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# Boundedness and Convergence Analysis of Stochastic Differential Equations with Hurst Brownian Motion

Davood Ahmadian* and Omid Farkhondeh Rouz

ABSTRACT: In this paper, we discuss about the boundedness and convergence analysis of the fractional Brownian motion (FBM) with Hurst parameter $H$. By the simple analysis and using the mean value theorem for stochastic integrals we conclude that in case of decreasing diffusion function, the solution of FBM is bounded for any $H \in (0,1)$. Also, we derive the convergence rate which shows efficiency and accuracy of the computed solutions.

Key Words: Fractional Brownian Motion, Hurst Parameter, Boundedness, Convergence.

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## 1. Introduction

The self-similarity and long-range dependence properties make the fractional Brownian motion a suitable driving noise in different applications like mathematical finance and network traffic analysis. he convergence analysis of stochastic differential equations in this paper is useful for fractional calculus. For example, in [3], [25], and [26] it is established suffient explicit conditions for globally asymptotic stability of linear fractional differential system with distributed delays. In addition recent achievements in mathematical finance theory [24], [14], and [15] could be extended with Hurst Brownian motion. As well some stochastic concepts of almost sure and exponential mean-square stability of Hurst Brownian motion can be investigated ([11], [12] and [13]).

Recently, there have been several attempts to construct a stochastic calculus with respect to the FBM (see [8], [19]). It is highly important to identify the value of the Hurst parameter in order to understand the structure of the process and its applications, since the calculations differ dramatically according to the

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value of $H$, therefore, some techniques have been developed for estimating the Hurst parameter, for example in [2] and [1], authors investigated to construct a stochastic integral with respect to the FBM with Hurst parameter $H \in (0, \frac{1}{2})$, by using Malliavin calculus, or in [7], authors present a path-wise approach towards a stochastic analysis for fractional Brownian motions. Also in some papers such as [21], authors study such a stochastic integrals to a smaller class of processes, namely the bounded sure processes, on finite time intervals. Moreover Duncan et al. (see [9]) study the FBM, in Hilbert space with the Hurst parameter in the interval $H \in (\frac{1}{2}, 1)$. As well some numerical methods are presented to estimate the Hurst parameter $H$ (see [5], [20], [23] and [18]) and some numerical solutions are presented to prove the convergence rate of this kind of equations (see in [16] and [17]).
In [22], the existence and uniqueness of the multi dimensional, time dependent FBM is driven with Hurst parameter $H > \frac{1}{2}$.
Based on these papers we first show that the solution of FBM is bounded and subsequently by following the [10], we prove the convergence solution of FBM with Hurst parameter $H \in (0, 1)$. To this aim, first we evaluate the boundedness of the solution of FBM. We use the elementary chain rule calculus and the mean value theorem for stochastic functions develepoed in [6], which cause to restrict the diffusion function $g(t, X(t))$ to a decreasing diffusion function.
The rest of the paper is organized as follows. Section 2 begins with notations and preliminaries of Hurst Brownian motion. Section 3 examines the conditions under which the solution of FBM is bounded. Section 4 describes the convergence analysis of this kind equations.

## 2. Preliminaries and Notations

In this section, we review some of the standard facts on the fractional calculus. Let $t \in (0, \infty)$ be a real number and $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space. The scalar stochastic differential equation with a standard Hurst parameter has the following general form

$$\begin{cases} dX(t) = f\big(t, X(t)\big)dt + g\big(t, X(t)\big)dW^H(t), \quad t > 0, \\ X(t_0) = X_0. \end{cases} \tag{2.1}$$

The following definition provides an infinite-dimensional analogue of a fractional Brownian motion in a finite-dimensional space with Hurst parameter $H \in (0, 1)$ (see [19]).

