$ is the solution to the initial value problem $y^{\triangle} = p(t) y$, $y(s) = 1$. Other properties of the exponential function are given by the following lemma.

Lemma 2.4 ([7, Theorem 2.36])**.** *Let $p, q \in \mathcal{R}$. Then*

(*i*) $e_0(t, s) = 1$ *and* $e_p(t, t) = 1$,

(*ii*) $e_p(\sigma(t), s) = (1 + \mu(t) p(t)) e_p(t, s)$,

(*iii*) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t, s)$, *where* $\ominus p(t) = -\frac{p(t)}{1+\mu(t)p(t)}$,

(*iv*) $e_p(t, s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s, t)$,

(*v*) $e_p(t, s) e_p(s, r) = e_p(t, r)$,

(*vi*) $e_p^{\triangle}(., s) = p e_p(., s)$ *and* $\left(\frac{1}{e_p(.,s)}\right)^{\triangle} = -\frac{p(t)}{e_p^{\sigma}(.,s)}$.

Lemma 2.5 ([1])**.** *If $p \in \mathcal{R}^+$, then*

$$0 < e_p(t, s) \leq \exp\left(\int_s^t p(u) \triangle u\right), \ \forall t \in \mathbb{T}.$$

3. Main results

For the convenience of the reader, let us recall the definition of asymptotic stability. For each $t_0 > 0$, we denote $C_{rd}(t_0)$ the space of rd-continuous functions on $[m(t_0), t_0] \cap \mathbb{T}$ with the supremum norm $\|.\|_{t_0}$. For each $(t_0, \phi) \in \mathbb{T} \times C_{rd}(t_0)$, denoted by $x(t) = x(t, t_0, \phi)$ the unique solution of Eq. (1.1).

Definition 3.1. *The zero solution of Eq. (1.1) is called:*

(i) stable if for each $\varepsilon > 0$*, there exists a* $\delta > 0$ *such that* $|x(t,t_0,\phi)| < \varepsilon$ *for all* $t \geq t_0$ *if* $\|\phi\|_{t_0} < \delta$*,*

(ii) asymptotically stable if it is stable and $\lim_{t\to\infty} |x(t,t_0,\phi)| = 0$.

Theorem 3.2. *Suppose that the following two conditions hold:*

$$A \in \mathcal{R}^+, \ \liminf_{t\to\infty} \int_0^t \frac{1}{\mu(\tau)} \log(1 + \mu(\tau)A(\tau))\Delta\tau > -\infty, \tag{3.1}$$

$$\sup_{t\geq 0} \int_0^t \omega(s) e_{\ominus A}(t,s)\Delta s = \alpha < 1, \tag{3.2}$$

where

$$A(\tau) = \int_{\tau - r(\tau)}^{\tau} a(\tau,s)\Delta s + b(\tau),$$

and

$$\omega(s) = \int_{s-r(s)}^{s} |a(s,w)| \int_w^{\sigma(s)} \left(\int_{u-r(u)}^{u} |a(u,v)| \, \Delta v + |b(u)| \right) \Delta u \Delta w$$
$$+ |b(s)| \int_{s-h(s)}^{\sigma(s)} \left(\int_{u-r(u)}^{u} |a(u,v)| \, \Delta v + |b(u)| \right) \Delta u.$$

Then the zero solution of (1.1) is asymptotically stable if and only if

$$\int_0^t \frac{1}{\mu(\tau)} \log(1 + \mu(\tau)A(\tau))\Delta\tau \to \infty \ \textit{as} \ t \to \infty. \tag{3.3}$$

Proof. In order to be able to construct a new fixed mapping, we transform the Levin-Nohel equation into an equivalent equation. Obviously, we have

$$x(s) = x^\sigma(t) - \int_s^{\sigma(t)} x^\Delta(u)\Delta u, \ x(t-h(t)) = x^\sigma(t) - \int_{t-h(t)}^{\sigma(t)} x^\Delta(u)\Delta u.$$

Insetting these relations into (1.1), we get

$$x^\Delta(t) + \int_{t-r(t)}^{t} a(t,s) \left(x^\sigma(t) - \int_s^{\sigma(t)} x^\Delta(u)\Delta u \right) \Delta s$$
$$+ b(t)x^\sigma(t) - b(t) \int_{t-h(t)}^{\sigma(t)} x^\Delta(u)\Delta u = 0, \ t \geq t_0,$$

or equivalently

$$x^\Delta(t) + x^\sigma(t) \left(\int_{t-r(t)}^{t} a(t,s)\Delta s + b(t) \right) - \int_{t-r(t)}^{t} a(t,s) \left(\int_s^{\sigma(t)} x^\Delta(u)\Delta u \right) \Delta s$$
$$- b(t) \int_{t-h(t)}^{\sigma(t)} x^\Delta(u)\Delta u = 0, \ t \geq t_0.$$

After substituting x^{Δ} from (1.1), we obtain

$$\begin{aligned}
&x^{\Delta}(t)+\left(\int_{t-r(t)}^{t} a(t,s)\Delta s+b(t)\right)x^{\sigma}(t)\\
&+\int_{t-r(t)}^{t} a(t,s)\left(\int_{s}^{\sigma(t)}\left(\int_{u-r(u)}^{u} a(u,v)x(v)\Delta v+b(u)x(u-h(u))\right)\Delta u\right)\Delta s\\
&+b(t)\int_{t-h(t)}^{\sigma(t)}\left(\int_{u-r(u)}^{u} a(u,v)x(v)\Delta v+b(u)x(u-h(u))\right)\Delta u \;=\; 0,\; t\;\geq\; t_0.
\end{aligned} \tag{3.4}$$

For the convenience of the statement, we put

$$\begin{aligned}
L_x(t) &= \int_{t-r(t)}^{t} a(t,s)\left(\int_{s}^{\sigma(t)}\left(\int_{u-r(u)}^{u} a(u,v)x(v)\Delta v\right.\right.\\
&\qquad \left.\left.+b(u)x(u-h(u))\right)\Delta u\right)\Delta s,\\
N_x(t) &= b(t)\int_{t-h(t)}^{\sigma(t)}\left(\int_{u-r(u)}^{u} a(u,v)x(v)\Delta v+b(u)x(u-h(u))\right)\Delta u.
\end{aligned}$$

Then, (3.4) now becomes

$$x^{\Delta}(t)+A(t)x^{\sigma}(t)+L_x(t)+N_x(t)=0,\; t\geq t_0,$$

which, by variation of constants formula, gives us

$$x(t)=\phi(t_0)e_{\ominus A}(t,t_0)-\int_{t_0}^{t} L_x(s)e_{\ominus A}(t,s)\Delta s-\int_{t_0}^{t} N_x(s)e_{\ominus A}(t,s)\Delta s,\; t\geq t_0. \tag{3.5}$$

Sufficient condition. Suppose that (3.3) holds. Denoted by C_{rd} the space of rd-continuous bounded functions $x:[m(t_0),\infty)\cap\mathbb{T}\to\mathbb{R}$ such that $x(t)=\phi(t)$, $t\in[m(t_0),t_0]\cap\mathbb{T}$. It is known that C_{rd} is a complete metric space endowed with a metric $\|x\|=\sup_{t\geq m(t_0)}|x(t)|$. Define the operator P on C_{rd} by $(Px)(t)=\phi(t)$, $t\in[m(t_0),t_0]\cap\mathbb{T}$ and

$$(Px)(t)=\phi(t_0)e_{\ominus A}(t,t_0)-\int_{t_0}^{t} L_x(s)e_{\ominus A}(t,s)\Delta s-\int_{t_0}^{t} N_x(s)e_{\ominus A}(t,s)\Delta s,\; t\geq t_0.$$

Obviously, Px is rd-continuous for each $x\in C_{rd}$. Moreover, it is a contraction operator. Indeed, let $x,y\in C_{rd}$,

$$\begin{aligned}
&|(Px)(t)-(Py)(t)|\\
&\leq \int_{t_0}^{t}|L_x(s)-L_y(s)|\,e_{\ominus A}(t,s)\Delta s+\int_{t_0}^{t}|N_x(s)-N_y(s)|\,e_{\ominus A}(t,s)\Delta s.
\end{aligned}$$