**Definition 2.1.** *A $U$-valued Gaussian process $(W^H(t), t \in \mathbb{R})$ on $(\Omega, \mathcal{F}, \mathbf{P})$ is called a fractional Brownian motion with Hurst parameter $H \in (0, 1)$ if*

$$\mathbb{E}\left(W_t^H\right) = 0,$$
$$\mathbb{E}\left(W_t^H W_s^H\right) = \frac{1}{2}\left\{|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right\}. \tag{2.2}$$

The constant $H$ determines the sign of the covariance of the future and past increments. This covariance is positive when $H > 1/2$ and negative when $H < 1/2$. The case $H = \frac{1}{2}$ corresponds to the ordinary Brownian motion.

**Assumption 1.** *Let the functions $f$ and $g$ satisfy the local Lipschitz condition, that is, for each $j > 0$ there exists a positive constant $K_j$ such that for any $X, \overline{X} \in \mathbb{R}^n$ with $|X| \vee |\overline{X}| \leq j$, and $t > 0$,*

$$|f(t, X) - f(t, \overline{X})| \vee |g(t, X) - g(t, \overline{X})| \leq K_j(|X - \overline{X}|). \tag{2.3}$$

## 3. Boundedness Analysis of the FBM with Hurst Index

In this section, let us firstly investigate boundedness of the solution of the FBM with Hurst parameter.

**Lemma 3.1.** *Let $T > 0$ and $X(t)$ be the solution of equation (2.1) at $t \in [0, T]$, then for any $H \in (0, 1)$ and dereasing diffusion function $g(t, X(t))$, there exists a finite positive constant $C$ such that*

$$\sup_{0 \leq t \leq T} \mathbb{E}(|X(t)|^2) \leq C \left(1 + \mathbb{E}|X(0)|^2\right). \tag{3.1}$$

**Proof:** From equation (2.1), we have

$$X(t) = X(0) + \int_0^t f(s, X(s))ds + \int_0^t g(s, X(s))dW^H(s) = X(0) + A + B. \tag{3.2}$$

By using the cauchy-schwarz inegaulty and linear growth condition for $f$, it is clear that:

$$\mathbb{E}(|A|^2) \leq \mathbb{E}\left(|\int_0^t f^2(s, X(s))ds|\right) \leq C\mathbb{E}\left(|\int_0^t (1 + X^2(s))ds|\right). \tag{3.3}$$

Now we want to obtain the similar result for the term $B$, so we have

$$\begin{aligned}
\mathbb{E}(|B|^2) &= \mathbb{E}\left(|\int_0^t g(s, X(s))dW^H(s)|^2\right) \\
&= \mathbb{E}\left(|\int_0^t g'(s, X(s))W^H(s)ds|^2\right) \\
&= \mathbb{E}\left(|\int_0^t \int_0^t g'(s, X(s))g'(u, X(u))W^H(s)W^H(u)|dsdu\right) \\
&= \int_0^t \int_0^t \mathbb{E}\big(|g'(s, X(s))g'(u, X(u))|\big)\mathbb{E}\left(W^H(s)W^H(u)\right) dsdu,
\end{aligned} \tag{3.4}$$

by the Definition 2.1 we have

$$\begin{aligned}
\mathbb{E}\left(|B|^2\right) &= \frac{1}{2}\int_0^t\int_0^t \mathbb{E}\big(|g'(s,X(s))g'(u,X(u))|\big)\left[|u|^{2H}+|s|^{2H}-|u-s|^{2H}\right] dsdu \\
&= \frac{1}{2}\int_0^t\int_0^t \mathbb{E}\big(|g'(s,X(s))g'(u,X(u))|\big)|u|^{2H} dsdu \\
&+ \frac{1}{2}\int_0^t\int_0^t \mathbb{E}\big(|g'(s,X(s))g'(u,X(u))|\big)|s|^{2H} dsdu \\
&- \frac{1}{2}\int_0^t\int_0^u \mathbb{E}\big(|g'(s,X(s))g'(u,X(u))|\big)(u-s)^{2H} dsdu \\
&- \frac{1}{2}\int_0^t\int_u^t \mathbb{E}\big(|g'(s,X(s))g'(u,X(u))|\big)(s-u)^{2H} dsdu \\
&= B_1+B_2+B_3+B_4, \qquad (3.5)
\end{aligned}$$