Since $x(t) = y(t) = \phi(t)$ for all $t \in [m(t_0), t_0] \cap \mathbb{T}$ this implies that

$$\begin{aligned} &|L_x(s) - L_y(s)| \\ &\leq \left(\int_{s-r(s)}^{s} |a(s,w)| \int_{w}^{\sigma(s)} \left(\int_{u-r(u)}^{u} |a(u,v)| \, \Delta v + |b(u)| \right) \Delta u \Delta w \right) \|x - y\| \,, \end{aligned}$$

and

$$\begin{aligned} &|N_x(s) - N_y(s)| \\ &\leq \left(|b(s)| \int_{s-h(s)}^{\sigma(s)} \left(\int_{u-r(u)}^{u} |a(u,v)| \, \Delta v + |b(u)| \right) \Delta u \right) \|x - y\| \,. \end{aligned}$$

Consequently, it holds for all $t \geq t_0$ that

$$\begin{aligned} &|(Px)(t) - (Py)(t)| \\ &\leq \left[\int_{t_0}^{t} \left(\int_{s-r(s)}^{s} |a(s,w)| \int_{w}^{\sigma(s)} \left(\int_{u-r(u)}^{u} |a(u,v)| \, \Delta v + |b(u)| \right) \Delta u \Delta w \right) e_{\ominus A}(t,s) \Delta s \right. \\ &\left. + \int_{t_0}^{t} |b(s)| \left(\int_{s-h(s)}^{\sigma(s)} \left(\int_{u-r(u)}^{u} |a(u,v)| \, \Delta v + |b(u)| \right) \Delta u \right) e_{\ominus A}(t,s) \Delta s \right] \|x - y\| \,. \end{aligned}$$

Hence, it follows from (3.2) that

$$|(Px)(t) - (Py)(t)| \leq \alpha \|x - y\| \,, \; t \geq t_0.$$

Thus P is a contraction operator on C_{rd}.

We now consider a closed subspace S of C_{rd} that is defined by

$$S = \{ x \in C_{rd} : |x(t)| \to 0 \text{ as } t \to \infty \} \,.$$

We will show that $P(S) \subset S$. To do this, we need to point out that for each $x \in S$, $|(Px)(t)| \to 0$ as $t \to \infty$. Let $x \in S$, by the definition of P we have

$$\begin{aligned} |(Px)(t)| &\leq |\phi(t_0)| \, e_{\ominus A}(t,t_0) + \left| \int_{t_0}^{t} L_x(s) e_{\ominus A}(t,s) \Delta s \right| + \left| \int_{t_0}^{t} N_x(s) e_{\ominus A}(t,s) \Delta s \right| \\ &= I_1 + I_2 + I_3, \; t \geq t_0. \end{aligned}$$

The first term I_1 tends to 0 by (3.3). For any $T \in (t_0, t)$, we have the following estimate for the second term

$$\begin{aligned}
I_2 &\le \left|\int_{t_0}^{T} L_x(s)e_{\ominus A}(t,s)\Delta s\right| + \left|\int_{T}^{t} L_x(s)e_{\ominus A}(t,s)\Delta s\right| \\
&\le \int_{t_0}^{T}\left(\int_{s-r(s)}^{s}|a(s,w)|\left(\int_{w}^{\sigma(s)}\left(\int_{u-r(u)}^{u}|a(u,v)|\,\|x\|\,\Delta v\right.\right.\right. \\
&\quad \left.\left.\left. + |b(u)|\,\|\phi\|_{t_0}\right)\Delta u\right)\Delta w\right) e_{\ominus A}(t,s)\Delta s \\
&\quad + \int_{T}^{t}\left(\int_{s-r(s)}^{s}|a(s,w)|\left(\int_{w}^{\sigma(s)}\left(\int_{u-r(u)}^{u}|a(u,v)|\,|x(v)|\,\Delta v\right.\right.\right. \\
&\quad \left.\left.\left. + |b(u)|\,|x(u-h(u))|\right)\Delta u\right)\Delta w\right) e_{\ominus A}(t,s)\Delta s \\
&\le \int_{t_0}^{T}\left(\int_{s-r(s)}^{s}|a(s,w)|\left(\int_{w}^{\sigma(s)}\left(\int_{u-r(u)}^{u}|a(u,v)|\,\Delta v\right.\right.\right. \\
&\quad \left.\left.\left. + |b(u)|\right)\Delta u\right)\Delta w\right) e_{\ominus A}(t,s)\Delta s\left(\|x\| + \|\phi\|_{t_0}\right) \\
&\quad + \int_{T}^{t}\left(\int_{s-r(s)}^{s}|a(s,w)|\left(\int_{w}^{\sigma(s)}\left(\int_{u-r(u)}^{u}|a(u,v)|\,|x(v)|\,\Delta v\right.\right.\right. \\
&\quad \left.\left.\left. + |b(u)|\,|x(u-h(u))|\right)\Delta u\right)\Delta w\right) e_{\ominus A}(t,s)\Delta s \\
&= I_{21} + I_{22}.
\end{aligned}$$