then we evaluate the terms of $B_1, B_2, B_3$ and $B_4$ separately. First we consider the value of $B_1$

$$\begin{aligned}
B_1 &= \frac{1}{2}\int_0^t\int_0^t \mathbb{E}\left(|g'(s,X(s))g'(u,X(u))|\right)|u|^{2H} dsdu \\
&= \frac{1}{2}\mathbb{E}|\int_0^t g'(s,X(s))\left(\int_0^t g'(u,X(u))|u|^{2H}du\right)|ds \\
&= \frac{1}{2}g(s,X(s))\mid_0^t \int_0^t \mathbb{E}\big(|g'(u,X(u))|\big)|u|^{2H}du. \qquad (3.6)
\end{aligned}$$

By using the chain rule in $B_1$ we have

$$\begin{aligned}
B_1 &= \frac{1}{2}\big(g(t,X(t))-g(0,X(0))\big)\left(|t|^{2H}g(t,X(t))-2H\mathbb{E}\left(\int_0^t (g(u,X(u)))|u|^{2H-1}du\right)\right) \\
&= \frac{t^{2H}}{2}g^2(t,X(t))-Hg(t,X(t))\mathbb{E}\left(\int_0^t g(u,X(u))|u|^{2H-1}du\right) \\
&- \frac{t^{2H}}{2}g(t,X(t))g(0,X(0))+Hg(0,X(0))\mathbb{E}\left(\int_0^t g(u,X(u))|u|^{2H-1}du\right). \qquad (3.7)
\end{aligned}$$

Similary for evaluating the term $B_2$ we can conclude that $B_2 = B_1$.

For the third term of (3.5), similarly by using the chain rule we derive

$$
\begin{aligned}
B_3 &= -\frac{1}{2}\int_0^t\int_0^u \mathbb{E}\big(|g'(s,X(s))g'(u,X(u))|\big)(u-s)^{2H}dsdu \\
&= -\frac{1}{2}\mathbb{E}\int_0^t g'(u,X(u))\left(\int_0^u (u-s)^{2H}g'(s,X(s))ds\right)du \\
&= -\frac{1}{2}\mathbb{E}\int_0^t g'(u,X(u))\left\{(u-s)^{2H}g(s,X(s))\mid_0^u + 2H\int_0^u (u-s)^{2H-1}g(s,X(s))ds\right\}du \\
&= -\frac{1}{2}\mathbb{E}\int_0^t g'(u,X(u))\left\{-u^{2H}g(0,X(0)) + 2H\int_0^u (u-s)^{2H-1}g(s,X(s))ds\right\}du \\
&= \frac{1}{2}\mathbb{E}\int_0^t u^{2H}g(0,X(0))g'(u,X(u))du \\
&- H\mathbb{E}\int_0^t g'(u,X(u))\left(\int_0^u (u-s)^{2H-1}g(s,X(s))ds\right)du \\
&= B_{31}+B_{32}.
\end{aligned} \tag{3.8}
$$

By using the chain rule for the first term of relation (3.8), we obtain

$$
\begin{aligned}
B_{31} &= \frac{1}{2}g(0,X(0))\mathbb{E}\left(\int_0^t u^{2H}g'(u,X(u))du\right) \\
&= \frac{1}{2}g(0,X(0))\left\{u^{2H}g(u,X(u))\mid_0^t - 2H\mathbb{E}\left(\int_0^t u^{2H-1}g(u,X(u))du\right)\right\} \\
&= \frac{1}{2}g(0,X(0))\left\{t^{2H}g(t,X(t)) - 2H\mathbb{E}\left(\int_0^t u^{2H-1}g(u,X(u))du\right)\right\} \\
&= \frac{1}{2}g(0,X(0))t^{2H}g(t,X(t)) - Hg(0,X(0))\mathbb{E}\left(\int_0^t u^{2H-1}g(u,X(u))du\right),
\end{aligned} \tag{3.9}
$$

and for the second term of (3.8), we obtain

$$
\begin{aligned}
B_{32} &= -H\mathbb{E}\int_0^t g'(u,X(u))\left(\int_0^u (u-s)^{2H-1}g(s,X(s))ds\right)du \\
&= -H\left(\mathbb{E}\int_0^u g(s,X(s))(u-s)^{2H-1}ds \times g(u,X(u))\mid_0^t\right. \\
&\left. -(2H-1)\mathbb{E}\int_0^t g(u,X(u))\left(\int_0^u (u-s)^{2H-2}g(s,X(s))ds\right)du\right) \\
&= -H\left(g(u,X(u))\mathbb{E}\int_0^u g(s,X(s))(u-s)^{2H-1}ds\mid_0^t\right. \\
&\left. +H(2H-1)\mathbb{E}\int_0^t g(u,X(u))\left(\int_0^u (u-s)^{2H-2}g(s,X(s))ds\right)du\right).
\end{aligned} \tag{3.10}
$$

For the term $B_{32}$ we can express similar evaluation as (3.10) too and derive that

$$B_{32} = -Hg(t, X(t))\mathbb{E}\left(\int_0^t g(s, X(s))(t-s)^{2H-1}ds\right)$$
$$+ H(2H-1)\mathbb{E}\int_0^t g(u, X(u))\left(\int_0^u (u-s)^{2H-2} g(s, X(s))ds\right)du. \quad (3.11)$$

Finally for the last term of (3.5), again we can compute that:

$$\begin{aligned}
B_4 &= -\frac{1}{2}\int_0^t\int_u^t \mathbb{E}\big(|g'(s, X(s))g'(u, X(u))|\big)(s-u)^{2H}dsdu \\
&= -\frac{1}{2}\mathbb{E}\int_0^t g'(u, X(u))\left(\int_u^t (s-u)^{2H} g'(s, X(s))ds\right)du \\
&= -\frac{1}{2}\mathbb{E}\int_0^t g'(u, X(u))\left\{(s-u)^{2H} g(s, X(s)) \mid_u^t -2H\int_u^t (s-u)^{2H-1} g(s, X(s))ds\right\}du \\
&= -\frac{1}{2}\mathbb{E}\int_0^t g'(u, X(u))\left\{(t-u)^{2H} g(t, X(t)) - 2H\int_u^t (s-u)^{2H-1} g(s, X(s))ds\right\}du \\
&= -\frac{1}{2}g(t, X(t))\mathbb{E}\int_0^t (t-u)^{2H} g'(u, X(u))du \\
&+ H\mathbb{E}\int_0^t g'(u, X(u))\left(\int_u^t (s-u)^{2H-1} g(s, X(s))ds\right)du \\
&= B_{41} + B_{42}. \qquad (3.12)
\end{aligned}$$

By using the chain rule for the term $B_{41}$, we obtain

$$\begin{aligned}
B_{41} &= -\frac{1}{2}g(t, X(t))\mathbb{E}\int_0^t (t-u)^{2H} g'(u, X(u))du \\
&= -\frac{1}{2}g(t, X(t))\mathbb{E}\left\{(t-u)^{2H} g(u, X(u)) \mid_0^t +2H\int_0^t (t-u)^{2H-1} g(u, X(u))du\right\} \\
&= \frac{1}{2}g(t, X(t))t^{2H} g(0, X(0)) - Hg(t, X(t))\mathbb{E}\left(\int_0^t (t-u)^{2H-1} g(u, X(u))du\right),
\end{aligned} \quad (3.13)$$

as well we can simply conclude that

$$B_{42} = -Hg(0, X(0))\mathbb{E}\left(\int_0^t s^{2H-1} g(s, X(s))ds\right)$$
$$- H(2H-1)\mathbb{E}\int_0^t g(u, X(u))\left(\int_u^t (s-u)^{2H-2} g(s, X(s))ds\right)du. \quad (3.14)$$