Since $t - r(t) \to \infty$ as $t \to \infty$, this implies that $u - r(u) \to \infty$ as $T \to \infty$. Thus, from the fact $|x(v)| \to 0$, $v \to \infty$ we can infer that for any $\varepsilon > 0$ there exists $T_1 = T > t_0$ such that

$$\begin{aligned}
I_{22} &< \frac{\varepsilon}{2\alpha}\int_{T_1}^{t}\left(\int_{s-r(s)}^{s}|a(s,w)|\int_{w}^{\sigma(s)}\left(\int_{u-r(u)}^{u}|a(u,v)|\,\Delta v\right.\right. \\
&\quad \left.\left. + |b(u)|\right)\Delta u\Delta w\right) e_{\ominus A}(t,s)\Delta s,
\end{aligned}$$

and hence, $I_{22} < \frac{\varepsilon}{2}$ for all $t \ge T_1$. On the other hand, $\|x\| < \infty$ because $x \in S$. This combined with (3.3) yields $I_{21} \to 0$ as $t \to \infty$. As a consequence, there exists $T_2 \ge T_1$ such that $I_{21} < \frac{\varepsilon}{2}$ for all $t \ge T_2$. Thus, $I_2 < \varepsilon$ for all $t \ge T_2$; that is, $I_2 \to 0$ as $t \to \infty$. Similarly, $I_3 \to 0$ as $t \to \infty$. So $P(S) \subset S$.

By the Contraction Mapping Principle, P has a unique fixed point x in S which is a solution of (1.1) with $x(t) = \phi(t)$ on $[m(t_0), t_0] \cap \mathbb{T}$ and $x(t) = x(t, t_0, \phi) \to 0$ as $t \to \infty$.

To obtain the asymptotic stability, we need to show that the zero solution of (1.1) is stable. By condition (3.1), we can define

$$K = \sup_{t \ge 0} e_{\ominus A}(t, 0) < \infty. \tag{3.6}$$

Using the formula (3.5) and condition (3.2), we can obtain

$$|x(t)| \le K\,\|\phi\|_{t_0}\, e_A(t_0, 0) + \alpha(\|x\| + \|\phi\|_{t_0}),\ t \ge t_0,$$

which leads us to

$$\|x\| \leq \frac{Ke^{\int_0^{t_0} A(\tau)\Delta\tau} + \alpha}{1-\alpha} \|\phi\|_{t_0}. \tag{3.7}$$

Thus for every $\varepsilon > 0$, we can find $\delta > 0$ such that $\|\phi\|_{t_0} < \delta$ implies that $\|x\| < \varepsilon$. This shows that the zero solution of (1.1) is stable and hence, it is asymptotically stable.

Necessary condition. Suppose that the zero solution of (1.1) is asymptotically stable and that the condition (3.3) fails. It follows from (3.1) that there exists a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ such that

$$\lim_{n\to\infty} \int_0^{t_n} \frac{1}{\mu(\tau)} \log(1+\mu(\tau)A(\tau))\Delta\tau,$$

exists and is finite. Hence, we can choose a positive constant L satisfying

$$-L < \lim_{n\to\infty} \int_0^{t_n} \frac{1}{\mu(\tau)} \log(1+\mu(\tau)A(\tau))\Delta\tau < L, \ \forall n \geq 1. \tag{3.8}$$

Then, condition (3.2) gives us

$$c_n = \int_0^{t_n} \omega(s)e_A(s,0)\Delta s \leq \alpha e_A(t_n,0) < e^L.$$

The sequence $\{c_n\}$ is increasing and bounded, so it has a finite limit. For any $\delta_0 > 0$, there exists $n_0 > 0$ such that

$$\int_{t_{n_0}}^{t_n} \omega(s)e_A(s,0)\Delta s < \frac{\delta_0}{2K}, \ \forall n \geq n_0, \tag{3.9}$$

where K is as in (3.6). We choose δ_0 such that $\delta_0 < \frac{1-\alpha}{Ke^L+1}$ and consider the solution $x(t) = x(t,t_n,\phi)$ of (1.1) with the initial data $\phi(t_{n_0}) = \delta_0$ and $|\phi(s)| \leq \delta_0$, $s \leq t_{n_0}$. It follows from(3.7) that

$$|x(t)| \leq 1-\delta_0, \ \forall t \geq t_{n_0}. \tag{3.10}$$

Applying the fundamental inequality $|a-b| \geq |a|-|b|$ and then using (3.10), (3.9) and (3.8), we get

$$\begin{aligned}
|x(t_n)| &\geq |\phi(t_{n_0})e_{\ominus A}(t_n,t_{n_0})| - \left| \int_{t_{n_0}}^{t_n} (L_x(s)+N_x(s))\, e_{\ominus A}(t_n,s)\Delta s \right| \\
&\geq \delta_0 e_{\ominus A}(t_n,t_{n_0}) - \int_{t_{n_0}}^{t_n} \omega(s)e_{\ominus A}(t_n,s)\Delta s \\
&\geq e_{\ominus A}(t_n,t_{n_0}) \left(\delta_0 - e_{\ominus A}(t_{n_0},0) \int_{t_{n_0}}^{t_n} \omega(s)e_A(s,0)\Delta s \right) \\
&\geq e_{\ominus A}(t_n,t_{n_0}) \left(\delta_0 - K \int_{t_{n_0}}^{t_n} \omega(s)e_A(s,0)\Delta s \right) \\
&\geq \frac{1}{2}\delta_0 e_{\ominus A}(t_n,t_{n_0}) \geq \frac{1}{2}\delta_0 e^{-2L} > 0,
\end{aligned}$$

which is a contradiction because $x(t_n) \to 0$ as $t_n \to \infty$. The proof is complete. □

For the special case $b(t) = 0$ we get the following.

Corollary 3.3. *Suppose that the following two conditions hold*

$$A_0 \in \mathcal{R}^+, \ \liminf_{t\to\infty} \int_0^t \frac{1}{\mu(\tau)} \log(1+\mu(\tau)A_0(\tau))\Delta\tau > -\infty, \tag{3.11}$$

$$\sup_{t\geq 0} \int_0^t \left(\int_{s-r(s)}^s |a(s,w)| \int_w^{\sigma(s)} \int_{u-r(u)}^u |a(u,v)| \Delta v \Delta u \Delta w \right) e_{\ominus A_0}(t,s)\Delta s = \alpha < 1, \tag{3.12}$$

where

$$A_0(\tau) = \int_{\tau-r(\tau)}^{\tau} a(\tau,s)\Delta s.$$

Then the zero solution of

$$x^{\Delta}(t) + \int_{t-r(t)}^t a(t,s)x(s)\Delta s = 0,$$

is asymptotically stable if and only if

$$\int_0^t \frac{1}{\mu(\tau)} \log(1+\mu(\tau)A_0(\tau))\Delta\tau \to \infty \ \text{as } t \to \infty. \tag{3.13}$$

Example 3.1. *Let $\mathbb{T} = \mathbb{R}$. Consider the following linear Levin-Nohel integro-differential equation with variable delay*

$$x'(t) + \int_{t-r(t)}^t a(t,s)x(s)ds = 0, \tag{3.14}$$

where $a(t,s) = \frac{10}{t^2+1}$, $r(t) = 0.2t$. Then the zero solution of (3.14) is asymptotically stable.