By summing up the terms $B_1, B_2, B_{31}, B_{32}, B_{41}$ and $B_{42}$ and ommitting the same terms we obtain

$$\begin{aligned}\mathbb{E}\left(|B|^2\right) &= t^{2H}\mathbb{E}\big(g^2(t,X(t))\big) \qquad (3.15)\\ &- 2H\mathbb{E}\big(g(t,X(t))\big)\mathbb{E}\left(\int_0^t g(u,X(u))\big(|t-u|^{2H-1}+|u|^{2H-1}\big)du\right)\\ &+ H(2H-1)\mathbb{E}\int_0^t g(u,X(u))\left(\int_0^u (u-s)^{2H-2}g(s,X(s))ds\right)du\\ &- H(2H-1)\mathbb{E}\int_0^t g(u,X(u))\left(\int_u^t (s-u)^{2H-2}g(s,X(s))ds\right)du. \qquad (3.16)\end{aligned}$$

By using the mean value theorem for stochastic functions discuused in [6], we can conclude for the the second term of (3.15)

$$\begin{aligned}&\mathbb{E}\big(g(t,X(t))\big)\mathbb{E}\left(\int_0^t g(u,X(u))\big(|t-u|^{2H-1}+|u|^{2H-1}\big)du\right)\\ &\quad=\mathbb{E}\big(g(t,X(t))\big)\mathbb{E}(g(c,X(c)))\mathbb{E}\left(\int_0^t \big(|t-u|^{2H-1}+|u^{2H-1}\big)du\right), \qquad (3.17)\end{aligned}$$

for ome constant $c\in(0,t)$. Now, if we impose $g(t,X(t))$ as a stochastic decreasing function, then we have

$$\begin{aligned}&2H\mathbb{E}\left(g(t,X(t))\right)\mathbb{E}\left(g(c,X(c))\right)\mathbb{E}\left(\int_0^t \big(|t-u|^{2H-1}+|u|^{2H-1}\big)du\right)\\ &\qquad\qquad\geq 2H\mathbb{E}\left(g^2(t,X(t))\right)\times\frac{t^{2H}}{H}, \qquad (3.18)\end{aligned}$$

so we can conclude that from (3.15), for some constant $C$ we have:

$$\begin{aligned}\mathbb{E}\left(|B|^2\right) &\leq -t^{2H}\mathbb{E}\big(g^2(t,X(t))\big)\\ &+ H(2H-1)\mathbb{E}\int_0^t g(u,X(u))\left(\int_0^u (u-s)^{2H-2}g(s,X(s))ds\right)du\\ &- H(2H-1)\mathbb{E}\int_0^t g(u,X(u))\left(\int_u^t (s-u)^{2H-2}g(s,X(s))ds\right)du\\ &\leq C\mathbb{E}\left(\int_0^t g(u,X(u))\left(\int_0^u (u-s)^{2H-2}g(s,X(s))\right.\right.\\ &\left.\left.+\int_u^t (u-s)^{2H-2}g(s,X(s))\right)ds\right)du\\ &= C\mathbb{E}\left(\int_0^t g(u,X(u))\big(\int_0^t (u-s)^{2H-2}g(s,X(s))\big)ds\right)du\\ &\leq C_1\mathbb{E}\left(\int_0^t g(u,X(u))\big(\int_0^t |u|^{2H-2}g(s,X(s))\big)ds\right)du. \qquad (3.19)\end{aligned}$$

Since $0 \leq u \leq t$, we can conclude that

$$\mathbb{E}\left(|B|^2\right) \leq C_1 \mathbb{E}\left(\int_0^t g(u, X(u))du\right)^2. \tag{3.20}$$