Proof. We have

$$A_0(t) = \int_{0.8t}^t a(t,s)ds = \int_{0.8t}^t \frac{10}{t^2+1}ds = \frac{2t}{t^2+1},$$

$$\int_0^t A_0(\tau)d\tau = \ln(t^2+1),$$

and

$$\begin{aligned} &\int_{s-r(s)}^s |a(s,w)| \int_w^s \int_{u-r(u)}^u |a(u,v)|\, dvdudw \\ &= \int_{0.8s}^s \frac{10}{s^2+1} \int_w^s \int_{0.8u}^u \frac{10}{u^2+1} dvdudw \\ &= \frac{1}{s^2+1} (4s + 2\arctan 0.8s - 2\arctan s \\ &+8s\ln\left(0.64s^2+1\right) - 8s\ln\left(s^2+1\right)). \end{aligned}$$

Then

$$\begin{aligned}
&\sup_{t\geq 0}\int_0^t\left(\int_{s-r(s)}^s |a(s,w)|\int_w^s\int_{u-r(u)}^u |a(u,v)|\,dvdudw\right)e^{-\int_s^t A_0(\tau)d\tau}ds\\
&=\sup_{t\geq 0}\left\{\frac{1}{t^2+1}\int_0^t\left(\int_{0.8s}^s\frac{10}{s^2+1}\int_w^s\int_{0.8u}^u\frac{10}{u^2+1}dvdudw\right)\left(s^2+1\right)ds\right\}\\
&\leq 0.216<1.
\end{aligned}$$

It is easy to see that all the conditions of Corollary 3.3 hold for $\alpha = 0.216 < 1$. Thus Corollary 3.3 implies that the zero solution of (3.14) is asymptotically stable. □

Letting $a(t,s) = 0$ in (1.1), we get the following.

Corollary 3.4. *Suppose that the following two conditions hold*

$$b\in\mathcal{R}^+,\ \liminf_{t\to\infty}\int_0^t\frac{1}{\mu(\tau)}\log(1+\mu(\tau)b(\tau))\Delta\tau > -\infty, \tag{3.15}$$

$$\sup_{t\geq 0}\int_0^t |b(s)|\left(\int_{s-h(s)}^{\sigma(s)}|b(u)|\,\Delta u\right)e_{\ominus b}(t,s)\Delta s = \alpha < 1. \tag{3.16}$$

Then the zero solution of

$$x^{\Delta}(t) + b(t)x(t-h(t)) = 0,$$

is asymptotically stable if and only if

$$\int_0^t\frac{1}{\mu(\tau)}\log(1+\mu(\tau)b(\tau))\Delta\tau\to\infty\ \text{as}\ t\to\infty. \tag{3.17}$$

4. Levin-Nohel equations with several delays

The method in Section 3 can be extended to the following mixed Levin-Nohel integro-dynamic equation with several delays

$$x^{\Delta}(t) + \sum_{k=1}^m\int_{t-r_k(t)}^t a_k(t,s)x(s)\Delta s + \sum_{k=1}^M b_k(t)x(t-h_k(t)) = 0. \tag{4.1}$$

In a same way as in Theorem 3.2, we can rewrite (4.1) as follows

$$\begin{aligned}
&x^{\Delta}(t) + x^{\sigma}(t)\left(\sum_{k=1}^m\int_{t-r_k(t)}^t a_k(t,s)\Delta s + \sum_{k=1}^M b_k(t)\right)\\
&-\sum_{k=1}^m\int_{t-r_k(t)}^t a_k(t,s)\left(\int_s^{\sigma(t)}x^{\Delta}(u)\Delta u\right)\Delta s\\
&-\sum_{k=1}^M b_k(t)\int_{t-h_k(t)}^{\sigma(t)}x^{\Delta}(u)\Delta u\ =\ 0,
\end{aligned}$$

or equivalently

$$\begin{aligned}
&x^{\Delta}(t) + x^{\sigma}(t)\left(\sum_{k=1}^{m}\int_{t-r_k(t)}^{t} a_k(t,s)\Delta s + \sum_{k=1}^{M} b_k(t)\right) \\
&+\sum_{k=1}^{m}\int_{t-r_k(t)}^{t} a_k(t,s)\left(\int_s^{\sigma(t)}\left(\sum_{i=1}^{m}\int_{u-r_i(u)}^{u} a_i(u,v)x(v)\Delta v\right.\right. \\
&\left.\left.+\sum_{i=1}^{M} b_i(u)x(u-h_i(u))\right)\Delta u\right)\Delta s \\
&+\sum_{k=1}^{M} b_k(t)\int_{t-h_k(t)}^{\sigma(t)}\left(\sum_{i=1}^{m}\int_{u-r_i(u)}^{u} a_i(u,v)x(v)\Delta v\right. \\
&\left.+\sum_{i=1}^{M} b_i(u)x(u-h_i(u))\right)\Delta u \ = \ 0, \ t \ \geq \ t_0.
\end{aligned}$$

Put

$$\overline{A}(t) = \sum_{k=1}^{m}\int_{t-r_k(t)}^{t} a_k(t,s)\Delta s + \sum_{k=1}^{M} b_k(t),$$

$$\begin{aligned}
\overline{L}_x(t) = &\sum_{k=1}^{m}\int_{t-r_k(t)}^{t} a_k(t,s)\left(\int_s^{\sigma(t)}\left(\sum_{i=1}^{m}\int_{u-r_i(u)}^{u} a_i(u,v)x(v)\Delta v\right.\right. \\
&\left.\left.+\sum_{i=1}^{M} b_i(u)x(u-h_i(u))\right)\Delta u\right)\Delta s,
\end{aligned}$$

and

$$\begin{aligned}
\overline{N}_x(t) = &\sum_{k=1}^{M} b_k(t)\int_{t-h_k(t)}^{\sigma(t)}\left(\sum_{i=1}^{m}\int_{u-r_i(u)}^{u} a_i(u,v)x(v)\Delta v\right. \\
&\left.+\sum_{i=1}^{M} b_i(u)x(u-h_i(u))\right)\Delta u.
\end{aligned}$$

Then, (4.1) now becomes

$$x^{\Delta}(t) + \overline{A}(t)x^{\sigma}(t) + \overline{L}_x(t) + \overline{N}_x(t) = 0.$$

The proof of the following theorem is similar to that of Theorem 3.2 and hence, we omit it.

Theorem 4.1. *Suppose that the following two conditions hold*

$$\overline{A} \in \mathcal{R}^{+}, \ \liminf_{t\to\infty}\int_0^t \frac{1}{\mu(\tau)}\log(1+\mu(\tau)\overline{A}(\tau))\Delta\tau > -\infty,$$

and

$$\sup_{t\geq 0}\int_0^t \overline{\omega}(s)e_{\ominus\overline{A}}(t,s)\Delta s = \alpha < 1,$$

where

$$\overline{\omega}(s)$$
$$= \sum_{k=1}^{m}\int_{s-r_k(s)}^{s}|a_k(s,w)|\left(\int_w^{\sigma(s)}\left(\sum_{i=1}^{m}\int_{u-r_i(u)}^{u}|a_i(u,v)|\,\Delta v + \sum_{i=1}^{M}|b_i(u)|\right)\Delta u\right)\Delta w$$
$$+\sum_{k=1}^{M}|b_k(s)|\int_{t-h_k(s)}^{\sigma(s)}\left(\sum_{i=1}^{m}\int_{u-r_i(u)}^{u}|a_i(u,v)|\,\Delta v + \sum_{i=1}^{M}|b_i(u)|\right)\Delta u.$$

Then the zero solution of (4.1) is asymptotically stable if and only if

$$\int_0^t \frac{1}{\mu(\tau)}\log(1+\mu(\tau)\overline{A}(\tau))\Delta\tau \to \infty \text{ as } t\to\infty.$$

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