By using the Cauchy-schwarz inequality, we obtain

$$\mathbb{E}\left(|B|^2\right) \leq C_1 \mathbb{E}\left(\int_0^t g^2(u, X(u))du\right), \tag{3.21}$$

now by relation (3.2) and using the Gronwall inequality we can obtain the desired inequality of (3.1). □

## 4. Convergence Analysis of the FBM with Hurst Index

In this section we follow the [10], and we consider the convergence analysis of FBM with Hurst parameter $H$.
Define

$$\begin{cases} X^0(t) := X(0), \\ X^{n+1}(t) := X(0) + \int_0^t f(s, X^n(s))ds + \int_0^t g(s, X^n(s))dW^H, \end{cases} \tag{4.1}$$

for $n = 0, 1, \cdots$ and $0 \leq t \leq T$. Define also

$$d^n(t) := E(|X^{n+1}(t) - X^n(t)|^2). \tag{4.2}$$

We claim that

$$d^n(t) \leq \frac{(Mt)^{n+1}}{(n+1)!} \quad \text{for all} \quad n = 0, 1, \cdots, \quad 0 \leq t \leq T, \tag{4.3}$$

for some constant $M$, depending on $L$, $T$ and $X(0)$. Indeed for $n = 0$, and by the relation (3.21) which has been proven in section 3 we have:

$$\begin{aligned} d^0(t) &= \mathbb{E}\left(|X^1(t) - X^0(t)|^2\right) \\ &= \mathbb{E}\left(\left|\int_0^t f(s, X(0))ds + \int_0^t g(s, X(0))dW^H\right|^2\right) \\ &\leq 2\mathbb{E}\left(\left|\int_0^t L(1 + X(0))ds\right|^2\right) + 2\mathbb{E}\left(\int_0^t L^2(1 + |X(0)|^2)ds\right) \\ &\leq tM, \end{aligned} \tag{4.4}$$

for some large enough constant $M$. This confirms the claim for $n = 0$.

Next, to complete the proof by induction on $n - 1$, and by implementing the lemma in section 3, we omit the proceding of the proof and refer the reader to follow the (Chapter 5, Pages $92 - 94$) in [10].

**Remark 4.1.** *We consider the Mandelbrot-Van Ness representation of fractional Brownian motion $W_t^H$ in terms of Wiener integrals*

$$W_t^H = C_H \left\{ \int_{-\infty}^{t} \frac{dW_s^{\mathbb{P}}}{(t-s)^{\gamma}} - \int_{-\infty}^{0} \frac{dW_s^{\mathbb{P}}}{(-s)^{\gamma}} \right\}, \tag{4.5}$$

*where* $\gamma = 1/2 - H$ *with the choice* $C_H = \sqrt{\frac{2H\Gamma(3/2-H)}{\Gamma(H+1/2)\Gamma(2-2H)}}$.

Since, the right hand side of (4.5) is Ito integral, and it is proved to be martingale, so we can state that $W_t^H$ is martingale (see [4]).

## 5. Conclusion

In this paper, we are interested to investigate whether the FBM with Hurst index has bounded solution or convergent. We imploy the Definition 2.1 and using the chain rule as well by the important remark of mean value theorem for stochastic functions which has been proven in [6], we get a conclusion of boundedness of solution of FBM, if the diffusion function $g(t, X(t))$ is a decrasing function under some measurable probability. Also by using the proposed conclusion and following [10], we directly prove the convergence of the solution of FBM with any Hurst index $H \in (0, 1)$.

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*D. Ahmadian,*
*Faculty of Mathematical Sciences,*
*University of Tabriz, Tabriz, Iran.*
*E-mail address:* d.ahmadian@tabrizu.ac.ir

*and*

*O. Farkhondeh Rouz,*
*Faculty of Mathematical Sciences,*
*University of Tabriz, Tabriz, Iran.*
*E-mail address:* omid_farkhonde_7088@yahoo.com